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DERIVATIONS WITH NILPOTENT VALUES ON Γ-RINGS

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Abstract. Let M be a prime Γ-ring and let d be a derivation of M. If there exists a fixed integer n such that $(d(x)\alpha)^n d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$, then we prove that $d(x) = 0$ for all $x \in M$. This result can be extended to semiprime Γ-rings.

1. Introduction

The notion of a Γ-ring was first introduced by Nobusuwa [10] as a generalization of a classical rings and then Barnes [2] generalized the same concepts in a broad sense. The concept of a derivation and a Jordan derivation of Γ-rings have been first introduced by Sapanci and Nakajima [13] and they proved that every Jordan derivation in a certain prime Γ-ring is a derivation. Afterwards many Mathematicians worked on derivations of Γ-rings and developed some fruitful results. Paul and Uddin [11, 12] studied on Jordan and Lie structures in Γ-rings and they proved the Levitzki's Theorem in Γ-rings. In [5], Halder and Paul proved that if d is a left derivation of a 2-torsion free semiprime Γ-ring such that $(d(x)\alpha)^n d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$, then $d = 0$, where n is a fixed integer. Giambruno and Herstien [4] proved a classical result in rings which is stated as follows: If d is a derivation of a prime ring R, such that $d(x)^n = 0$ for all $x \in R$, then $d(x) = 0$, where n is a fixed integer. He also extended this result to semiprime rings. Feng Wei [15] proved it in generalized derivations of semiprime rings. Then, Ali, Ali and Fillips [1] worked on a nilpotent and invertible values on semiprime rings with generalized derivations and they developed some remarkable results. By the same motivations as in Giambruno and Herstein [4], we develop the following result in this paper. If d is a derivation of a prime Γ-ring such that $(d(x)\alpha)^n d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$, then $d = 0$, where n is a fixed integer. We also extend this result in semiprime Γ-rings.

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2. Γ-rings and Derivations

Let M and Γ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$ which satisfies the conditions:

(1) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$,

(2) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring in the sense of Barnes [2]. A Γ-ring M is prime if $x\Gamma M\Gamma y = 0$ implies that $x = 0$ or $y = 0$, and is semiprime if $x\Gamma M\Gamma x = 0$ implies $x = 0$. A subring A of a Γ-ring M is said to be an ideal of M if $A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$. Let M be a Γ-ring. An additive mapping $d : M \to M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$, and d is called a Jordan derivation if $d(x\alpha x) = d(x)\alpha x + x\alpha d(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$. An ideal P of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. A Γ-ring M is said to be prime if the zero ideal is prime.

Theorem 2.1 ([10])**.** *If M is a Γ-ring, the following conditions are equivalent:*

(1) *M is a prime Γ-ring.*

(2) *If $a, b \in M$ and $a\Gamma M\Gamma b = \langle 0\rangle$, then $a = 0$ or $b = 0$.*

(3) *If $\langle a\rangle$ and $\langle b\rangle$ are principal ideals of M such that $\langle a\rangle\Gamma\langle b\rangle = \langle 0\rangle$, then $a = 0$ or $b = 0$.*

(4) *If A and B are right ideals of M such that $A\Gamma B = \langle 0\rangle$, then $A = \langle 0\rangle$ or $B = \langle 0\rangle$.*

(5) *If A and B are left ideals of M such that $A\Gamma B = \langle 0\rangle$, then $A = \langle 0\rangle$ or $B = \langle 0\rangle$.*

3. Derivations with Nilpotent Values on Γ-rings

We begin with the following lemmas which are essential for proving our main results.

Lemma 3.1 ([14, Lemma 3])**.** *If $d \neq 0$ is a derivation of M, then d does not vanish on a non-zero one-sided ideal of M.*

Proof. Let $L \neq 0$ be the left ideal of M. Suppose that $d(L) = 0$. For all $x \in L$, $m \in M$ and $\alpha \in \Gamma$, we have $m\alpha x \in L$. Therefore, $0 = d(m\alpha x) = d(m)\alpha x + m\alpha d(x) = d(m)\alpha x$. Since $d \neq 0$, we have $x = 0$, a contradiction to the fact that $L \neq 0$. □

Lemma 3.2. *If $L \neq 0$ is a left ideal of M and $T = \{x \in L \mid L\Gamma x = x\Gamma L = 0\}$. Then, L/T is a prime Γ-ring.*

Proof. It is sufficient to prove that T is a prime ideal of L. Let U and V be ideals of L such that $U\Gamma V \subseteq T$. Then, $L\Gamma U\Gamma V\Gamma L \subseteq L\Gamma T\Gamma L = 0$. But $L\Gamma U$ and $V\Gamma L$ are ideals of M. Since M is prime, either $L\Gamma U = 0$ or $V\Gamma L = 0$. If $L\Gamma U = 0$, then $U \subseteq T$. If $V\Gamma L = 0$, then $V \subseteq T$. Therefore, we have either $U \subseteq T$ and $V \subseteq T$. □

In [11, Theorem 3.1], Paul and Uddin proved the Levitzki Theorem in Γ-rings. In this paper we will frequent used of its special case.

Lemma 3.3 ([11, Theorem 3.1]). *If L is a left ideal of M and $(x\alpha)^n x = 0$ for all $x \in L$ and $\alpha \in \Gamma$, where n is a fixed integer, then $L = 0$.*

We shall also use an easy variant of Lemma 3.3.

Lemma 3.4. *If $x, y \in M$ and $((x\alpha m\beta y))^n(x\alpha m\beta y) = 0$ for all $m \in M$ and $\alpha, \beta \in \Gamma$, where n is a fixed integer, then $y\alpha x = 0$ for all $\alpha \in \Gamma$.*

Definition 3.5. Let M be a Γ-ring and let R be a subset of M. Define $L(R) = \{x \in M \mid x\alpha r = 0,$ for all $r \in R$ and $\alpha \in \Gamma\}$, and $T(R) = \{x \in M \mid r\alpha x = 0,$ for all $r \in R$ and $\alpha \in \Gamma\}$.

It is clear that $L(R)$ is a left ideal and $T(R)$ is a right ideal of M.

Let M be a prime Γ-ring and d be a derivation of M such that $(d(x)\alpha)^n d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$. We have to show that $d = 0$.

We begin with assuming that $d = 0$. Our first result is:

Lemma 3.6. *For $x \in M$, $d(L(x)) \subseteq L(x)$ and $d(T(x)) \subseteq T(x)$.*

Proof. If $y \in L(x)$, then $y\alpha x = 0$ for all $\alpha \in \Gamma$. Therefore,

$$\begin{aligned} 0 &= y\alpha d(x\alpha y)\alpha d(x\alpha y) \\ &= y\alpha(d(x)\alpha y + x\alpha d(y)\alpha d(x\alpha y) \\ &= y\alpha d(x)\alpha y\alpha d(x\alpha y) + y\alpha x\alpha d(y))\alpha d(x\alpha y) \\ &= y\alpha d(x)\alpha y\alpha(d(x)\alpha y + x\alpha d(y)) \\ &= y\alpha d(x)\alpha y\alpha d(x)\alpha y. \end{aligned}$$

Now, we have

$$\begin{aligned} 0 &= y\alpha(d(x\alpha y)\alpha)2d(x\alpha y) \\ &= y\alpha d(x\alpha y)\alpha d(x\alpha y)\alpha d(x\alpha y) \\ &= y\alpha d(x)\alpha y\alpha d(x)\alpha y\alpha(d(x)\alpha y + x\alpha d(y)) \\ &= y\alpha((d(x)\alpha y\alpha)2d(x)\alpha y \end{aligned}$$

Therefore, we have

$$0 = y\alpha(d(x\alpha y)\alpha)^n d(x\alpha y) = y\alpha((d(x)\alpha y)\alpha)^n d(x)\alpha y.$$

Thus, $d(x)\alpha y\alpha(d(x\alpha y)\alpha)nd(x)\alpha y = 0$. This implies that $((d(x)\alpha y)\alpha)^{n+1}d(x)\alpha y = 0$ for all $y \in L(x)$ and $\alpha \in \Gamma$. But then $L(x)\Gamma d(x)$ is a left ideal of M in which every element is nilpotent. Therefore, by Lemma 3.3, $L(x)\Gamma d(x) = 0$.

For $y \in L(x)$, we have $0 = d(y\alpha x) = d(y)\alpha x + y\alpha d(x) = d(y)\alpha x$. Now, we have $d(L(x)) \subseteq L(x)$. On the other hand, the analogous argument yields $d(T(x)) \subseteq T(x)$. □

Lemma 3.7. *If $x \in M$, then either $d(x\Gamma M)\Gamma x = 0$ or $L(x)\Gamma d(L(x))$. Similarly, either $x\Gamma d(M\Gamma x) = 0$ or $d(T(x))\Gamma T(x) = 0$.*

Proof. Let $a, b \in L(x)$. Then, $a\alpha x = 0$ and $b\alpha x = 0$ for all $\alpha \in \Gamma$. Now, we obtain that $d(b)\alpha x\alpha a = 0$, and so, $d(b)\alpha d(x\alpha a) = 0$. Since $x\alpha a \in L(x)$, we have $d(x\alpha a)\alpha d(x\alpha a) = 0$. Now, we have

$$\begin{aligned} 0 &= d(x\alpha a + b)\alpha d(x\alpha a + b) \\ &= (d(x\alpha a) + d(b))\alpha(d(x\alpha a) + d(b)) \\ &= d(x\alpha a)\alpha d(x\alpha a) + d(b)\alpha) d(x\alpha a) + d(x\alpha a)\alpha d(b) + d(b)\alpha d(b) \\ &= d(x\alpha a)\alpha d(b). \end{aligned}$$

Hence,

$$\begin{aligned} 0 &= d(x\alpha a + b)\alpha d(x\alpha a + b)\alpha d(x\alpha a + b) \\ &= (d(x\alpha a)\alpha d(b))\alpha(d(x\alpha a) + d(b)) \\ &= d(x\alpha a)\alpha d(b)\alpha d(x\alpha a) + d(x\alpha a)\alpha d(b)\alpha d(b) \\ &= d(x\alpha a)\alpha d(b)\alpha d(b). \end{aligned}$$

Using the same argument, we obtain,

$$0 = (d(x\alpha a + b)\alpha)^n d(x\alpha a + b) = d(x\alpha a)\alpha(d(b)\alpha)^{n-1}d(b). \tag{1}$$

Let $m \in M$, $a, b \in L(x)$, $a\alpha x = 0$ and $b\alpha x = 0$ for all $\alpha \in \Gamma$. Therefore, $a\alpha x\alpha m\alpha a = b\alpha x\alpha m\alpha a = 0$. Hence, the result of (2) gives us

$$\begin{aligned} 0 &= d(a\alpha m\alpha x\alpha a)\alpha(d(b)\alpha)^{n-1}d(b) \\ &= (d(a\alpha m)\alpha(x\alpha a) + a\alpha m\alpha d(x\alpha a)\alpha d(b)\alpha)^{n-1}d(b) \\ &= d(a\alpha m)\alpha(x\alpha a)\alpha d(b)\alpha)^{n-1}d(b) + a\alpha m\alpha d(x\alpha a)\alpha(d(b)\alpha)^{n-1}d(b) \\ &= d(a\alpha m)\alpha(x\alpha a)\alpha d(b)\alpha)^{n-1}d(b), \text{ using (2)}. \end{aligned}$$

In other words, we write the above relation as $d(a\alpha m)\alpha x\alpha L(x)\alpha d(b)\alpha)^{n-1}d(b) = 0$ for all $m \in M$, $b \in L(x)$ and $\alpha \in \Gamma$. If $L(x)\alpha d(b)\alpha)^{n-1}d(b) \neq 0$, then by the primeness of M, we obtain $d(a\alpha m)\alpha x = 0$ for all $m \in M$ and $\alpha \in \Gamma$. Hence, we have $d(a\Gamma M)\Gamma x = 0$. On the other hand, suppose that $L(x)\alpha(d(b)\alpha)^{n-1}d(b) = 0$ for all $b \in L(x)$ and $\alpha \in \Gamma$. Since $d(L(x)) \subseteq L(x)$ and $d(T) \subseteq T$ where $T = \{c \in L(x) \mid L(x)\alpha c = 0\}$, d induces a derivation which we write as d on $B = L(x)/T$.

By Lemma 3.2, B is a prime Γ-ring. The fact that $L(x)\alpha(d(b)\alpha)^{n-1}d(b) = 0$ for all $b \in L(x)$ translates into $(d(c)\alpha)^{n-1}d(c) = 0$ for all $c \in B$. Thus, $d(c) = 0$ for all $c \in B$ (by induction). This yields us that $d(L(x)) \subseteq T$ and so $L(x)\alpha d(L(x)) = 0$ for all $\alpha \in \Gamma$. The same argument yields the right-handed version of what we have just proved. Thus, the proof is completed. □

Lemma 3.6 gives us two sets of elements which have rather particular properties, and which yield the following definition:

We set $A = \{x \in M \mid x\Gamma d(M\Gamma x) = 0\}$ and $B = \{x \in M \mid d(x\Gamma M)\Gamma x = 0\}$. These two subsets A and B play a key role in which what is to follow. Their basic algebraic characterizations are expressed in the following.

Lemma 3.8. *A is a non-zero left ideal of M, B is a nonzero right ideal of M and $A\Gamma B = 0$. Furthermore, $d(A) \subseteq A$, $d(B) \subseteq B$ and $A\Gamma d(A) = d(B)\Gamma B = 0$.*

Proof. The stated properties of A and B are the same, so we have to show that $A \neq 0$ is a left ideal of M, $d(A) \subseteq A$ and $d(A)\Gamma A = 0$. If $x, y \in M$ are such that $(x)\Gamma d(L(x)) = 0$ and $L(y)\Gamma d(L(y)) = 0$. Then, we shall prove that $L(x)\Gamma d(L(y)) = 0$. In order to see this, let $a, b \in L(x)$ and $t, z \in L(y)$. By our assumption on $L(x)$ and $L(y)$, we obtain $d(a\alpha b) = d(a)\alpha b$ and $d(t\alpha z) = d(t)\alpha z$. Therefore,

$$\begin{aligned} 0 &= b\alpha(d(a\alpha b + t\alpha z)\alpha)2nd(a\alpha b + t\alpha z) \\ &= b\alpha((d(a)\alpha b + d(t)\alpha z)\alpha)2n(d(a)\alpha b + d(t)\alpha z) \\ &= b\alpha d(t)\alpha z\alpha((d(a)\alpha b + d(t)\alpha z)\alpha)^{2n-1}(d(a)\alpha b + d(t)\alpha z) \\ &\dots \\ &= (b\alpha((d(t)\alpha z\alpha d(a)\alpha b)\alpha)nd(t)\alpha z\alpha d(a)\alpha b. \end{aligned}$$

Therefore, $(b\alpha(d(t)\alpha z\alpha((d(a)\alpha)^{n+1}b\alpha d(t)\alpha z\alpha d(a) = 0$ for all $a, b \in L(x)$, $t, z \in L(y)$ and $\alpha \in \Gamma$. Making several uses of Lemma 3.3, we obtain from the above relation that $L(x)\alpha d(L(y))\alpha L(y)\alpha d(L(x)) = 0$ for all $\alpha \in \Gamma$. Since M is prime, we have $L(x)\alpha d(L(y)) = 0$ or $L(y)\alpha d(L(x)) = 0$ for all $\alpha \in \Gamma$. Suppose that $L(y)\alpha d(L(x)) = 0$ for all $\alpha \in \Gamma$. Then, for all $b \in L(y)$, $z, t \in L(x)$. Since $L(x)\alpha d(L(x)) = 0$, $0 = z\alpha d(b\alpha t) = z\alpha(d(b)\alpha t + b\alpha d(t)) = z\alpha d(b)\alpha t$ and $b\alpha d(t) \in L(y)\alpha d(L(x)) = 0$. Thus, $z\alpha d(b)\alpha L(x) = 0$ and so $z\alpha d(b) = 0$. This says that $L(x)\alpha d(L(y)) = 0$. Thus, our assertion has been verified. We shall now show that $A \neq 0$. Suppose that $A = 0$. By Lemma 3.6, we get that $L(x)\alpha d(L(x)) = 0$ for all $x \in M$. Take $y \in M$ such that $L(y) \neq 0$. By Lemma 3.1, $d(L(y)) \neq 0$. Since $(d(x)\alpha)^n d(x) = 0$ for all $x \in M$, $\alpha \in \Gamma$, $d(x) \in L(d(x)\alpha)^{n-1}d(x)$, hence $d(x)\alpha d(L(y)) = 0$. Since $d(L(y)) \neq 0$, $d(x) = 0$ which is a contradiction to the fact that $d \neq 0$. Thus, indeed, $A \neq 0$.

Our next goal is to show that A is a left ideal of M. From the definition of $A = \{x \in M \mid x\Gamma d(M\Gamma x) = 0\}$. It is clear that $x \in A$, $t \in M$, $\alpha \in \Gamma$ forces $t\alpha x \in A$. So, all we need to show that if $x, y \in A$, then $x + y \in A$. If $a, b, z, t \in M$, then $d(a\alpha x\alpha b\alpha x) = d(a\alpha x)\alpha b\alpha x + a\alpha x\alpha d(b\alpha x) = d(a\alpha x)\alpha b\alpha x$, since $x \in A$. Similarly, $d(z\alpha y\alpha t\alpha y) = d(z\alpha y)\alpha b\alpha y$. Now, we have

$$\begin{aligned} 0 &= ((d(a\alpha x)\alpha b\alpha x + z\alpha y\alpha t\alpha y)\alpha)^{2n}(d(a\alpha x\alpha b\alpha x + z\alpha y\alpha t\alpha y)\alpha d(a\alpha x)\alpha b\alpha x \\ &= ((d(a\alpha x)\alpha b\alpha x + d(z\alpha y)\alpha t\alpha y)\alpha)^{2n}(d(a\alpha x)\alpha b\alpha x + d(z\alpha y)\alpha t\alpha y)\alpha d(a\alpha x)\alpha b\alpha x \\ &= d(a\alpha x)\alpha b\alpha x\alpha((d(z\alpha y)\alpha b\alpha y\alpha d(a\alpha x)\mu)^n(d(z\alpha y)\alpha t\alpha y\alpha d(a\alpha x))\alpha b\alpha x. \end{aligned}$$

Since $a\alpha x\alpha d(b\alpha x)$. Thus, we get that

$$((d(a\alpha x)\alpha b\alpha x\alpha d(z\alpha y)\alpha t\alpha y)\alpha)^{n+1}(d(a\alpha x)\alpha b\alpha x\alpha d(z\alpha y)\alpha t\alpha y = 0,$$

for all $a, b, z, t \in M$ and $\alpha \in \Gamma$. In view of Lemma 3.3, we obtain that $x\alpha d(z\alpha y) = 0$ for all $x, y, z \in M$ and $\alpha \in \Gamma$. This yields that $x\alpha d(m\alpha y) = 0$ or $y\alpha d(m\alpha x) = 0$ for all $m \in M$ and $\alpha \in \Gamma$. If $x\alpha d(m\alpha y) = 0$, then since $x\alpha d(m\alpha x) = 0$, we have

$$\begin{aligned} 0 &= x\alpha d(m\alpha y\alpha m\alpha x) \\ &= x\alpha d(m\alpha y)\alpha m\alpha x + x\alpha m\alpha y\alpha d(m\alpha x) \\ &= x\alpha m\alpha y\alpha d(m\alpha x) \end{aligned}$$

and so $y\alpha d(m\alpha x) = 0$ by the primeness of M. Thus, we obtain that

$$\begin{aligned} (x + y)\alpha d(m\alpha x + d(m\alpha y)) &= x\alpha d(m\alpha x) \\ &= x\alpha d(m\alpha x + y\alpha d(m\alpha x) + y\alpha d(m\alpha y) = 0, \end{aligned}$$

this implies that $x + y \in A$. Therefore, so far, we have seen that $A \neq 0$ is a left ideal of M. Of course, we also now have by the same argument that $B \neq 0$ is a right ideal of M. In view of the definition of A and Lemma 3.5, twice yields that $d(A) \subseteq A$. Now, we want to prove that $A\Gamma d(A) = 0$. Let $x, y \in A$. We have seen that $y\alpha d(m\alpha x) = 0$, hence $y\alpha d(m\alpha x\alpha y) = 0$. This gives that $0 = y\alpha d(m\alpha x)\alpha y + y\alpha m\alpha x\alpha d(y) = y\alpha M\alpha x\alpha d(y)$. The primeness of M gives that $x\alpha d(y) = 0$ and so, $A\Gamma d(A) = 0$. Similarly, we can prove that $d(B)\Gamma B = 0$. Now, we also have to show that $A\Gamma B = 0$. Let $x \in A$, $y \in B$, $z \in M$ and $\alpha \in \Gamma$,

$$\begin{aligned} 0 &= (d(x\alpha z\alpha x + y)\alpha)^{2n}(d(x\alpha z\alpha x + y) \\ &= (d(x\alpha z\alpha x) + d(y))\alpha)^{2n}(d(x\alpha z\alpha x) + d(y)) \\ &= ((d(x)\alpha z\alpha x + x\alpha d(z\alpha x) + d(y))\alpha)^{2n}(d(x)\alpha z\alpha x + x\alpha d(z\alpha x) + d(y)) \\ &= ((d(x)\alpha z\alpha x + d(y))\alpha)^{2n}(d(x)\alpha z\alpha x + d(y)), \text{ since } x\alpha d(z\alpha x) = 0. \end{aligned}$$

Therefore,

$$((d(x)\alpha z\alpha x + d(y))\alpha)^{2n}(d(x)\alpha z\alpha x + d(y)) = 0.$$

This gives that

$$0 = (d(x)\alpha z\alpha x + x\alpha d(z\alpha x) + d(y))^{2n}(d(x)\alpha z\alpha x + x\alpha d(z\alpha x) + d(y))$$

and since $d(y)\alpha d(y) = 0$ and $x\alpha d(x) = 0$. We obtain $(d(x)\alpha z\alpha x+d(y))^{2n}(d(x)\alpha z\alpha x+d(y)) = 0$. Therefore, $(d(x)\alpha z\alpha x + d(y))^{2n}(d(x)\alpha z\alpha x + d(y)) = 0$. By Lemma 3.4, we obtain that $x\alpha d(y)\alpha d(x) = 0$ for all $x \in A$, $y \in B$ and $\alpha \in \Gamma$. So, $d(x\alpha y)\alpha d(x) = d(x)\alpha y\alpha d(x) + x\alpha d(x)\alpha d(x) = d(x)\alpha y\alpha d(x)$. But,

$$\begin{aligned} 0 &= (d(x\alpha y)\alpha)^n d(x\alpha y)\alpha d(x) \\ &= (d(x\alpha y)\alpha)^n (d(x)\alpha y + x\alpha d(y))\alpha d(x) \\ &= (d(x\alpha y)\alpha)^n d(x)\alpha y\alpha d(x) \\ &\dots \\ &= (d(x)\alpha y)\alpha)^{n+1} d(x). \end{aligned}$$

Therefore, $((d(x)\alpha y\alpha)^{n+1}d(x)\alpha y = 0$. This shows that $d(x)\alpha y$ is a nilpotent element of a nil right ideal $d(A)\Gamma B$ of bounded index of nilpotent $n + 1$. By Lemma 3.3, $d(x)\alpha y = 0$. This gives that $d(A)\Gamma B = 0$ since A is a left ideal of M, $0 = d(M\Gamma A)\Gamma B = d(M)\Gamma A\Gamma B + M\Gamma d(A)\Gamma B = d(M)\Gamma A\Gamma d(B)$. We conclude that $A\Gamma B = 0$. □

Lemma 3.9. *If $x \in M$ and $x\alpha x = 0$ for all $\alpha \in \Gamma$, then $x \in A \cup B$.*

Proof. Suppose that $x \notin B$, by Lemma 3.6, $L(x)\Gamma d(L(x)) = 0$. Since $x\alpha x = 0$, $x \in L(x)$ and $M\Gamma d(L(x)) \subseteq M\Gamma L(x) \subseteq L(x)$. Now, we have $d(M\Gamma x) \subseteq d(L(x))$. Therefore, $x\Gamma d(M\Gamma x) \subseteq L(x)\Gamma d(L(x)) = 0$. Hence, $x\Gamma d(M\Gamma x) = 0$ and by the definition of A, we obtain $x \in A$. □

Now, we are in a position to prove our main result.

Theorem 3.10. *If M is a prime Γ-ring and $d : M \to M$ a derivation such that $(d(x)\alpha)^n d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$, where $n \geq 1$ is a fixed integer, then $d = 0$.*

Proof. Let $C = A \cup B \supseteq B\Gamma A \neq 0$, where $A \neq 0$ and $B \neq 0$ are respectively, left and right ideals of the prime Γ-ring M. Let $p \in C$, since $d(x)$ is nilpotent, we have $0 = p\alpha d(x)\alpha p = p\alpha d(x)\alpha d(x)\alpha p$. Because $p\alpha p \in C\Gamma C \subseteq A\Gamma B = 0$, we get that $(p\alpha d(x) - d(x)\alpha p)\alpha(p\alpha d(x) - d(x)\alpha p) = 0$. But then, by Lemma 3.8, $p\alpha d(x) - d(x)\alpha p \in A \cup B$ for all $x \in M$. Suppose that $p\alpha d(x) - d(x)\alpha p \in A$, say, since $p \in C \subseteq A$, $d(x)\alpha p \in A$. Hence, $p\alpha d(x) \in A$. If $p\alpha d(x) - d(x)\alpha p \in B$, then the similar argument end up with $d(x)\alpha p \in B$, since $p\alpha d(x) \in B$. So, for every $x \in M$, either $p\alpha d(x) \in A$ or $d(x)\alpha p \in B$. This implies that $p\Gamma d(M) \subseteq A$ or $d(M)\Gamma p \subseteq B$.

If $p\Gamma d(M) \subseteq A$, then since $p\Gamma C \subseteq B$, $p\Gamma d(M) \subseteq B$, hence $p\Gamma d(M) \subseteq C$. Similarly, if $d(M)\Gamma p \subseteq B$, then we get $d(M)\Gamma p \subseteq C$. So, for every $p \in C$, $p\Gamma d(M) \subseteq C$ or $d(M)\Gamma p \subseteq C$. This implies that $C\Gamma d(M) \subseteq C$ or $d(M)\Gamma C \subseteq C$. Suppose that $C\Gamma d(M) \subseteq C$. Hence, $C\Gamma d(M)\Gamma d(A) \subseteq C\Gamma d(A) \subseteq A\Gamma d(A) = 0$. Now, $B\Gamma A \subseteq C$, thus $B\Gamma A \subseteq d(M)\Gamma d(A) \subseteq C\Gamma d(M)\Gamma d(A) = 0$. Since B is a right ideal of M and $B \neq 0$, the primeness of M forces that $A\Gamma d(M)\Gamma d(A) = 0$.

Consider the left ideal $A\Gamma d(M)$ of M. Let $t = \sum a_i\alpha_i d(m_i)$, $a_i \in A$, $\alpha_i \in \Gamma$, $m_i \in M$, be any element of $A\Gamma d(M)$. Thus, if $v = \sum a_i\alpha_i m_i$, then

$$d(v) = \sum d(a_i)\alpha_i m_i + \sum a_i\alpha_i d(m_i) = t + w,$$

where $w = \sum d(a_i)\alpha_i m_i$. Furthermore,

$$t\Gamma w = \sum a_i\alpha_i d(m_i)\Gamma d(a_i)\alpha_i m_i \in A\Gamma d(M)\Gamma d(A)\Gamma M = 0,$$

so $t\Gamma w = 0$. Now, we have

$$\begin{aligned} 0 = (d(v)\alpha)^n d(v) &= ((t+w)\alpha)^n(t+w) \\ &= (t\alpha)^n t + (w\alpha)^n w + (w\alpha)^{n-1}w\alpha t + \ldots + w\alpha(t\alpha)^{n-1}t, \end{aligned}$$

since $t\alpha w = 0$ for every $\alpha \in \Gamma$. Therefore, $0 = t\alpha(t\alpha)^n t + (w\alpha)^n w + \ldots + w\alpha(t\alpha)^{n-1}t) = (t\alpha)^{n+1}t$. In other words, every element in $A\Gamma d(M)$ is nilpotent of degree at most $n+1$. Therefore, by Lemma 3.3, $A\Gamma d(M) = 0$. Since $A \neq 0$, we have $d(M) = 0$. Similarly, if $d(M)\Gamma C \subseteq C$, then we have $d(M)\Gamma B = 0$. Since $B \neq 0$, $d(M) = 0$. This proves the theorem. □

Now we prove the more general result.

Theorem 3.11. *Let $I \neq 0$ be an ideal of a prime Γ-ring M and d be a derivation of M. If $(d(x)\alpha)^n d(x) = 0$ for all $x \in I$, where $n \geq 1$ is a fixed integer, then $d = 0$.*

Proof. If $d(I) \subseteq I$, the result is obvious. But even if $d(I) \nsubseteq I$, our proof is easily adjusted to prove the result. □

Theorem 3.11 can be extended to semiprime Γ-rings which is given in the following theorem.

Theorem 3.12. *Let M be a semiprime Γ-ring and d be a derivation of M such that $(d(x)\alpha)^n d(x) = 0$ for all $x \in M$. Then, $d = 0$.*

Proof. Since M is semiprime, $\bigcap J = 0$ where the intersection runs over all prime ideals J of M. Now, we claim that $d(J) \subseteq J$ for every prime ideal J. Let $a \in J$, $x \in M$ and $\alpha \in \Gamma$. Then, $0 = (d(a\alpha x))^n d(a\alpha x) = ((d(a)\alpha x + a\alpha d(x)))^n(d(a)\alpha x +$

$a\alpha d(x)) = (d(a\alpha x)\alpha)^n d(a)\alpha x \mod J$. Thus, in the prime Γ-ring $\overline{M} = M/J$, $((\overline{d(a)}\alpha\overline{x})\alpha)^n = 0$ for all $\overline{x} \in \overline{M}$, $\alpha \in \Gamma$. Hence, the right ideal $\overline{d(a)}\Gamma\overline{M}$ is a nil of bounded index n. Therefore, by Theorem 3.1 of [11] has a nilpotent ideal-which it can not, since it is prime unless $\overline{d(a)} = 0$. But $d(a) \in J$. So, $d(J) \subseteq J$. Hence, $d(J) \subseteq J$ for all prime ideals J of M and so d induces a derivation on the prime Γ-ring $\overline{M} = M/J$ such that $(\overline{d(x)}\alpha)^n\overline{d(x)} = 0$ for all $\overline{x} \in \overline{M}$, $\alpha \in \Gamma$. By Theorem 3.11, $\overline{d(x)} = 0$. Hence, $\overline{d(M)} = 0$, that is $d(M) \subseteq J$ for all prime ideals J of M. Since $\cap J = 0$, we get $d(M) = 0$. Hence, $d = 0$. □

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