

# Characterization of New Two Parametric Generalized Useful Information Measure

**Ashiq Hussain Bhat \***

Post Graduate Department of Statistics  
University of Kashmir, India  
E-mail: [ashiqhb14@gmail.com](mailto:ashiqhb14@gmail.com)

**M. A. K. Baig**

Post Graduate Department of Statistics  
University of Kashmir, India  
E-mail: [baigmak@gmail.com](mailto:baigmak@gmail.com)

## ABSTRACT

In this paper we define a two parametric new generalized useful average code-word length  $L_{\alpha}^{\beta}(P; U)$  and its relationship with two parametric new generalized useful information measure  $H_{\alpha}^{\beta}(P; U)$  has been discussed. The lower and upper bound of  $L_{\alpha}^{\beta}(P; U)$ , in terms of  $H_{\alpha}^{\beta}(P; U)$  are derived for a discrete noiseless channel. The measures defined in this communication are not only new but some well known measures are the particular cases of our proposed measures that already exist in the literature of useful information theory. The noiseless coding theorems for discrete channel proved in this paper are verified by considering Huffman and Shannon-Fano coding schemes on taking empirical data. Also we study the monotonic behavior of  $H_{\alpha}^{\beta}(P; U)$  with respect to parameters  $\alpha$  and  $\beta$ . The important properties of  $H_{\alpha}^{\beta}(P; U)$  have also been studied.

**Keywords:** Shannon's entropy, codeword length, useful information measure, Kraft inequality, Holder's inequality, Huffman codes, Shannon-Fano codes, Noiseless coding theorem

---

## Open Access

**Accepted date:** November 28, 2016  
**Received date:** July 3, 2016

**\*Corresponding Author:** Ashiq Hussain Bhat  
Research scholar  
Post Graduate Department of Statistics  
University of Kashmir, India  
E-mail: [ashiqhb14@gmail.com](mailto:ashiqhb14@gmail.com)

All JISTaP content is Open Access, meaning it is accessible online to everyone, without fee and authors' permission. All JISTaP content is published and distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/3.0/>). Under this license, authors reserve the copyright for their content; however, they permit anyone to unrestrictedly use, distribute, and reproduce the content in any medium as far as the original authors and source are cited. For any reuse, redistribution, or reproduction of a work, users must clarify the license terms under which the work was produced.

## 1. INTRODUCTION / LITERATURE REVIEW

The growth of telecommunication in the early twentieth century led several researchers to study the information control of signals; the seminal work of Shannon (1948), based on papers by Nyquists (1924; 1928) and Hartley (1928), rationalized these early efforts into a coherent mathematical theory of communication and initiated the area of research now known as information theory. The central paradigm of classical information theory is the engineering problem of the transmission of information over a noisy channel. The most fundamental results of this theory are Shannon's source coding theorem which establishes that on average the number of bits needed to represent the result of an uncertain event is given by its entropy; and Shannon's noisy-channel coding theorem which states that reliable communication is possible over noisy channels provided that the rate of communication is below a certain threshold, called the channel capacity. Information theory is a broad and deep mathematical theory with equally broad and deep applications, amongst which is the vital field of coding theory. Information theory is a new branch of probability and statistics with extensive potential application to communication systems. The term information theory does not possess a unique definition. Broadly speaking, information theory deals with the study of problems concerning any system. This includes information processing, information storage, and decision making. In a narrow sense, information theory studies all theoretical problems connected with the transmission of information over communication channels. This includes the study of uncertainty (information) measure and various practical and economical methods of coding information for transmission.

It is a well-known fact that information measures are important for practical applications of information processing. For measuring information, a general approach is provided in a statistical framework based on information entropy introduced by Shannon (1948) as a measure of information. The Shannon entropy satisfies some desirable axiomatic requirements and also it can be assigned operational significance in important practical problems, for instance in coding and telecommunication. In coding theory, usually we come across the problem of efficient coding of messages to be sent over a noiseless channel where our concern is to maximize the

number of messages that can be sent through a channel in a given time. Therefore, we find the minimum value of a mean codeword length subject to a given constraint on codeword lengths. As the codeword lengths are integers, the minimum value lies between two bounds, so a noiseless coding theorem seeks to find these bounds which are in terms of some measure of entropy for a given mean and a given constraint. Shannon (1948) found the lower bounds for the arithmetic mean by using his own entropy. Campbell (1965) defined his own exponentiated mean and by applying Kraft's (1949) inequality, found lower bounds for his mean in terms of Renyi's (1961) measure of entropy. Longo (1976) developed lower bound for useful mean codeword length in terms of weighted entropy introduced by Belis and Guiasu (1968). Guiasu and Picard (1971) proved a noiseless coding theorem by obtaining lower bounds for another useful mean code-word length. Gurdial and Pessoa (1977) extended the theorem by finding lower bounds for useful mean codeword length of order  $\alpha$ ; also various authors like Jain and Tuteja (1989), Taneja et al (1985), Hooda and Bhaker (1997), and Khan et al (2005) have studied generalized coding theorems by considering different generalized 'useful' information measures under the condition of unique decipherability.

In this paper we define a new two parametric generalized useful average code-word length  $L_{\alpha}^{\beta}(P; U)$  and discuss its relationship with new two parametric generalized useful information measure  $H_{\alpha}^{\beta}(P; U)$ . The lower and upper bound of  $L_{\alpha}^{\beta}(P; U)$ , in terms of  $H_{\alpha}^{\beta}(P; U)$  are derived for a discrete noiseless channel in Section 3. The measures defined in this communication are not only new but also generalizations of certain well known measures in the literature of useful information theory. In Section 4, the noiseless coding theorems for discrete channels proved in this paper are verified by considering Huffman and Shannon-Fano coding schemes using empirical data. In Section 5, we study the monotonic behavior of  $H_{\alpha}^{\beta}(P; U)$  with respect to parameters  $\alpha$  and  $\beta$ . Several other properties of  $H_{\alpha}^{\beta}(P; U)$  are studied in Section 6.

## 2. BASIC CONCEPTS

Let  $X$  be a finite discrete random variable or finite source taking values  $x_1, x_2, \dots, x_n$  with respective probabilities  $P = (p_1, p_2, \dots, p_n), p_i \geq 0 \forall i = 1, 2, \dots, n$  and

$\sum_{i=1}^n p_i = 1$ . Shannon (1948) gives the following measure of information and calls it entropy.

$$H(P) = -\sum_{i=1}^n p_i \log p_i \quad (1.1)$$

The measure (1.1) serves as a suitable measure of entropy. Let  $p_1, p_2, p_3, \dots, p_n$  be the probabilities of  $n$  code-words to be transmitted and let their lengths  $l_1, l_2, \dots, l_n$  satisfy Kraft (1949) inequality,

$$\sum_{i=1}^n D^{-l_i} \leq 1 \quad (1.2)$$

For uniquely decipherable codes, Shannon (1948) showed that for all codes satisfying (1.2), the lower bound of the mean codeword length,

$$L(P) = \sum_{i=1}^n p_i l_i \quad (1.3)$$

lies between  $H(P)$  and  $H(P)+1$ , where  $D$  is the size of code alphabet.

Shannon's entropy (1.1) is indeed a measure of uncertainty and is treated as information supplied by a probabilistic experiment. This formula gives us the measure of information as a function of the probabilities only in which various events occur without considering the effectiveness or importance of the events. Belis and Guiasu (1968) remarked that a source is not completely specified by the probability distribution  $P$  over the source alphabet  $X$  in the absence of quality character. They enriched the usual description of the information source (i.e., a finite source alphabet and finite probability distribution) by introducing an additional parameter measuring the utility associated with an event according to their importance or utilities in view of the experimenter.

Let  $U = (u_1, u_2, \dots, u_n)$  be the set of positive real numbers, where  $u_i$  is the utility or importance of outcome  $x_i$ . The utility, in general, is independent of  $p_i$ , i.e., the probability of encoding of source symbol  $x_i$ . The information source is thus given by

$$S = \begin{bmatrix} X_1 & X_2 & \dots & X_n \\ p_1 & p_2 & \dots & p_n \\ u_1 & u_2 & \dots & u_n \end{bmatrix}, u_i > 0, p_i \geq 0, \sum_{i=1}^n p_i = 1 \quad (1.4)$$

We call (1.4) a Utility Information Scheme. Belis and Guiasu (1968) introduced the following quantitative - qualitative measure of information for this scheme.

$$H(P, U) = -\sum_{i=1}^n u_i p_i \log p_i \quad (1.5)$$

and call it as 'useful' entropy. The measure (1.5) can be taken as satisfactory measure for the average quantity of 'valuable' or 'useful' information provided by the information source (1.4). Guiasu and Picard (1971) considered the problem of encoding the letter output by the source (1.4) by means of a single letter prefix code whose codeword's  $c_1, c_2, \dots, c_n$  have lengths  $l_1, l_2, \dots, l_n$  respectively and satisfy the Kraft's inequality (1.2), they introduced the following quantity

$$L(P; U) = \frac{\sum_{i=1}^n u_i p_i l_i}{\sum_{i=1}^n u_i p_i} \quad (1.6)$$

and call it as 'useful' mean length of the code. Further, they derived a lower bound for (1.6). However, Longo (1976) interpreted (1.6) as the average transmission cost of the letters  $x_i$  with probabilities  $p_i$  and utility  $u_i$  and gave some practical interpretations of this length; bounds for the cost function (1.6) in terms of (1.5) are derived by him.

### 3. NOISELESS CODING THEOREMS FOR 'USEFUL' CODES

Define a two parametric new generalized useful information measure for the incomplete power distribution as:

$$H_\alpha^\beta(P; U) = \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right], \quad (2.1)$$

Where  $0 < \alpha < 1, 0 < \beta \leq 1, p_i \geq 0 \forall i=1, 2, \dots, n, \sum_{i=1}^n p_i \leq 1$

#### Remarks for (2.1)

- I. When  $\beta=1$ , (2.1) reduces to 'useful' information measure studied by Taneja, Hooda, and Tuteja (1985), i.e.,

$$H_\alpha(P; U) = \frac{1}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i} \right] \quad (2.2)$$

- II. When  $\beta=1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored and  $\sum_{i=1}^n p_i=1$ , (2.1) reduces to Reyni's (1961) entropy, i.e.,

$$H_\alpha(P) = \frac{1}{1-\alpha} \log_D \left[ \sum_{i=1}^n p_i^\alpha \right] \quad (2.3)$$

- III. When  $\beta=1$  and  $\alpha \rightarrow 1$ , (2.1) reduces to ‘useful’ information measure for the incomplete distribution due to Bhakar and Hooda (1993), i.e.,

$$H(P, U) = -\frac{\sum_{i=1}^n u_i p_i \log p_i}{\sum_{i=1}^n u_i p_i} \quad (2.4)$$

- IV. When  $\beta=1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored,  $\sum_{i=1}^n p_i=1$  and  $\alpha \rightarrow 1$ , the measure (2.1) reduces Shannon’s (1948) entropy, i.e.,

$$H(P) = -\sum_{i=1}^n p_i \log p_i \quad (2.5)$$

- V. When  $\beta=1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored,  $\sum_{i=1}^n p_i=1$ ,  $\alpha \rightarrow 1$ , and  $p_i = \frac{1}{n} \forall i=1, 2, \dots, n$ , the measure (2.1) reduces to maximum entropy, i.e.,

$$H\left(\frac{1}{n}\right) = \log_D n \quad (2.6)$$

- VI. When  $\alpha \rightarrow 1$ , the measure (2.1) reduces to useful information measure for the incomplete power distribution  $p^\beta$  due to Sharma, Man Mohan, and Mitter (1978), i.e.,

$$H^\beta(P; U) = -\frac{\sum_{i=1}^n u_i p_i^\beta \log p_i^\beta}{\sum_{i=1}^n u_i p_i^\beta} \quad (2.7)$$

- VII. When  $\alpha \rightarrow 1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored, (2.1) reduces to a measure of incomplete power probability distribution due to Mitter and Mathur (1972), i.e.,

$$H^\beta(P) = -\frac{\sum_{i=1}^n p_i^\beta \log p_i^\beta}{\sum_{i=1}^n p_i^\beta} \quad (2.8)$$

Further, we define a two parametric new generalized useful average code-word length corresponding to (2.1) and is given by

$$L_\alpha^\beta(P; U) = \frac{\alpha\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^n u_i p_i^\beta} \right], \quad (2.9)$$

Where  $0 < \alpha < 1, 0 < \beta \leq 1, p_i \geq 0 \forall i = 1, 2, \dots, n, \sum_{i=1}^n p_i \leq 1$  and  $D$  is the size of code alphabet.

#### Remarks for (2.9)

- I. When  $\beta=1$ , (2.9) reduces to ‘useful’ average codeword length due to Taneja, Hooda, and Tuteja (1985), i.e.,

$$L_\alpha(P; U) = \frac{\alpha}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^n u_i p_i} \right] \quad (2.10)$$

- II. When  $\beta=1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored and  $\sum_{i=1}^n p_i=1$ , (2.9) reduces to exponentiated mean codeword length due to Campbell (1965) entropy, i.e.,

$$L_\alpha(P) = \frac{\alpha}{1-\alpha} \log_D \left[ \sum_{i=1}^n p_i D^{-l_i(\frac{\alpha-1}{\alpha})} \right] \quad (2.11)$$

- III. When  $\beta=1$  and  $\alpha \rightarrow 1$ , (2.9) reduces to ‘useful’ code-word length due to Guiasu and Picard (1971), i.e.,

$$L(P; U) = \frac{\sum_{i=1}^n u_i p_i l_i}{\sum_{i=1}^n u_i p_i} \quad (2.12)$$

- IV. When  $\beta=1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored,  $\sum_{i=1}^n p_i=1$  and  $\alpha \rightarrow 1$ , (2.9) reduces to optimal codeword length defined by Shannon (1948), i.e.,

$$L(P) = \sum_{i=1}^n p_i l_i \quad (2.13)$$

- IV. When  $\beta=1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored,  $\sum_{i=1}^n p_i=1, \alpha \rightarrow 1$ , and  $l_1 = l_2 = \dots = l_n = 1$ , then (2.9) reduces to 1.

Now we derive the lower and upper bound of (2.9) in terms of (2.1) under the condition

$$\frac{\sum_{i=1}^n u_i D^{-l_i}}{\sum_{i=1}^n u_i p_i^\beta} \leq 1. \quad (2.14)$$

This is a generalization of Kraft’s inequality (1.2). It is easy to see that when  $\beta=1, u_i=1, \forall i=1, 2, \dots, n$ , i.e., when the utility aspect is ignored and  $\sum_{i=1}^n p_i=1$ , then the inequality (2.14) reduces to Kraft’s (1949) inequality (1.2). A code satisfying (2.14) would be termed as a ‘useful’ personal probability code.

**Theorem 3.1:** Let  $\{u_i\}_{i=1}^n, \{p_i\}_{i=1}^n$  and  $\{l_i\}_{i=1}^n$ , satisfies the inequality (2.14), then the two parametric generalized ‘useful’ code-word length (2.9) satisfies the inequality

$$L_\alpha^\beta(P; U) \geq H_\alpha^\beta(P; U), 0 < \alpha < 1, 0 < \beta \leq 1, \quad (2.15)$$

Where  $H_\alpha^\beta(P; U)$  and  $L_\alpha^\beta(P; U)$  are defined in (2.1) and (2.9) respectively. Furthermore, equality holds good if

$$l_i = -\log_D \left[ \frac{p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^{\alpha\beta}} \right] \quad (2.16)$$

**Proof:** By Holder's inequality, we have

$$(\sum_{i=1}^n x_i^p)^{\frac{1}{p}} (\sum_{i=1}^n y_i^q)^{\frac{1}{q}} \leq \sum_{i=1}^n x_i y_i \quad (2.17)$$

For all  $x_i, y_i > 0$ ,  $i=1,2,3,\dots,n$  and  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p < 1$  ( $\neq 0$ ),  $q < 0$  or  $q < 1$  ( $\neq 0$ ),  $p < 0$ . We see the equality holds if there exists a positive constant  $c$  such that

$$x_i^p = c y_i^q \quad (2.18)$$

Making the substitution

$$x_i = \frac{u_i^{\frac{\alpha}{\alpha-1}} p_i^{\frac{\alpha\beta}{\alpha-1}}}{(\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}})^{\frac{\alpha}{\alpha-1}}} D^{-l_i}, \quad p = \frac{\alpha-1}{\alpha}$$

$$y_i = \frac{u_i^{\frac{1}{1-\alpha}} p_i^{\frac{\alpha\beta}{1-\alpha}}}{(\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}})^{\frac{1}{1-\alpha}}} \quad \text{and} \quad q = 1 - \alpha.$$

Using these values in (2.17), and after suitable simplification, we get

$$\left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}}} \right]^{\frac{\alpha}{\alpha-1}} \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]^{\frac{1}{1-\alpha}} \leq \frac{\sum_{i=1}^n u_i D^{-l_i}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \quad (2.19)$$

Now using the inequality (2.14), we get

$$\left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}}} \right]^{\frac{\alpha}{\alpha-1}} \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]^{\frac{1}{1-\alpha}} \leq 1 \quad (2.20)$$

Or equation (2.20), can be written as

$$\left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}}} \right]^{\frac{\alpha}{\alpha-1}} \leq \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]^{\frac{1}{1-\alpha}} \quad (2.21)$$

Taking logarithms to both sides with base  $D$  to equation (2.21), we get

$$\frac{\alpha}{\alpha-1} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{\alpha-1}}} \right] \leq \frac{1}{\alpha-1} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right] \quad (2.22)$$

Or equivalently we can write equation (2.22), as

$$\frac{\alpha}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right] \geq \frac{1}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right] \quad (2.23)$$

As  $0 < \beta \leq 1$ , multiply equation (2.23) both sides by  $\beta$ , we get

$$\frac{\alpha\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right] \geq \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right] \quad (2.24)$$

This implies

$$L_{\alpha}^{\beta}(P; U) \geq H_{\alpha}^{\beta}(P; U).$$

Hence the result for  $0 < \alpha < 1, 0 < \beta \leq 1$ .

Now we will show that the equality in (2.15) holds if and only if

$$l_i = -\log_D \left[ \frac{p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right], \quad 0 < \alpha < 1, 0 < \beta \leq 1.$$

Or equivalently we can write

$$D^{-l_i} = \left[ \frac{p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]$$

Or we can write

$$D^{-l_i} = p_i^{\alpha\beta} \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]^{-1} \quad (2.25)$$

Raising both sides to the power  $\frac{\alpha-1}{\alpha}$ , to equation (2.25), and after simplification we get

$$D^{-l_i \frac{\alpha-1}{\alpha}} = p_i^{\beta(\alpha-1)} \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]^{\frac{1-\alpha}{\alpha}} \quad (2.26)$$

Multiply equation (2.26) both sides by  $\frac{u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}$ , and then summing over  $i=1,2,\dots,n$ , both sides to the resultant expression and after suitable simplification, we get

$$\frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} = \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right] \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]^{\frac{1-\alpha}{\alpha}}$$

Or equivalently we can write

$$\frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}} D^{-l_i \frac{\alpha-1}{\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} = \left[ \frac{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}}{\sum_{i=1}^n u_i p_i^{\frac{\alpha\beta}{1-\alpha}}} \right]^{\frac{1}{\alpha}} \quad (2.27)$$

Taking logarithms both sides with base  $D$  to equa-

tion (2.27), then multiply both sides by  $\frac{\alpha\beta}{(1-\alpha)}$ , we get

$$\frac{\alpha\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^n u_i p_i^\beta} \right] = \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] \quad (2.28)$$

This implies

$$L_\alpha^\beta(P; U) = H_\alpha^\beta(P; U). \text{ Hence the result}$$

**Theorem 3.2:** For every code with lengths  $l_1, l_2, \dots, l_n$  satisfies the condition (2.14),  $L_\alpha^\beta(P; U)$  can be made to satisfy the inequality

$$L_\alpha^\beta(P; U) < H_\alpha^\beta(P; U) + \beta. \text{ Where, } 0 < \alpha < 1, 0 < \beta \leq 1. \quad (2.29)$$

**Proof:** From the theorem (2.1) we have

$$L_\alpha^\beta = H_\alpha^\beta(P), \text{ holds if and only if}$$

$$l_i = -\log_D \left[ \frac{p_i^{\alpha\beta}}{\frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta}} \right], 0 < \alpha < 1, 0 < \beta \leq 1.$$

Or equivalently we can write

$$-\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right]$$

Now we choose the code-word lengths  $l_1, l_2, \dots, l_n$  in such a way that they satisfy the inequality,

$$-\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] \leq l_i < -\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + 1$$

Consider the interval

$$\delta_i = \left[ -\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right], -\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + 1 \right]$$

of length unity. In every  $\delta_i$ , there lies exactly one positive integer  $l_i$  such that,

$$0 < -\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] \leq l_i < -\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + 1 \quad (2.30)$$

Now we will first show that the sequence  $l_1, l_2, \dots, l_n$  thus defined satisfies the inequality (2.14), which is a generalization of Kraft inequality.

From the left inequality of (2.30), we have

$$-\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] \leq l_i$$

Or equivalently we can write

$$D^{-l_i} \leq \left[ \frac{p_i^{\alpha\beta}}{\frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta}} \right] \quad (2.31)$$

Multiply equation (2.31) both sides by  $u_i$  then summing over  $i=1, 2, \dots, n$ , both sides to the resulted expression, and after suitable operations, we get the required result (2.14), i.e.,

$$\frac{\sum_{i=1}^n u_i D^{-l_i}}{\sum_{i=1}^n u_i p_i^\beta} \leq 1.$$

Now the last inequality of (2.30), gives

$$l_i < -\log_D p_i^{\alpha\beta} + \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + 1$$

Or equivalently we can write

$$D^{l_i} < p_i^{-\alpha\beta} \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] D \quad (2.32)$$

As  $0 < \alpha < 1$ , then  $(1-\alpha) > 0$ , and  $(\frac{1-\alpha}{\alpha}) > 0$ , raising both sides to the power  $(\frac{1-\alpha}{\alpha}) > 0$ , to equation (2.32), and after suitable operations, we get

$$D^{l_i(\frac{1-\alpha}{\alpha})} < p_i^{\beta(\alpha-1)} \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right]^{\frac{1-\alpha}{\alpha}} D^{\frac{1-\alpha}{\alpha}}$$

Or equivalently we can write

$$D^{-l_i(\frac{\alpha-1}{\alpha})} < p_i^{\beta(\alpha-1)} \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right]^{\frac{1-\alpha}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \quad (2.33)$$

Multiply equation (2.33) both sides by  $\frac{u_i p_i^\beta}{\sum_{i=1}^n u_i p_i^\beta}$ , and then summing over  $i=1, 2, \dots, n$ , both sides to the resulted expression and after suitable simplification, we get

$$\frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^n u_i p_i^\beta} < \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right]^{\frac{1-\alpha}{\alpha}} D^{\frac{1-\alpha}{\alpha}}$$

Or equivalently we can write

$$\frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^n u_i p_i^\beta} < \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right]^{\frac{1}{\alpha}} D^{\frac{1-\alpha}{\alpha}} \quad (2.34)$$

Taking logarithms with base  $D$  both sides to the equation (2.34), we get

$$\log_D \left[ \frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^n u_i p_i^\beta} \right] < \frac{1}{\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + \frac{1-\alpha}{\alpha} \quad (2.35)$$

As  $0 < \alpha < 1, 0 < \beta \leq 1$  then  $(1-\alpha) > 0$  and  $\frac{\alpha\beta}{1-\alpha} > 0$ , multiply equation (2.35), both sides by  $\frac{\alpha\beta}{1-\alpha} > 0$ , we get

$$\frac{\alpha\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i(\frac{\alpha-1}{\alpha})}}{\sum_{i=1}^n u_i p_i^\beta} \right] < \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + \beta$$

This implies

$$L_\alpha^\beta(P; U) < H_\alpha^\beta(P; U) + \beta.$$

Hence the result for  $0 < \alpha < 1, 0 < \beta \leq 1$ .

Thus from above two coding theorems, we have shown that

$$H_\alpha^\beta(P; U) \leq L_\alpha^\beta(P; U) < H_\alpha^\beta(P; U) + \beta.$$

Where  $0 < \alpha < 1, 0 < \beta \leq 1$ .

In the next section we verify the noiseless coding theorems by considering the Shannon-Fano coding scheme and Huffman coding scheme by taking an empirical dataset.

#### 4. ILLUSTRATION

In this section we illustrate the veracity of the theo-

**Table 1.** Using Huffman coding scheme the values of  $H_\alpha^\beta(P; U)$ ,  $H_\alpha^\beta(P; U) + \beta$ ,  $L_\alpha^\beta(P; U)$  and  $\eta$  for different values of  $\alpha$  and  $\beta$

Probabilities $p_i$	Huffman Codewords	$l_i$	$u_i$	$\alpha$	$\beta$	$H_\alpha^\beta(P; U)$	$L_\alpha^\beta(P; U)$	$\eta = \frac{H_\alpha^\beta(P; U)}{L_\alpha^\beta(P; U)} \times 100$	$H_\alpha^\beta(P; U) + \beta$
0.41	1	1	6	0.9	1	1.987	2.012	98.757%	2.987
0.18	000	3	5	0.9	0.9	1.654	1.874	88.260%	2.554
0.15	001	3	1	0.8	1	2.016	2.079	96.969%	3.016
0.13	010	3	2						
0.1	0110	4	4						
0.03	0111	4	3						

**Table 2.** Using Shannon-Fano coding scheme the values of  $H_\alpha^\beta(P; U)$ ,  $H_\alpha^\beta(P; U) + \beta$ ,  $L_\alpha^\beta(P; U)$  and  $\eta$  for different values of  $\alpha$  and  $\beta$

Probabilities $p_i$	Shannon-Fano Codewords	$l_i$	$u_i$	$\alpha$	$\beta$	$H_\alpha^\beta(P; U)$	$L_\alpha^\beta(P; U)$	$\eta = \frac{H_\alpha^\beta(P; U)}{L_\alpha^\beta(P; U)} \times 100$	$H_\alpha^\beta(P; U) + \beta$
0.41	00	2	6	0.9	1	1.987	2.217	89.625%	2.987
0.18	01	2	5	0.9	0.9	1.654	2.014	82.125%	2.554
0.15	100	3	1	0.8	1	2.016	2.226	90.566%	3.016
0.13	101	3	2						
0.1	110	3	4						
0.03	111	3	3						

rems 3.1 and 3.2 by taking empirical data as given in Tables 1 and 2 on the lines of A. H. Bhat and M. A. K. Baig (2016).

Using Huffman coding scheme the values of  $H_\alpha^\beta(P; U)$ ,  $H_\alpha^\beta(P; U) + \beta$ ,  $L_\alpha^\beta(P; U)$  and  $\eta$  for different values of  $\alpha$  and  $\beta$  are shown in the following table 1.

Now using Shannon-Fano coding scheme the values of  $H_\alpha^\beta(P; U)$ ,  $H_\alpha^\beta(P; U) + \beta$ ,  $L_\alpha^\beta(P; U)$  and  $\eta$  for different values of  $\alpha$  and  $\beta$  are shown in the following table 2.

From Tables 1 and 2 we infer the following:

- I. Theorems 3.1 and 3.2 hold both the cases of Shannon-Fano codes and Huffman codes, i.e.  $H_\alpha^\beta(P; U) \leq L_\alpha^\beta(P; U) < H_\alpha^\beta(P; U) + \beta$ , where  $0 < \alpha < 1, 0 < \beta \leq 1$ .
- II. Huffman mean code-word length is less than Shannon-Fano mean code-word length.
- III. Coefficient of efficiency of Huffman codes is greater than coefficient of efficiency of Shan-

non-Fano codes; i.e., it is concluded that Huffman coding scheme is more efficient than Shannon-Fano coding scheme.

## 5. MONOTONIC BEHAVIOR OF THE TWO PARAMETRIC NEW GENERALIZED 'USEFUL' INFORMATION MEASURE $H_\alpha^\beta(P; U)$

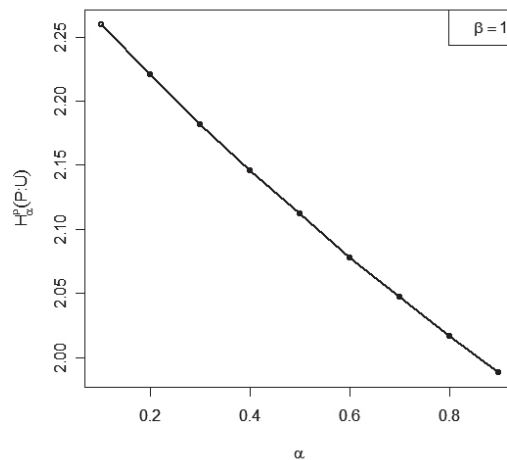
In this section we study the monotonic behavior of the new two parametric generalized 'useful' information measure  $H_\alpha^\beta(P; U)$  given in (2.1) with respect to the parameters  $\alpha$  and  $\beta$ .

Let  $P=(0.41, 0.18, 0.15, 0.13, 0.10, 0.03)$  be a set of probabilities. Assuming  $\beta=1$  we calculate the values of  $H_\alpha^\beta(P; U)$  for different values of  $\alpha$  as shown in the following table 3.

Next we draw the graph of the table (3) and illustrate from Figure 1 that  $H_\alpha^\beta(P; U)$  is monotonic decreasing

**Table 3.** Monotonic Behavior of  $H_\alpha^\beta(P; U)$  with Respect to  $\alpha$  for fixed  $\beta=1$

$\alpha$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$H_\alpha^\beta(P; U)$	2.260	2.221	2.182	2.146	2.112	2.078	2.047	2.017	1.988

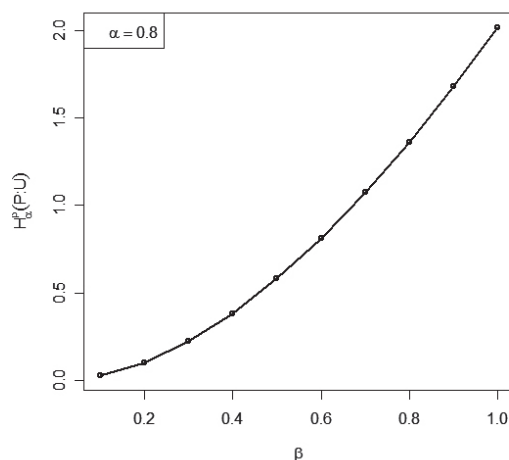


**Fig. 1** Monotonic behavior of  $H_\alpha^\beta(P)$  with respect to  $\alpha$  for fixed  $\beta=1$



**Table 4.** Monotonic Behavior of  $H_{\alpha}^{\beta}(P;U)$  with Respect to  $\beta$  for fixed  $\alpha=0.8$ 

$\beta$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$H_{\alpha}^{\beta}(P;U)$	0.026	0.102	0.222	0.383	0.580	0.811	1.072	1.361	1.677	2.017

**Fig. 2** Monotonic behavior of  $H_{\alpha}^{\beta}(P;U)$  with respect to  $\beta$  for fixed  $\alpha=0.8$ 

with increasing values of  $\alpha$ .

Now assuming  $\alpha=0.8$  we calculate the values of  $H_{\alpha}^{\beta}(P;U)$  for different values of  $\beta$  as shown in the following table 4.

Next we draw the graph of the table (4) and illustrate from Figure 2 that  $H_{\alpha}^{\beta}(P;U)$  is monotonic increasing with increasing values of  $\beta$ .

## 6. PROPERTIES OF THE NEW TWO PARAMETRIC GENERALIZED 'USEFUL' INFORMATION MEASURE $H_{\alpha}^{\beta}(P;U)$

In this section we will discuss some properties of the two parametric new generalized 'useful' information measure  $H_{\alpha}^{\beta}(P;U)$  given in (2.1):

**Property 6.1:**  $H_{\alpha}^{\beta}(P;U)$  is non-negative.

**Proof:** From (2.1), we have

$$H_{\alpha}^{\beta}(P;U) = H_{\alpha}^{\beta}(P;U) = \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^{\beta}} \right], 0 < \alpha < 1, 0 < \beta \leq 1.$$

From Tables 1 and 2, it is observed that  $H_{\alpha}^{\beta}(P;U)$  is non-negative for given values of  $\alpha$  and  $\beta$ .

**Property 6.2:**  $H_{\alpha}^{\beta}(P;U)$  is a symmetric function on every  $p_i, i=1,2,3,\dots,n$ .

**Proof:** It is obvious that  $H_{\alpha}^{\beta}(P;U)$  is a symmetric function on every  $p_i, i=1,2,3,\dots,n$ . i.e.,  $H_{\alpha}^{\beta}(p_1 u_1, p_2 u_2, \dots, p_{n-1} u_{n-1}, p_n u_n) = H_{\alpha}^{\beta}(p_n u_n, p_1 u_1, p_2 u_2, \dots, p_{n-1} u_{n-1})$

**Property 6.3:**  $H_{\alpha}^{\beta}(P;U)$  is maximum when all the events have equal probabilities.

**Proof:** When  $p_i = \frac{1}{n} \forall i = 1, 2, \dots, n$  and  $\beta=1, \alpha \rightarrow 1$ . and  $u_i = 1, \forall i = 1, 2, \dots, n$ , i.e., when the utility aspect is ignored,  $\sum_{i=1}^n p_i = 1$ . Then  $H_\alpha^\beta(P; U) = \log_D n$ , which is maximum entropy.

**Property 6.4:**  $H_\alpha^\beta(P; U)$  satisfies the additivity of the following form:

$$H_\alpha^\beta(P * Q; U) = H_\alpha^\beta(P; U) + H_\alpha^\beta(Q; U)$$

Where  $(P * Q; U) = (p_1 q_1, \dots, p_1 q_m, p_2 q_1, \dots, p_2 q_m, \dots, p_n q_1, \dots, p_n q_m; U)$

**Proof:** Let  $H_\alpha^\beta(P * Q; U) = H_\alpha^\beta(P; U) + H_\alpha^\beta(Q; U)$

Taking R.H.S =  $H_\alpha^\beta(P; U) + H_\alpha^\beta(Q; U)$

$$\begin{aligned} &= \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{j=1}^m u_j q_j^{\alpha\beta}}{\sum_{j=1}^m u_j q_j^\beta} \right] \\ &= \frac{\beta}{1-\alpha} \left[ \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] + \log_D \left[ \frac{\sum_{j=1}^m u_j q_j^{\alpha\beta}}{\sum_{j=1}^m u_j q_j^\beta} \right] \right] \\ &= \frac{\beta}{1-\alpha} \left[ \log_D \left( \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right) \left( \frac{\sum_{j=1}^m u_j q_j^{\alpha\beta}}{\sum_{j=1}^m u_j q_j^\beta} \right) \right] \\ &= \frac{\beta}{1-\alpha} \left[ \log_D \left( \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta} \sum_{j=1}^m u_j q_j^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta \sum_{j=1}^m u_j q_j^\beta} \right) \right] \\ &= \frac{\beta}{1-\alpha} \left[ \log_D \left( \frac{\sum_{i=1}^n \sum_{j=1}^m (u_i p_i^{\alpha\beta}) (u_j q_j^{\alpha\beta})}{\sum_{i=1}^n \sum_{j=1}^m (u_i p_i^\beta) (u_j q_j^\beta)} \right) \right] \\ &= \frac{\beta}{1-\alpha} \left[ \log_D \left( \frac{\sum_{i=1}^n \sum_{j=1}^m (u_i u_j) (p_i q_j)^{\alpha\beta}}{\sum_{i=1}^n \sum_{j=1}^m (u_i u_j) (p_i q_j)^\beta} \right) \right] \\ &= H_\alpha^\beta(P * Q; U) = \text{L.H.S} \end{aligned}$$

**Property 6.5:**  $H_\alpha^\beta(P)$  is concave function for  $p_1, p_2, \dots, p_n$ .

**Proof:** From (2.1), we have

$$H_\alpha^\beta(P; U) = \frac{\beta}{1-\alpha} \log_D \left[ \frac{\sum_{i=1}^n u_i p_i^{\alpha\beta}}{\sum_{i=1}^n u_i p_i^\beta} \right] \quad 0 < \alpha < 1, 0 < \beta \leq 1.$$

If  $\beta = 1, \alpha \rightarrow 1, u_i = 1, \forall i = 1, 2, \dots, n$ , i.e., when the

utility aspect is ignored, and  $\sum_{i=1}^n p_i = 1$ , then the first derivative of (2.1) with respect  $p_i$  is given by

$$\left[ \frac{d}{dp_i} H_\alpha^\beta(P; U) \right]_{\substack{\beta=1 \\ \alpha \rightarrow 1 \\ u_i=1}} = -n - \sum_{i=1}^n \log_D p_i$$

And the second derivative is given by

$$\left[ \frac{d^2}{dp_i^2} H_\alpha^\beta(P; U) \right]_{\substack{\beta=1 \\ \alpha \rightarrow 1 \\ u_i=1}} = -\sum_{i=1}^n \left( \frac{1}{p_i} \right) < 0.$$

For all  $p_i \in [0, 1]$  and  $i = 1, 2, \dots, n$ .

Since the second derivative of  $H_\alpha^\beta(P; U)$  with respect to  $p_i$  is negative on given interval  $p_i \in [0, 1], i = 1, 2, \dots, n$ . as  $\beta = 1, \alpha \rightarrow 1, u_i = 1, \forall i = 1, 2, \dots, n$ , i.e., when the utility aspect is ignored, and  $\sum_{i=1}^n p_i = 1$ , therefore,

$H_\alpha^\beta(P; U)$  is concave function for  $p_1, p_2, \dots, p_n$ .

## 7. CONCLUSION

In this paper we define a new two parametric generalized 'useful' entropy measure, i.e.,  $H_\alpha^\beta(P; U)$ . This measure also generalizes some well-known information measures already existing in the literature of 'useful' information theory. Also we define a new two parametric generalized 'useful' code-word mean lengths, i.e.,  $L_\alpha^\beta(P; U)$  corresponding to  $H_\alpha^\beta(P; U)$ , and then we characterize  $L_\alpha^\beta(P; U)$  in terms of  $H_\alpha^\beta(P; U)$  and showed that

$$H_\alpha^\beta(P; U) \leq L_\alpha^\beta < H_\alpha^\beta(P; U) + \beta.$$

where  $0 < \alpha < 1, 0 < \beta \leq 1$

Further we have established the noiseless coding theorems proved in this paper with the help of two different techniques by taking experimental data and showed that Huffman coding scheme is more efficient than Shannon-Fano coding scheme. We have also studied the monotonic behavior of  $H_\alpha^\beta(P; U)$  with respect to parameters  $\alpha$  and  $\beta$ . The important properties of  $H_\alpha^\beta(P; U)$  have also been studied.

## REFERENCES

Belis, M., & Guiasu S. (1968). A quantitative-qualitative

- measure of information in cybernetic System. *IEEE Transaction on Information Theory*, 14, 593-594.
- Bhaker, U.S., & Hooda, D.S. (1993). Mean value characterization of 'useful' information measure. *Tamkang Journal of Mathematics*, 24, 283-294.
- Bhat, A. H., & Baig, M. A. K. (2016). Some coding theorems on generalized Renyi's entropy of order  $\alpha$  and type  $\beta$ . *International Journal of Applied Mathematics and Information Sciences Letters*, 5, 1-5.
- Campbell L. L. (1965). A coding theorem and Renyi's entropy. *Information and Control*, 8, 423-429.
- Guiasu, S., & Picard, C. F. (1971). *Borne inferieure de la longueur de certain codes* (pp. 248-251). Paris: C.R. Academic Sciences, t. 273.
- Gurdial, P. F. (1977). On 'useful' Information of order  $\alpha$ . *Journal of Combinatorics Information and System Sciences*, 2, 158-162.
- Hartley, R. V. L. (1928). Transmission of information. *Bell System Technical Journal*, 7, 535-563.
- Hooda, D. S., & Bhaker, U. S. (1997). A generalized 'useful' information measure and coding theorems. *Soochow Journal of Mathematics*, 23, 53-62.
- Jain P., & Tuteja, R. K. (1989). On coding theorem connected with 'useful' entropy of order  $\beta$ . *International Journal of Mathematics and Mathematical Sciences*, 12, 193-198.
- Khan, A. B., Bhat, B. A., & Pirzada, S. (2005). Some results on a generalized 'useful' information measure. *Journal of Inequalities in Pure and Applied Mathematics*, 6, 117.
- Kraft, L. G. (1949). *A device for quantizing grouping and coding amplitude modulates pulses* (M. S. thesis). Department of Electrical Engineering, MIT, Cambridge.
- Longo G. (1976). A noiseless coding theorem for source having utilities. *SIAM Journal of Applied Mathematics*, 30, 739-748.
- Mitter, J., & Mathur, Y. D. (1972). Comparison of entropies of power distributions. *ZAMM*, 52, 239-240.
- Nyquist, H. (1924). Certain factors affecting telegraph speed. *Bell System Technical Journal*, 3, 324-346.
- Nyquist, H. (1928). Certain topics in telegraphy transmission theory. *Journal of the American Institute of Electrical Engineers*, 47, 617.
- Renyi, A. (1961). On measure of entropy and information. *Proceedings Fourth Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1, 547-561.
- Shannon, C. E. A mathematical theory of communication. *Bell System Technical Journal*, 27, 379-423, 623-659.
- Sharma, B. D, Mohan, M., & Mitter, J. On measure of 'useful' information. *Information and Control*, 39, 323-336.
- Taneja, H. C., Hooda, D. S., & Tuteja, R. K. Coding theorems on a generalized 'useful' information. *Soochow Journal of Mathematics*, 11, 123-131.