

Logic of Quantum Mechanics for Information Technology Field

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ABSTRACT

Quantum mechanics is a branch of physics for a mathematical description of the particle wave, and it is applied to information technology such as quantum computer, quantum information, quantum network and quantum cryptography, etc. In 1936, Garrett Birkhoff and John von Neumann introduced the logic of quantum mechanics (quantum logic) in order to investigate projections on a Hilbert space. As another type of quantum logic, orthomodular implication algebra was introduced by Chajda et al. This algebra has the logical implication as a binary operation. In pure mathematics, there are many algebras such as Hilbert algebras, implicative models, implication algebras and dual BCK-algebras (DBCK-algebras), which have the logical implication as a binary operation. In this paper, we introduce the definitions and some properties of those algebras and clarify the relations between those algebras. Also, we define the implicative poset which is a generalization of orthomodular implication algebras and DBCK-algebras, and research properties of this algebraic structure.

Keywords: Hilbert algebras, implication algebras, DBCK-algebras, orthomodular implication algebras, implicative posets.

1. INTRODUCTION

Quantum mechanics is a branch of physics for a mathematical description of the particle wave, and it is applied to information technology such as quantum computer, quantum information, quantum network and quantum cryptography, etc. G. Birkhoff and J. von Neumann [1] introduced the quantum logic in order to investigate projections on a Hilbert space as a formulation of quantum mechanics. Husimi [2] proposed orthomodular law and orthomodular lattices [3] were studied to complement the quantum logic.

An *orthomodular lattice* is a lattice L which has an orthocomplementation $'$ and satisfies the orthomodular law:

$$x \leq y \Rightarrow x \vee (x' \wedge y) = y.$$

Finch introduced logical conjunctions and implications that are defined on an orthomodular lattice [4], [5], and some logical structures with implication " \rightarrow " were considered to describe the quantum logic. Chajda et al. [6] proposed orthomodular implication algebras as another type of quantum logic.

An *orthomodular implication algebra* is an algebraic system $(X, ;, 1)$ satisfying the following axioms.

$$(OMI1) \quad xx = 1,$$

$$(OMI2) \quad x(yx) = 1,$$

$$(OMI3) \quad (xy)x = x,$$

$$(OMI4) \quad (xy)y = (yx)x,$$

$$(OMI5) \quad (((xy)y)z)(xz) = 1,$$

$$(OMI6) \quad (((((((((xy)y)z)z)z)x)x)x)x)x)x = (((xy)y)z)z,$$

The binary operation " \cdot " in orthomodular implication algebra is an implication between propositions. There are many mathematical structures such as Hilbert algebras, implicative models, implication algebras and DBCK-algebras, which have logical implication as a binary operation, and those algebras have the similar notions and definitions.

Orthomodular lattices and Boolean algebras have a binary operation \vee and a unary operation $'$ (negation). We can define an implication " \rightarrow " using these operations. In orthomodular lattice L , if one define two implications " \rightarrow_1 " and " \rightarrow_2 " by

$$x \rightarrow_1 y = y \vee (x' \wedge y'),$$

$$x \rightarrow_2 y = x' \vee y,$$

then $(L, \rightarrow_1, 1)$ becomes an orthomodular implication algebra, but $(L, \rightarrow_2, 1)$ does not. In Boolean algebra B , both $(B, \rightarrow_1, 1)$ and $(B, \rightarrow_2, 1)$ are orthomodular implication algebra. Also, $(B, \rightarrow_2, 1)$ becomes a Hilbert algebra, an implication algebra and a DBCK-algebra. Hence those algebras can be viewed as generalizations of orthomodular lattices or Boolean algebras.

A *DBCK-algebra* is an algebraic system $(X, ;, 1)$ satisfying the following axioms.

$$(DBCK1) \quad (xy)((yz)(xz)) = 1,$$

$$(DBCK2) \quad x((xy)y) = 1,$$

$$(DBCK3) \quad xx = 1,$$

$$(DBCK4) \quad xy = 1 \text{ and } yx = 1 \text{ imply } x = y,$$

$$(DBCK5) \quad x1 = 1.$$

The notion of DBCK-algebra was studied and generalized in [7]-[9] as the dual concept of BCK-algebra [10], [11].

In this paper, we research the relations of algebras with implication which mentioned above. In section 2, we introduce the definition of implicative model [12]-[14] and two different definitions of Hilbert algebra [15]-[18], and show that those

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definitions are equivalent. Also, we show that every Hilbert algebra is a DBCK-algebra. In section 3, we show that commutative Hilbert algebras are equivalent to implication algebras. In section 4, we introduce an example of orthomodular implication algebra which is not DBCK-algebra, and we research the relation between orthomodular implication algebras and DBCK-algebras. In section 5, we define the implicative poset as a new algebraic system, and show that this algebra is a generalization of orthomodular implication algebras and DBCK-algebras.

2. HILBERT ALGEBRAS

The algebraic systems, named as Hilbert algebra or implicative model, have slightly different three definitions which were introduced in [12]-[14], [15]-[18].

Definition 2.1. [15], [18] A Hilbert algebra is an algebraic system $(H, \cdot, 1)$ of type $(2,0)$ satisfying the following axioms.

- (H1) $x(yx) = 1$,
- (H2) $(x(yz))(xy)(xz) = 1$,
- (H3) $xy = 1$ and $yx = 1$ imply $x = y$.

Lemma 2.2. [18] A Hilbert algebra $(H, \cdot, 1)$ has the following properties.

- (1) $x1 = 1$,
- (2) $xx = 1$,
- (3) $1x = x$.

If we define a binary relation “ \leq ” on a Hilbert algebra H by $x \leq y \iff xy = 1$

for any $x, y \in H$, then (H, \leq) is a poset with the greatest element 1 [18].

Lemma 2.3. [18] A Hilbert algebra $(H, \cdot, 1)$ has the following properties.

- (1) $x \leq y \implies zx \leq zy$ and $yz \leq xz$,
- (2) $x(yz) = y(xz)$,
- (3) $x(yz) = (xy)(xz)$.

Example 1. Let $H = \{1, a, b, c, 0\}$ on which the binary operation “ \cdot ” is defined by Table 1. Then $(H, \cdot, 1)$ is a Hilbert algebra, and the Hasse diagram of the poset (H, \leq) is given by Figure 1.

Table 1. Binary operation table of Example 1

\cdot	1	a	b	c	0
1	1	a	b	c	0
a	1	1	b	c	c
b	1	a	1	c	0
c	1	a	1	1	a
0	1	1	1	1	1

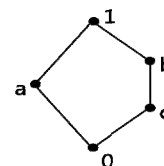


Fig. 1. Hasse diagram of Example 1

Lemma 2.4. A Hilbert algebra $(H, \cdot, 1)$ has the following properties.

- (1) $x \leq yx$,
- (2) $xy = x(xy)$,
- (3) $x \leq (xy)y$,
- (4) $xy \leq x((yz)z) = (yz)(xz)$,
- (5) $((xy)y)y = xy$,
- (6) $x \leq y$ and $x \leq yz \implies x \leq z$,
- (7) $x \leq y \implies x(yz) = y(xz) = xz$.

Proof. (1) is trivial from (H1).

(2) Let $x, y \in H$. Then by Lemma 2.3(3) and (2) and (3) of Lemma 2.2, we have

$$x(xy) = (xx)(xy) = 1(xy) = xy.$$

(3) Let $x, y \in H$. Then by Lemma 2.3(2) and Lemma 2.2(2), we have

$$x((xy)y) = (xy)(xy) = 1.$$

Hence $x \leq (xy)y$.

(4) Let $x, y, z \in H$. Then $y \leq (yz)z$ by (3) of this lemma, and we have

$$xy \leq x((yz)z) = (yz)(xz)$$

by (1) and (2) of Lemma 2.3.

(5) Let $x, y \in H$. Then $xy \leq ((xy)y)y$ by (3) of this lemma. Also, since $x \leq (xy)y$, we have

$$((xy)y)y \leq xy$$

by Lemma 2.3(1). Hence $((xy)y)y = xy$.

(6) Let $x \leq y$ and $x \leq yz$. Then $xy = 1$ and $x(yz) = 1$. This implies

$$xz = 1(xz) = (xy)(xz) = x(yz) = 1$$

by Lemma 2.2(3) and Lemma 2.3(3). Hence $x \leq z$.

(7) Let $x \leq y$. Then $xy = 1$. Hence by Lemma 2.3(3) and Lemma 2.2(3), we have

$$x(yz) = (xy)(xz) = 1(xz) = xz.$$

The following is a different definition of a Hilbert algebra.

Definition 2.5. [16], [17] A Hilbert algebra is an algebraic system $(H, \cdot, 1)$ of the type $(2,0)$ satisfying the following axioms.

- (D1) $xx = 1$,
- (D2) $1x = x$,
- (D3) $x(yz) = (xy)(xz)$,
- (D4) $(xy)((yx)x) = (yx)((xy)y)$.

Lemma 2.6. [16] Let $(H, \cdot, 1)$ be a Hilbert algebra in Definition 2.5. Then it has the following properties.

- (1) $x1 = 1$,
- (2) $xy = 1$ and $yx = 1$ imply $x = y$.

Theorem 2.7. Hilbert algebras of Definition 2.1 are equivalent to those of Definition 2.5.

Proof. Suppose that algebraic system $(H, \cdot, 1)$ satisfies the axioms of Definition 2.1. Then H satisfies (D1), (D2) and (D3) by (2) and (3) of Lemma 2.2 and Lemma 2.3(3) respectively. Let $x, y \in H$. Then by Lemma 2.3(3),

$$((yx)y)((yx)x) = (yx)(yx) = 1.$$

This implies $(yx)y \leq (yx)x$, and we have

$$\begin{aligned} (yx)((xy)y) &= (xy)((yx)y) && \text{(by Lemma 2.3(2))} \\ &\leq (xy)((yx)x) && \text{(by Lemma 2.3(1)).} \end{aligned}$$

Hence $(yx)((xy)y) \leq (xy)((yx)x)$. Interchanging the role of x and y , we have

$$(xy)((yx)x) \leq (yx)((xy)y).$$

Hence H satisfies the axiom (D4):

$$(xy)((yx)x) = (yx)((xy)y).$$

Conversely, suppose that algebraic system $(H, \cdot, 1)$ satisfies the axioms of Definition 2.5. Then by (D3), (D1) and Lemma 2.6(1), we have (H1):

$$x(yx) = (xy)(xx) = (xy)1 = 1.$$

Since $x(yz) \leq (xy)(xz)$ by (D3), we have (H2):

$$(x(yz))((xy)(xz)) = 1.$$

Also, H satisfies (H3) by Lemma 2.6(2).

The following implicative model is a different name for a Hilbert algebra.

Definition 2.8. [12]-[14] An *implicative model* is an algebraic system $(H, \cdot, 1)$ of the type (2,0) satisfying the following axioms.

- (I1) $x(yx) = 1$,
- (I2) $(x(yz))((xy)(xz)) = 1$,
- (I3) $x1 = 1$,
- (I4) $xy = 1$ and $yx = 1$ imply $x = y$.

The axioms for implicative model of Definition 2.8 are not independent. As (1) of Lemma 2.2, (I3) can be induced from (I1) and (I4). Hence Hilbert algebra of Definition 2.1 and implicative model have the same axioms.

We introduced the definition of DBCK-algebra in section 1. Hilbert algebras and DBCK-algebras have the following relation.

Theorem 2.9. Every Hilbert algebra is a DBCK-algebra.

Proof. Suppose that $(H, \cdot, 1)$ is a Hilbert algebra. Then it satisfies (DBCK1) and (DBCK2) by (4) and (3) of Lemma 2.4, and satisfies (DBCK3) and (DBCK5) by (2) and (1) of Lemma 2.2. Also, (DBCK4) is the same axiom with (H3). Hence H is a DBCK-algebra.

Definition 2.10. A DBCK-algebra $(X, \cdot, 1)$ is said to be *positive implicative* if it satisfies $x(yz) = (xy)(xz)$ for every $x, y, z \in X$, and said to be *implicative* if it satisfies $(xy)x = x$ for every $x, y \in X$.

We will denote the class of all Hilbert algebras by **HIL**, the class of all DBCK-algebras by **DBCK**, the class of all positive implicative DBCK-algebras by **piDBCK** and the class of all implicative DBCK-algebras by **iDBCK**.

A. Dudek [19] showed that an algebra $(H, \cdot, 1)$ is a Hilbert algebra if and only if it a positive implicative DBCK-algebra.

Hence from Theorem 2.9, we have a following relation:

$$\mathbf{HIL} = \mathbf{piDBCK} \subsetneq \mathbf{DBCK}.$$

There is an example of DBCK-algebra which is not positive implicative.

Example 2. Let $X = \{1, a, b, 0\}$ on which “ \cdot ” is defined by Table 2. Then $(X, \cdot, 1)$ is a DBCK-algebra, and the Hasse diagram of the poset (X, \leq) is given by Figure 2. This algebra is not positive implicative, because

$$b(a0) = bb = 1 \neq a = 1a = (ba)(b0).$$

Table 2. Binary operation table of Example 2

\cdot	1	a	b	0
1	1	a	b	0
a	1	1	b	b
b	1	1	1	a
0	1	1	1	1



Fig. 2. Hasse diagram of Example 2

3. COMMUTATIVE HILBERT ALGEBRAS AND IMPLICATION ALGEBRAS

Lemma 3.1. A Hilbert algebra $(H, \cdot, 1)$ has the following properties.

- (1) $x \leq (xy)y$ and $y \leq (xy)y$,
- (2) $x \leq y$ implies $(xy)y = y$.

Proof. (1) It is clear from (3) and (1) of Lemma 2.4.

(2) Let $x \leq y$. Then $xy = 1$, and $(xy)y = 1y = y$ by Lemma 2.2(3).

Let $(H, \cdot, 1)$ be a Hilbert algebra. Then $(xy)y$ is an upper bound of x and y for any $x, y \in H$ by Lemma 3.1(1), but it is not the least upper bound of x and y . Because in Example 1,

$$(b0)0 = 00 = 1 \neq b = b \vee 0.$$

Definition 3.2. A Hilbert algebra H is said to be *commutative* if it satisfies $(xy)y = (yx)x$ for every $x, y \in H$.

Example 3. The Hilbert algebras $(H, \cdot, 1)$ of Example 1 is not commutative, because

$$(cb)b = 1b = b \neq 1 = cc = (bc)c.$$

Theorem 3.3. Every commutative Hilbert algebra is a \vee -semilattice with $x \vee y = (xy)y$.

Proof. Suppose that $(H, \cdot, 1)$ is a commutative Hilbert algebra and $x, y \in H$. Then $(xy)y$ is an upper bound of x and y .

To show that $(xy)y$ is the least upper bound of x and y , let z be an upper bound of x and y , that is, $x \leq z$ and $y \leq z$. Then

$$xz = yz = 1.$$

Since $x \leq (zx)x$ by Lemma 3.1(1), we have $((zx)x)y \leq xy$ and $(xy)y \leq (((zx)x)y)y$ by Lemma 2.3(1). Also, since H is commutative, we have $((zx)x)y = (((zx)z)y)y = ((1z)y)y = (zy)y = (yz)z = 1z = z$. This implies $(xy)y \leq z$. Hence $(xy)y$ is the least upper bound of x and y .

Definition 3.4. [20] An *implication algebra* is an algebraic system (A, \cdot) of type 2 satisfying the following axioms.

- (IA1) $(xy)x = x$,
- (IA2) $(xy)y = (yx)x$,
- (IA3) $x(yz) = y(xz)$.

Lemma 3.5. [20] An implication algebra (A, \cdot) has the following properties.

- (1) $x(xy) = xy$,
- (2) $xx = (xy)(xy)$,
- (3) $xx = 1$,
- (4) $1x = x$,
- (5) $x1 = 1$,
- (6) $x(yx) = 1$,
- (7) $((xy)y)y = xy$,
- (8) $((xy)y)x = yx$,
- (9) $x((xy)y) = y((xy)y) = 1$,
- (10) $(xy)(yx) = yx$.

Let (A, \cdot) be an implication algebra. If we define a binary relation " \leq " on A by

$$x \leq y \iff xy = 1,$$

then (A, \leq) is a poset with the greatest element 1 [20].

Lemma 3.6. [20] An implication algebra (A, \cdot) has the following properties.

$$x \leq y \implies zx \leq zy \text{ and } yz \leq xz.$$

Lemma 3.7. An implication algebra (A, \cdot) has the following properties.

- (1) $xy \leq (zx)(zy)$,
- (2) $xy \leq (yz)(xz)$,
- (3) $x \leq y \iff y = ux$ for some $u \in A$,
- (4) $x(yz) = (xy)(xz)$.

Proof. (1) Let $x, y, z \in A$. Then we have

$$\begin{aligned} (xy)((zx)(zy)) &= (zx)((xy)(zy)) \quad (\text{by (IA3)}) \\ &= (zx)(z((xy)y)) \quad (\text{by (IA2)}) \\ &= (zx)((yx)x) \quad (\text{by (IA2)}) \\ &= (zx)((yx)(zx)) \quad (\text{by (IA3)}) \\ &= 1 \quad (\text{by Lemma 3.5(6)}). \end{aligned}$$

Hence $xy \leq (zx)(zy)$.

(2) Let $x, y, z \in A$. Then we have

$$(xy)((yz)(xz)) = (yz)((xy)(xz)) = 1$$

by (IA3) and (1) of this lemma. Hence $xy \leq (yz)(xz)$.

(3) Let $x \leq y$ and $u = yx$. Then $xy = 1$, and by (IA2), we have $ux = (yx)x = (xy)y = 1y = y$.

Conversely, let $y = ux$ for some $u \in A$. Then $xy = x(ux) = 1$ by Lemma 3.5(6). Hence $x \leq y$.

(4) Let $x, y, z \in A$. Then $y(xy) = 1$ by Lemma 3.5(6). That is, $y \leq xy$. This implies $(xy)z \leq yz$ by Lemma 3.6, and we have

$$(xy)(xz) = x((xy)z) \leq x(yz)$$

by (IA3) and Lemma 3.6. Also, since $(x(xz))(xz) = (xz)(xz) = 1$ by Lemma 3.5(1), we have

$$\begin{aligned} (xy)(xz) &= 1((xy)(xz)) \\ &= ((x(xz))(xz))((xy)(xz)) \\ &\geq (xy)(x(xz)) \quad (\text{by Lemma 3.7(2)}) \\ &\geq y(xz) \quad (\text{by Lemma 3.7(1)}) \\ &= x(yz), \end{aligned}$$

and this implies $x(yz) \leq (xy)(xz)$. Hence $x(yz) = (xy)(xz)$.

Theorem 3.8. Every implication algebra is a commutative Hilbert algebra.

Proof. Let (A, \cdot) be an implication algebra. Then it satisfies properties (H1) and (H2) by Lemma 3.5(6) and Lemma 3.7(4) respectively. Since (A, \leq) is a poset satisfying $x \leq y \iff xy = 1$, it satisfies (H3). Also, it is commutative by (IA2).

According to the paper [18], every commutative Hilbert algebra is an implication algebra. Hence implication algebras are equivalent to commutative Hilbert algebras by Theorem 3.8.

If we denote the class of all implication algebras by **IA** and the class of all commutative Hilbert algebras by **cHIL**, then we have a relation:

$$\mathbf{IA} = \mathbf{cHIL}.$$

In [7], [21], the authors proved implication algebras are equivalent to implicative DBCK-algebras. Hence we have a following relation:

$$\mathbf{iDBCK} = \mathbf{IA} = \mathbf{cHIL} \subseteq \mathbf{HIL} = \mathbf{piDBCK} \subseteq \mathbf{DBCK}.$$

The inequality of **cHIL** and **HIL** follows from Example 3.

4. ORTHOMODULAR IMPLICATION ALGEBRAS AND DBCK-ALGEBRAS

The definition of orthomodular implication algebra was introduced in section 1. Orthomodular implication algebra is a generalization of implication algebras, and DBCK-algebra is a generalization of Hilbert algebras. In this section, we research the relation between orthomodular implication algebra and DBCK-algebra.

In an orthomodular implication algebra $(X, \cdot, 1)$, a binary relation " \leq " is defined by

$$x \leq y \iff xy = 1,$$

and it satisfies the following lemma.

Lemma 4.1. [6] An orthomodular implication algebra has the following properties.

- (1) $x \leq x$,
- (2) $x \leq y$ and $y \leq x$ imply $x = y$,
- (3) $x \leq y$ and $y \leq z$ imply $x \leq z$,
- (4) $x \leq 1$,
- (5) $1x = x$,
- (6) $x \leq (xy)y$ and $y \leq (xy)x$,
- (7) $x \leq z$ and $y \leq z$ imply $(xy)y \leq z$,
- (8) $x \leq y$ implies $yz \leq xz$.

From (1)-(4) of Lemma 4.1, an orthomodular implication algebra $(X, \cdot, 1)$ is a poset with the partial order \leq and the greatest element 1, and from (6) and (7) of Lemma 4.1, (X, \vee)

is a join-semilattice with $x \vee y = (xy)y$.

Lemma 4.2. [6] (1) Every implication algebra is an orthomodular implication algebra.

(2) An algebra $(X, \cdot, 1)$ is an implication algebra if and only if it is an orthomodular implication algebra satisfying the condition (IA3): $x(yz) = y(xz)$.

If we denote the class of all orthomodular implication algebras by **OMI**, then we have a relation:

$$\mathbf{IA} \subsetneq \mathbf{OMI}.$$

In a DBCK-algebra $(X, \cdot, 1)$, a binary relation “ \leq ” is defined by

$$x \leq y \Leftrightarrow xy = 1,$$

and it satisfies the following lemma.

Lemma 4.3. [8] A DBCK-algebra has the following properties.

- (1) $xy \leq (yz)(xz)$,
- (2) $x \leq (xy)y$,
- (3) $x \leq x$,
- (4) $x \leq y$ and $y \leq x$ imply $x = y$,
- (5) $x \leq 1$,
- (6) $x \leq y$ and $y \leq z$ imply $x \leq z$,
- (7) $1x = x$,
- (8) $x(yz) = y(xz)$,
- (9) $((xy)y)y = xy$.

From (3)-(6) of Lemma 4.3, a DBCK-algebra X is also a poset with the partial order \leq and the greatest element 1.

There is an example of orthomodular implication algebra which is not DBCK-algebra.

Example 4. Let $X = \{0, a, b, c, d, e, f, g, h, 1\}$ be a set with a binary operation “ \cdot ” defined by Table 3. Then $(X, \cdot, 1)$ is an orthomodular implication algebra with Hasse diagram of Figure 3, but this algebra $(X, \cdot, 1)$ is not DBCK-algebra, because

$$(a0)((0g)(ag)) = b(1g) = bg = g \neq 1.$$

Table 3. Operation table of Example 4.

\cdot	1	a	b	c	d	e	f	g	h	0
1	1	a	b	c	d	e	f	g	h	0
a	1	1	b	b	1	b	1	g	h	b
b	1	a	1	f	d	d	f	g	h	a
c	1	d	1	1	d	d	1	g	h	d
d	1	f	b	c	1	b	f	g	h	c
e	1	f	1	f	1	1	f	g	h	f
f	1	d	b	b	d	e	1	g	h	e
g	1	a	b	c	d	e	f	1	h	h
h	1	a	b	c	d	e	f	g	1	g
0	1	1	1	1	1	1	1	1	1	1

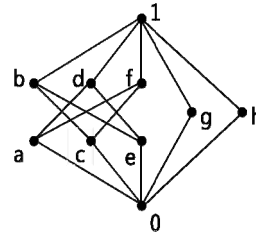


Fig. 3. Hasse diagram of Example 4.

There is a DBCK-algebra which is not orthomodular implication algebra.

Example 5. The Hilbert algebra $(H, \cdot, 1)$ of Example 1 is a DBCK-algebra, but not orthomodular implication algebra since it does not satisfy the axiom (OMI4). In fact,

$$(cb)b = 1b = b \neq 1 = cc = (bc)c.$$

From Example 4 and Example 5, we have a relation:

$$\mathbf{OMI} \not\subseteq \mathbf{DBCK} \text{ and } \mathbf{DBCK} \not\subseteq \mathbf{OMI}.$$

Definition 4.4. A DBCK-algebra $(X, \cdot, 1)$ is said to be commutative if it satisfies $(xy)y = (yx)x$ for every $x, y \in X$.

Theorem 4.5. An algebra is an implication algebra if and only if it is an implicative and commutative DBCK-algebra.

Proof. Suppose that $(X, \cdot, 1)$ is an implication algebra. Then X is a commutative Hilbert algebra by Theorem 3. 8. Hence it is a commutative DBCK-algebra by Theorem 2. 9. Also, by (IA1), it is implicative. Conversely, suppose that $(X, \cdot, 1)$ is an implicative and commutative DBCK-algebra. Then it satisfies (IA1) and (IA2). Also, it satisfies (IA3) from Lemma 4.3(8). Hence $(X, \cdot, 1)$ is an implication algebra.

Corollary 4.6. Every implicative and commutative DBCK-algebra is an orthomodular implication algebra.

Proof. It is clear from Theorem 4.5 and Lemma 4.2(1).

Corollary 4.7. An algebra $(X, \cdot, 1)$ is an implicative and commutative DBCK-algebra if and only if it is an orthomodular implication algebra satisfying the condition (IA3): $x(yz) = y(xz)$.

Proof. It is clear from Theorem 4.5 and Lemma 4.2(2).

Theorem 4.8. An algebra $(X, \cdot, 1)$ is an implication algebra if and only if it is an orthomodular implication algebra and a DBCK-algebra.

Proof. It is clear that an implication algebra is an orthomodular implication algebra and DBCK-algebra by Lemma 4.2 and Theorem 4.5. Conversely, let $(X, \cdot, 1)$ be an orthomodular implication algebra and a DBCK-algebra. Then X is an implicative and commutative DBCK-algebra by (OMI3) and (OMI4). Hence it is an implication algebra by Theorem 4.5.

From Theorem 4.8, we have a relation:

$$\text{iDBCK} = \text{IA} = \text{cHIL} = \text{OMI} \cap \text{DBCK}.$$

5. IMPLICATIVE POSETS

In this section, we define the implicative poset which is a generalization of orthomodular implication algebras and DBCK-algebras.

Definition 5.1. An *implicative poset* is an algebraic system $(X, \cdot, 1)$ of type $(2,0)$ satisfying the following axioms.

- (PA1) $x1 = 1,$
- (PA2) $1x = x,$
- (PA3) $xy = 1$ and $yx = 1$ imply $x = y,$
- (PA4) $xy = 1$ implies $(yz)(xz) = 1.$

The axioms (PA1)-(PA4) of Definition 5.1 are independent as the following example.

Example 6. Let $X = \{1, a, 0\}$ and define four binary operations $\circ_1, \circ_2, \circ_3$ and \circ_4 on X by Table4. Then (X, \circ_1) satisfies (PA1), (PA2) and (PA3), but not (PA4) since

$$11 = 1, \text{ but } (10)(10) = 00 = 0 \neq 1.$$

(X, \circ_2) satisfies (PA1), (PA2) and (PA4), but not (PA3) since

$$0a = 1 \text{ and } a0 = 1, \text{ but } 0 \neq a.$$

(X, \circ_3) satisfies (PA1), (PA3) and (PA4), but not (PA2) since

$$1a = 0 \neq a.$$

(X, \circ_4) satisfies (PA2), (PA3) and (PA4), but not (PA1) since

$$a1 = a \neq 1.$$

Table 4. Four binary operations $\circ_1, \circ_2, \circ_3$ and \circ_4 on X

\circ_1	1	a	0	\circ_2	1	a	0
1	1	a	0	1	1	a	0
a	1	1	0	a	1	1	1
0	1	1	0	0	1	1	1
\circ_3	1	a	0	\circ_4	1	a	0
1	1	0	0	1	1	a	0
a	1	1	0	a	a	1	a
0	1	0	1	0	0	a	1

Lemma 5.2. Let $(X, \cdot, 1)$ be a implicative poset. If a binary relation " \leq " on X is defined by

$$x \leq y \iff xy = 1$$

for any $x, y \in X$, then the relation \leq satisfies the following properties.

- (1) $x \leq 1,$
- (2) $x \leq y$ and $y \leq x$ imply $x = y,$
- (3) $x \leq x,$
- (4) $x \leq y$ and $y \leq z$ imply $x \leq z,$
- (5) $x \leq yx,$
- (6) $x \leq y \implies yz \leq xz.$

Proof. (1) and (2) are clear from (PA1) and (PA3).

(3) Let $x \in X$. Since $11 = 1$ by (PA1), we have

$$xx = (1x)(1x) = 1$$

by (PA2) and (PA4). Hence $x \leq x$.

(4) Let $x \leq y$ and $y \leq z$. Then $xy = 1$ and $yz = 1$. This implies

$$xz = 1(xz) = (yz)(xz) = 1$$

by (PA2) and (PA4). Hence $x \leq z$.

(5) Let $x, y \in X$. Since $y1 = 1$ by (PA1), we have

$$x(yx) = (1x)(yx) = 1$$

by (PA4). Hence $x \leq yx$.

(6) Let $x \leq y$. Then $xy = 1$. This implies $(yz)(xz) = 1$ by (PA4). Hence $yz \leq xz$.

The binary relation \leq defined in Lemma 5.2 is a partial order and 1 is the greatest element in X by (1)-(4) of Lemma 5.2. That is, (X, \leq) is a poset with the greatest element 1.

Theorem 5.3. Let (P, \leq) be a poset with a greatest element 1. If we define a binary operation " \circ " on P by

$$x \circ y = \begin{cases} 1, & x \leq y, \\ y, & \text{otherwise,} \end{cases}$$

then (P, \circ) is an implicative poset.

Proof. (PA1) and (PA2) are clear from the definition of binary operation \circ . Let $x \circ y = 1$ and $y \circ x = 1$. Then $x \leq y$ and $y \leq x$. Since (P, \leq) is a poset, $x = y$. To show (PA4), suppose that $x \circ y = 1$. Then $x \leq y$. For any $z \in P$, let $x \leq z$. Then $x \circ z = 1$, and we have

$$(y \circ z) \circ (x \circ z) = (y \circ z) \circ 1 = 1$$

since $y \circ z \leq 1$. Let $x \not\leq z$. If $y \leq z$, then $x \leq z$ since $x \leq y$. It is impossible. This implies $y \not\leq z$, and we have

$$(y \circ z) \circ (x \circ z) = z \circ z = 1.$$

Hence it satisfies (PA4): $x \circ y = 1$ implies $(y \circ z) \circ (x \circ z) = 1$.

Theorem 5.4. Every DBCK-algebra is an implicative poset.

Proof. Suppose that $(X, \cdot, 1)$ is a DBCK-algebra. Then it satisfies (PA1) and (PA3) by (DBCK5) and (DBCK4) respectively, and satisfies (PA2) by Lemma 4.3(7). To show (PA4), let $xy = 1$. Then by Lemma 4.3(7) and (DBCK1),

$$(yz)(xz) = 1(yz)(xz) = (xy)((yz)(xz)) = 1.$$

Hence it satisfies (PA4), and $(X, \cdot, 1)$ is an implicative poset.

Theorem 5.5. Every orthomodular implication algebra is an implicative poset.

Proof. Suppose that $(X, \cdot, 1)$ is an orthomodular implication algebra. Then (PA1), (PA2), (PA3) and (PA4) are directly derived from (4), (5), (2) and (8) of Lemma 4.1 respectively.

If we denote the class of all implicative posets by **IP**, then we have a relation:

$$\text{DBCK} \subsetneq \text{IP} \text{ and } \text{OMI} \subsetneq \text{IP}.$$

There is an example of implicative poset which are neither DBCK-algebra nor orthomodular implication algebra.

Example 7. Let $X = \{1, a, b, 0\}$, which has a binary operation " \cdot " defined by Table 5. Then $(X, \cdot, 1)$ is an implicative poset, which is not DBCK-algebra since

$$(ba)((a0)(b0)) = a(ab) = ab = b \neq 1,$$

and not orthomodular implication algebra since

$$(a0)a = aa = 1 \neq a.$$

Table 5. Binary operation of Example 7

·	1	a	b	0
1	1	a	b	0
a	1	1	b	a
b	1	a	1	b
0	1	1	1	1

Theorem 5.6. An algebra $(X, \cdot, 1)$ is a DBCK-algebra if and only if it is an implicative poset satisfying the following conditions.

- (E) $x(yz) = y(xz)$,
- (O) $xy = 1$ implies $(zx)(zy) = 1$.

Proof. Let $(X, \cdot, 1)$ be a DBCK-algebra. Then it is an implicative poset by Theorem 5.4, and it satisfies the condition (E) by Lemma 4.3(8). To show the condition (O), let $xy = 1$. Then

$$(zx)(zy) = (xy)((zx)(zy)) = (zx)((xy)(zy)) = 1$$

by (7) and (8) of Lemma 4.3 and (DBCK1). Hence it satisfies (O). Conversely, let $(X, \cdot, 1)$ be an implicative poset satisfying the given conditions, and let $x, y, z \in X$. Then we have

$$x((xy)y) = (xy)(xy) = 1$$

by (E). Hence it satisfies (DBCK2): $x \leq (xy)y$. This implies $y \leq (yz)z$, that is, $y((yz)z) = 1$. Hence we have (DBCK1):

$$(xy)((yz)(xz)) = (xy)x((yz)z) = 1$$

by (E) and (O). Also, (DBCK3), (DBCK4) and (DBCK5) are derived from by Lemma 5.2(3), (PA3) and (PA1) respectively. Hence $(X, \cdot, 1)$ is a DBCK-algebra.

6. CONCLUSION

Orthomodular lattices were studied for the logic of quantum mechanics, and an orthomodular implication algebra was introduced as a generalization of orthomodular lattices. This algebra has the implication as a binary operation. We researched some properties and the relations of algebras with implication as a binary operation such as Hilbert algebras, implicative models, implication algebras and DBCK-algebras. Figure 4 depicts the relations of those algebras. As DBCK-algebras have the dual notion of BCK-algebras, BCK-algebras and Hilbert algebras have been studied in many literatures. Hence those algebras seem to be used for the study of orthomodular implication algebras for quantum logic. Also, we define the implicative poset as generalizations of DBCK-algebras and orthomodular implication algebras. Hence the implicative posets can be used to characterize algebras with implication, containing the algebras mentioned above. Also, as a new logical system, the implicative posets can be applied to the information technology using quantum mechanics.

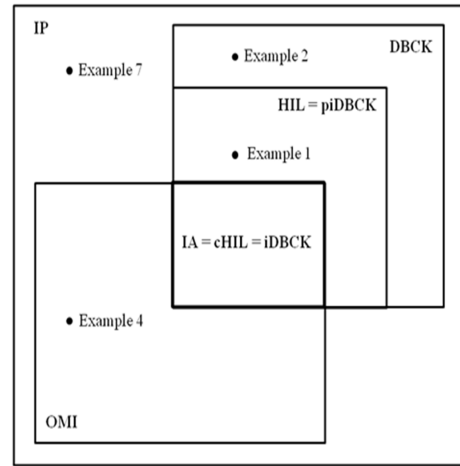


Fig. 4. the relations of algebras with implication.

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