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FUZZY RELATIONS AND ALEXANDROV L-TOPOLOGIES

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Abstract. In this paper, we investigate the relationships between fuzzy relations and Alexandrov L-topologies in complete residuated lattices. Moreover, we give their examples.

1. Introduction

Pawlak [9, 10] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [3] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Radzikowska [11] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Lai [7, 8] introduced Alexandrov L-topologies induced by fuzzy rough sets. Algebraic structures of fuzzy rough sets are developed in many directions [1-13].

In this paper, we investigate the relationships between fuzzy relations and Alexandrov L-topologies in complete residuated lattices. Moreover, we give their examples.

2. Preliminaries

Definition 2.1 ([1, 3])**.** An algebra $(L, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ is called a *complete residuated lattice* if it satisfies the following conditions:

(L1) $L = (L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;

(L2) $(L, \odot, \top)$ is a commutative monoid;

(L3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

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In this paper, we assume $(L,\wedge,\vee,\odot,\rightarrow,^*\bot,\top)$ is a complete residuated lattice with the law of negation;i.e. $x^{**}=x$. For $\alpha\in L, A,\top_x\in L^X$, $(\alpha\rightarrow A)(x)=\alpha\rightarrow A(x)$, $(\alpha\odot A)(x)=\alpha\odot A(x)$ and $\top_x(x)=\top,\top_x(x)=\bot$, otherwise.

Definition 2.2 ([1, 7])**.** Let X be a set. A function $R:X\times X\rightarrow L$ is called a *fuzzy relation.* A fuzzy relation R is called a *fuzzy preorder* if satisfies (R1) and (R2).

(R1) *reflexive* if $R(x,x)=\top$ for all $x\in X$,

(R2) *transitive* if $R(x,y)\odot R(y,z)\le R(x,z)$, for all $x,y,z\in X$.

We denote $R^2(x,z)=(R\circ R)(x,z)=\bigvee_{y\in X}(R(x,y)\odot R(y,z))$.

Lemma 2.3 ([1, 3])**.** *Let* $(L,\vee,\wedge,\odot,\rightarrow,^*,\bot,\top)$ *be a complete residuated lattice with a negation* *. *For each* $x,y,z,x_i,y_i\in L$, *the following properties hold.*

(1) *If* $y\le z$, *then* $x\odot y\le x\odot z$.

(2) *If* $y\le z$, *then* $x\rightarrow y\le x\rightarrow z$ *and* $z\rightarrow x\le y\rightarrow x$.

(3) $x\odot(\bigvee_{i\in\Gamma}y_i)=\bigvee_{i\in\Gamma}(x\odot y_i)$ *and* $(\bigvee_{i\in\Gamma}x_i)\odot y=\bigvee_{i\in\Gamma}(x_i\odot y)$.

(4) $x\rightarrow(\bigwedge_{i\in\Gamma}y_i)=\bigwedge_{i\in\Gamma}(x\rightarrow y_i)$ *and* $(\bigvee_{i\in\Gamma}x_i)\rightarrow y=\bigwedge_{i\in\Gamma}(x_i\rightarrow y)$.

(5) $(x\rightarrow y)\odot x\le y$ *and* $(y\rightarrow z)\odot(x\rightarrow y)\le(x\rightarrow z)$.

(6) $(x\odot y)\rightarrow z=x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$ *and* $(x\odot y)^*=x\rightarrow y^*$.

(7) $x^*\rightarrow y^*=y\rightarrow x$ *and* $(x\rightarrow y)^*=x\odot y^*$.

Definition 2.4 ([5-7])**.** A subset $\tau\subset L^X$ is called an *Alexandrov topology* if it satisfies satisfies the following conditions.

(T1) $\bot_X,\top_X\in\tau$ where $\top_X(x)=\top$ and $\bot_X(x)=\bot$ for $x\in X$.

(T2) If $A_i\in\tau$ for $i\in\Gamma$, $\bigvee_{i\in\Gamma}A_i,\bigwedge_{i\in\Gamma}A_i\in\tau$.

(T3) $\alpha\odot A\in\tau$ for all $\alpha\in L$ and $A\in\tau$.

(T4) $\alpha\rightarrow A\in\tau$ for all $\alpha\in L$ and $A\in\tau$.

Definition 2.5 ([7])**.** Let $R\in L^{X\times X}$ be a fuzzy relation. A set $A\in L^X$ is called *extensional* if $A(x)\odot R(x,y)\le A(y)$ for all $x,y\in X$.

3. Fuzzy Relations and Alexandrov L-topologies

Theorem 3.1. *Let* $R\in L^{X\times X}$ *and* $R^{-1}\in L^{X\times X}$ *with* $R^{-1}(x,y)=R(x,y)$.

(1) τ *is an Alexandrov topology on* X *iff* $\tau^*=\{A^*\in L^X\mid A\in\tau\}$ *is an Alexandrov topology on* X.

(2) $\tau_R=\{A\in L^X\mid A(x)\odot R(x,y)\le A(y),x,y\in X\}$ *is an Alexandrov topology on* X. *Moreover,* $\tau_{R^{-1}}=\{A^*\mid A\in\tau_R\}=\tau_R^*$.

(3) *If* $\overline{R}$ *is the smallest fuzzy preorder such that* $R\le\overline{R}$, *then*

$$\overline{R}(x,y) = \bigwedge_{A\in\tau_R}(A(x)\to A(y)) = \bigvee_{n\in N}(R^r)^n(x,y),$$

where $R^r(x,y) = \triangle \vee R(x,y)$ *and* $\triangle(x,y) = \top$ *if* $x = y$ *and* $\triangle(x,y) = \bot$ *if* $x \neq y$. *Moreover,*

$$\overline{R}^{-1}(x,y) = \bigwedge_{A\in\tau_R^*}(A(x)\to A(y)) = \overline{R^{-1}}(x,y).$$

(4) $\tau_R = \{A \in L^X \mid \bigvee_{x\in X}(A(x)\odot\overline{R}(x,-)) = A\} = \{\bigvee_{x\in X}(a_x\odot\overline{R}(x,-)\}$ *where* $\overline{R}(x,-)(y) = \overline{R}(x,y)$ *for each* $y \in X$.

(5) $\tau_R = \{A \in L^X \mid A = \bigwedge_{y\in X}(\overline{R}(-,y)\to A(y))\} = \{\bigwedge_{y\in X}(\overline{R}(-,y)\to b_y)\}$ *where* $\overline{R}(-,y)(x) = \overline{R}(x,y)$ *for each* $x \in X$.

(6) $\tau_R^* = \{A \in L^X \mid \bigvee_{x\in X}(A(x)\odot\overline{R}(-,x)) = A\} = \{\bigvee_{x\in X}(a_x\odot\overline{R}(-,x)\}$ *where* $\overline{R}(-,x)(y) = \overline{R}(y,x)$ *for each* $y \in X$.

(7) $\tau_R^* = \{A \in L^X \mid A = \bigwedge_{y\in X}(\overline{R}(y,-)\to A(y))\} = \{\bigwedge_{y\in X}(\overline{R}(y,-)\to b_y)\}$ *where* $\overline{R}(y,-)(x) = \overline{R}(y,x)$ *for each* $x \in X$.

(8) $C_{\tau_R}(A) = \bigwedge\{B \in L^X \mid A \leq B, B \in \tau_R\} = \bigvee_{x\in X}(A(x)\odot\overline{R}(x,-))$. *Moreover,* $C_{\tau_R}(A) \in \tau_R$.

(9) $I_{\tau_R}(A) = \bigvee\{B \in L^X \mid B \leq A, B \in \tau_R\} = \bigwedge_{x\in X}(\overline{R}(-,x)\to A(x))$. *Moreover,* $I_{\tau_R}(A) \in \tau_R$.

(10) $A \in \tau_R$ *iff* $A = C_{\tau_R}(A) = I_{\tau_R}(A)$.

(11) $C_{\tau_R}(A) = (I_{\tau_{R^{-1}}}(A^*))^*$ *for all* $A \in L^X$.

Proof. (1) Let $A^* \in \tau^*$ for $A \in \tau$. Since $\alpha\odot A^* = (\alpha\to A)^*$ and $\alpha\to A^* = (\alpha\odot A)^*$, τ^* is an Alexandrov topology on X.

(2) (T1) Since $\top_X(x)\odot R(x,y) \leq \top_X(y) = \top$ and $\bot_X(x)\odot R(x,y) = \bot = \bot_X(y)$, Then $\bot_X, \top_X \in \tau_R$.

(T2) For $A_i \in \tau_R$ for each $i \in \Gamma$, since $(\bigvee_{i\in\Gamma}A_i(x))\odot R(x,y) = \bigvee_{i\in\Gamma}(A_i(x)\odot R(x,y)) \leq \bigvee_{i\in\Gamma}A_i(y)$, $\bigvee_{i\in\Gamma}A_i \in \tau_R$. Similarly, $\bigwedge_{i\in\Gamma}A_i \in \tau_R$.

(T3) For $A \in \tau_R$, $\alpha\odot A \in \tau_R$.

(T4) For $A \in \tau_R$, by Lemma 2.3(5), since $\alpha\odot(\alpha\to A(x))\odot R(x,y) \leq A(x)\odot R(x,y) \leq A(y)$, $(\alpha\to A(x))\odot R(x,y) \leq \alpha\to A(y)$. Then $\alpha\to A \in \tau_R$. Moreover $A \in \tau_R$ iff $A^* \in \tau_{R^{-1}}$ from:

$$\begin{aligned}&A(x)\odot R(x,y) \leq A(y) \text{ iff } R(x,y)\to A^* \geq A^*(y)\\ &\text{iff } A^*(y)\odot R(x,y) \leq A^*(x) \text{ iff } A^*(y)\odot R^{-1}(y,x) \leq A^*(x).\end{aligned}$$

(3) Define $R_{\tau_R}(x,y) = \bigwedge_{B\in\tau_R}(B(x)\to B(y))$. Then R_{τ_R} is a fuzzy preorder. Since $B \in \tau_R$ and $B(x)\odot R(x,y) \leq B(y)$, then $R(x,y) \leq B(x)\to B(y)$. Hence

$R(x,y) \le R_{\tau_R}$. If P is a fuzzy preorder with $R \le P$, for $P_w(x) = P(w,x)$, then $P_w(x) \odot R(x,y) \le P_w(x) \odot P(x,y) \le P_w(y)$. Hence $P_w \in \tau_R$. Thus $R_{\tau_R}(x,y) = \bigwedge_{B\in\tau_R}(B(x) \to B(y)) \le P_x(x) \to P_x(y) = P(x,y)$. Thus,

$$\overline{R}(x,y) = \bigwedge_{A\in\tau_R} (A(x) \to A(y))$$

Since $R^r(x,y) = \triangle \vee R(x,y)$, we have $(R^r)^n(x,x) = \top$ for each $n \in N$. So $\bigvee_{n\in N}(R^r)^n(x,x) = \top$. Since

$$\bigvee_{y\in X} ((R^r)^k(x,y) \odot (R^r)^m(y,z) \le (R^r)^{k+m}(x,z) \le \bigvee_{n\in N} (R^r)^n(x,z),$$

then $\bigvee_{n\in N}(R^r)^n(x,y) \circ \bigvee_{n\in N}(R^r)^n(y,z) \le \bigvee_{n\in N}(R^r)^n(x,z)$. Hence $\bigvee_{n\in N}(R^r)^n$ is a fuzzy preorder. If $R \le P$ and P is fuzzy preorder, then $R^r \le P$ and $(R^r)^n \le P^n = P$, thus, $\bigvee_{n\in N}(R^r)^n \le P$. Hence $\overline{R} = \bigvee_{n\in N}(R^r)^n$.

$$\overline{R}^{-1}(x,y) = \bigwedge_{A\in\tau_R^*} (A(x) \to A(y)) = \overline{R^{-1}}(x,y).$$

$$\begin{aligned}&\overline{R}^{-1}(x,y) = \overline{R}(y,x) = \textstyle\bigwedge_{A\in\tau_R}(A(y) \to A(x))\\ &= \textstyle\bigwedge_{A^*\in\tau_R^*}(A^*(x) \to A^*(y)) = \bigwedge_{A\in\tau_{R^{-1}}}(A(x) \to A(y))\\ &= \overline{R^{-1}}(x,y).\end{aligned}$$

(4) Put $\tau = \{A \in L^X \mid \bigvee_{x\in X}(A(x) \odot \overline{R}(x,-)) = A\}$ and $\tau_1 = \{\bigvee_{x\in X}(a_x \odot \overline{R}(x,-))\}$. Since $A \in \tau_R$, $R_{\tau_R}(x,y) \odot A(x) = \bigwedge_{B\in\tau}(B(x) \to B(y)) \odot A(x) \le (A(x) \to A(y)) \odot A(x) \le A(y)$. Hence $\bigvee_{x\in X}(A(x) \odot \overline{R}(x,y)) \le A(y)$. Since $A(y) = A(y) \odot \overline{R}(y,y) \le \bigvee_{x\in X}(A(x) \odot \overline{R}(x,y))$, $\bigvee_{x\in X}(A(x) \odot \overline{R}(x,y)) = A(y)$. Thus, $A \in \tau$.

Let $A \in \tau$. Since $R \le \overline{R}$, $A(x) \odot R(x,y) \le A(x) \odot \overline{R}(x,y) = A(y)$. Thus, $A \in \tau_R$.

Let $A \in \tau$. Then $\bigvee_{x\in X}(A(x) \odot \overline{R}(x,y)) = A(y)$. Put $A(x) = a_x$. Then $\bigvee_{x\in X}(a_x \odot \overline{R}(x,-)) \in \tau_1$.

Let $D = \bigvee_{x\in X}(a_x \odot \overline{R}(x,-)) \in \tau_1$. Then

$$\begin{aligned}&\textstyle\bigvee_{w\in X}(D(w) \odot \overline{R}(w,y))\\ &= \textstyle\bigvee_{w\in X}\Big(\bigvee_{x\in X}(A(x) \odot \overline{R}(x,w)) \odot \overline{R}(w,y)\Big)\\ &= \textstyle\bigvee_{x\in X}(A(x) \odot \bigvee_{w\in X}(\overline{R}(x,w) \odot \overline{R}(w,y)))\\ &= \textstyle\bigvee_{x\in X}(A(x) \odot \overline{R}(x,y)) = D(y).\end{aligned}$$

Thus, $D \in \tau$. Hence $\tau_R = \tau = \tau_1$.

(5) Put $\eta = \{A \in L^X \mid A = \bigwedge_{y\in X}(\overline{R}(-,y) \to A(y))\}$ and $\eta_1 = \{\bigwedge_{y\in X}(\overline{R}(-,y) \to b_y)\}$. Since $A \in \tau_R$, $R_{\tau_R}(x,y) \to A(y) = \bigwedge_{B\in\tau}(B(x) \to B(y)) \to A(y) \ge (A(x) \to A(y)) \to A(y) \ge A(x)$. Hence $A(x) \le \bigwedge_{y\in X}(\overline{R}(x,y) \to A(y))$. Since $A(y) =$

$\overline{R}(y,y) \to A(y) \geq \bigwedge_{y\in X}(\overline{R}(x,y) \to A(y))$, $A(x) = \bigwedge_{y\in X}(\overline{R}(x,y) \to A(y))$. Thus, $A \in \eta$.

Let $A \in \eta$. Since $R \leq \overline{R}$, $\bigwedge_{y\in X}(R(x,y) \to A(y)) \geq \bigwedge_{y\in X}(\overline{R}(x,y) \to A(y)) = A(x)$. Thus, $R(x,y) \to A(y) \geq A(x)$ iff $A(x) \odot R(x,y) \leq A(y)$. So, $A \in \tau_R$.

Let $A \in \eta$. Then $A = \bigwedge_{y\in X}(\overline{R}(-,y) \to A(y))$. Put $A(y) = b_y$. Then $A = \bigwedge_{y\in X}(\overline{R}(-,y) \to b_y) \in \eta_1$.

Let $A = \bigwedge_{y\in X}(\overline{R}(-,y) \to b_y) \in \eta_1$. Then

$$\begin{aligned}
&\textstyle\bigwedge_{w\in X}(\overline{R}(x,w) \to A(w))\\
&= \textstyle\bigwedge_{w\in X}(\overline{R}(x,w) \to \bigwedge_{y\in X}(\overline{R}(w,y) \to b_y))\\
&= \textstyle\bigwedge_{w\in X}\bigwedge_{y\in X}(\overline{R}(x,w) \to (\overline{R}(w,y) \to b_y))\\
&= \textstyle\bigwedge_{w\in X}\bigwedge_{y\in X}((\overline{R}(x,w) \odot \overline{R}(w,y)) \to b_y)\\
&= \textstyle\bigwedge_{y\in X}(\bigvee_{w\in X}(\overline{R}(x,w) \odot \overline{R}(w,y)) \to b_y)\\
&= \textstyle\bigwedge_{y\in X}(\overline{R}(x,y) \to b_y) = A(x).
\end{aligned}$$

Thus, $A \in \eta$. Hence $\tau_R = \eta = \eta_1$.

(6) It follows from $\bigvee_{x\in X}(A(x) \odot \overline{R}(-,x)) = \bigvee_{x\in X}(A(x) \odot \overline{R}^{-1}(x,-)) = A$ iff $A \in \tau_{R^{-1}} = \tau_R^*$.

(7) It follows from $\bigwedge_{x\in X}(\overline{R}(x,-) \to A(x)) = \bigwedge_{x\in X}(\overline{R}^{-1}(-,x) \to A(x)) = A$ iff $A \in \tau_{R^{-1}} = \tau_R^*$.

(8) Put $B = \bigvee_{x\in X}(A(x) \odot \overline{R}(x,-))$. Then $B \in \tau_R$ from:

$$\begin{aligned}
&\textstyle\bigvee_{w\in X}(B(w) \odot \overline{R}(w,y))\\
&= \textstyle\bigvee_{w\in X}\left(\bigvee_{x\in X}(A(x) \odot \overline{R}(x,w)) \odot \overline{R}(w,y)\right)\\
&= \textstyle\bigvee_{x\in X}(A(x) \odot \bigvee_{w\in X}(\overline{R}(x,w) \odot \overline{R}(w,y)))\\
&= \textstyle\bigvee_{x\in X}(A(x) \odot \overline{R}(x,y)) = B(y).
\end{aligned}$$

If $A \leq E$ and $E \in \tau_R$, then $B \leq E$ from:

$$B(y) = \bigvee_{x\in X}(A(x) \odot \overline{R}(x,y)) \leq \bigvee_{x\in X}(E(x) \odot \overline{R}(x,y)) = E(y).$$

Hence $C_{\tau_R} = B$.

(9) Let $B = \bigwedge_{y\in X}(\overline{R}(-,y) \to A(y)) \in \tau_R$ from

$$\begin{aligned}
&\textstyle\bigwedge_{w\in X}(\overline{R}(x,w) \to B(w))\\
&= \textstyle\bigwedge_{w\in X}(\overline{R}(x,w) \to \bigwedge_{y\in X}(\overline{R}(w,y) \to A(y)))\\
&= \textstyle\bigwedge_{w\in X}\bigwedge_{y\in X}(\overline{R}(x,w) \to (\overline{R}(w,y) \to A(y)))\\
&= \textstyle\bigwedge_{w\in X}\bigwedge_{y\in X}((\overline{R}(x,w) \odot \overline{R}(w,y)) \to A(y))\\
&= \textstyle\bigwedge_{y\in X}(\bigvee_{w\in X}(\overline{R}(x,w) \odot \overline{R}(w,y)) \to A(y))\\
&= \textstyle\bigwedge_{y\in X}(\overline{R}(x,y) \to A(y)) = B(x).
\end{aligned}$$

If $E \le A$ and $E \in \tau_R$, then $E \le B$ from:

$$E(x) = \bigwedge_{y\in X} (\overline{R}(x,y) \to E(y)) \le \bigwedge_{y\in X} (\overline{R}(x,y) \to A(y)) = B(x).$$

Hence $I_{\tau_R} = B$.

(11)

$$\begin{array}{ll} C_{\tau_R}(A) & = \bigwedge\{B \mid A \le B,\ B \in \tau_{R_X}\} \\ & = \bigwedge\{B \mid B^* \le A^*,\ B^* \in \tau_{R_X^{-1}}\} \\ & = \Big(\bigvee\{B^* \mid B^* \le A^*,\ B^* \in \tau_{R_X^{-1}}\}\Big)^* \\ & = (I_{\tau_{R^{-1}}}(A^*))^*. \end{array}$$

$$\begin{array}{l} (I_{\tau_{R^{-1}}}(A^*))^* = \Big(\bigwedge_{x\in X}(\overline{R}(x,-) \to A^*(x))\Big)^* \\ = \bigvee_{x\in X}(\overline{R}(x,-) \odot A(x)) = C_{\tau_R}(A). \end{array}$$

□

Theorem 3.2. *Let R_X and R_Y be fuzzy relations and $f : X \to Y$ a map with $R_X(x,y) \le R_Y(f(x), f(y))$ for all $x,y \in X$. Then the following equivalent conditions hold.*

(1) $f^{-1}(B) \in \tau_{R_X}$ *for all* $B \in \tau_{R_Y}$.

(2) $f^{-1}(B) \in \tau^*_{R_X}$ *for all* $B \in \tau^*_{R_Y}$.

(3) $R_{\tau_{R_X}}(x,y) \le R_{\tau_{R_Y}}(f(x), f(y))$ *for all* $x,y \in X$.

(4) $R_{\tau^*_{R_X}}(x,y) = R^{-1}_{\tau_{R_X}}(y,x) \le R^{-1}_{\tau_{R_Y}}(f(y), f(x)) = R_{\tau^*_{R_Y}}(f(x), f(y))$ *for all* $x,y \in X$.

(5) $f(C_{\tau_{R_X}}(A)) \le C_{\tau_{R_Y}}(f(A))$ *for all* $A \in L^X$.

(6) $f(C_{\tau_{R_X^{-1}}}(A)) \le C_{\tau_{R_Y^{-1}}}(f(A))$ *for all* $A \in L^X$.

(7) $C_{\tau_{R_X}}(f^{-1}(B)) \le f^{-1}(C_{\tau_{R_X}}(B))$ *for all* $B \in L^Y$.

(8) $C_{\tau_{R_X^{-1}}}(f^{-1}(B)) \le f^{-1}(C_{\tau_{R_Y^{-1}}}(B))$ *for all* $B \in L^Y$.

(9) $f^{-1}(I_{\tau_{R_X}}(B)) \le I_{\tau_{R_Y}}(f^{-1}(B))$ *for all* $B \in L^Y$.

(10) $f^{-1}(I_{\tau_{R_Y^{-1}}}(B)) \le I^{-1}_{\tau_{R_X^{-1}}}(f^{-1}(B))$ *for all* $B \in L^Y$.

Proof. (1) For all $B \in \tau_{R_Y}$, $f^{-1}(B) \in \tau_{R_X}$ from:

$$\begin{array}{ll} f^{-1}(B)(x) \odot R_X(x,y) & \le B(f(x)) \odot R_Y(f(x), f(y)) \\ & \le B(f(y)) = f^{-1}(B)(y). \end{array}$$

(1) $\Leftrightarrow$ (2) It follows from (1) and Theorem 3.1(2).

(1) ⇒ (3)

$$\begin{aligned} R_{\tau_{R_Y}}(f(x),f(y)) &= \bigwedge_{B\in\tau_{R_Y}}(B(f(x))\to B(f(y))) \\ &= \bigwedge_{B\in\tau_{R_Y}}(f^{-1}(B)(x)\to f^{-1}(B)(y)) \\ &\geq \bigwedge_{A\in\tau_{R_X}}(A(x)\to A(y)) = R_{\tau_{R_X}}(x,y) \end{aligned}$$

(1) ⇒ (5)

$$\begin{aligned} C_{R_Y}(f(A)) &= \bigwedge\{B \mid f(A)\leq B,\ B\in\tau_{R_Y}\} \\ &\geq \bigwedge\{B \mid A\leq f^{-1}(B),\ f^{-1}(B)\in\tau_{R_X}\} \\ &\geq \bigwedge\{f(f^{-1}(B)) \mid A\leq f^{-1}(B),\ f^{-1}(B)\in\tau_{R_X}\} \\ &\geq f\Big(\bigwedge\{f^{-1}(B) \mid A\leq f^{-1}(B),\ f^{-1}(B)\in\tau_{R_X}\}\Big) \\ &\geq f(C_{R_Y}(A)). \end{aligned}$$

(3) ⇒ (5)

$$\begin{aligned} C_{R_Y}(f(A))(f(x)) &= \bigvee_{w\in Y}(f(A)(w)\odot R_Y(w,f(x))) \\ &\geq \bigvee_{z\in X}(f(A)(f(z))\odot R_Y(f(z),f(x))) \\ &\geq \bigvee_{z\in X}(A(z)\odot R_X(z,x)) = C_{R_X}(A)(x) \end{aligned}$$

(5) ⇒ (7) By (5), put $A=f^{-1}(B)$. Since $f(C_{\tau_{R_X}}(f^{-1}(B)))\leq C_{\tau_{R_Y}}(f(f^{-1}(B)))\leq C_{\tau_{R_Y}}(B)$, we have $C_{\tau_{R_X}}(f^{-1}(B))\leq f^{-1}(C_{\tau_{R_X}}(B))$.

(7) ⇒ (1) For all $B\in\tau_{R_Y}$, $C_{\tau_Y}(B)=B$. Since $C_{\tau_{R_X}}(f^{-1}(B))\leq f^{-1}(C_{\tau_{R_X}}(B))=f^{-1}(B)$, $f^{-1}(B)\in\tau_{R_X}$.

(1) ⇒ (9)

$$\begin{aligned} f^{-1}(I_{R_Y}(B))(x) &= I_{R_Y}(B)(f(x) \\ &= \bigvee\{D(f(x)) \mid D\leq B,\ D\in\tau_{R_Y}\} \\ &= \bigvee\{f^{-1}(D)(x) \mid f^{-1}(D)\leq f^{-1}(B),\ f^{-1}(D)\in\tau_{R_X}\} \\ &\leq \bigvee\{E(x) \mid E\leq f^{-1}(B),\ E\in\tau_{R_X}\} \\ &= I_{R_X}(f^{-1}(B)). \end{aligned}$$

$$\begin{aligned} f^{-1}(I_{R_Y}(B))(x) &= I_{R_Y}(B)(f(x)) \\ &= \bigwedge_{w\in Y}(R_Y(f(x),w)\to B(w)) \\ &\leq \bigwedge_{z\in X}(R_Y(f(x),f(z))\to B(f(z))) \\ &\leq \bigwedge_{z\in X}(R_X(x,z)\to f^{-1}(B)(z)) \\ &= I_{R_X}(f^{-1}(B))(x) \end{aligned}$$

(9) ⇒ (1) For all $B\in\tau_{R_Y}$, $I_{\tau_Y}(B)=B$. Since $I_{\tau_{R_X}}(f^{-1}(B))\geq f^{-1}(I_{\tau_{R_X}}(B))=f^{-1}(B)$, $f^{-1}(B)\in\tau_{R_X}$.

Other cases are similarly proved. □

Example 3.3. Let $(L=[0,1],\odot,\to,^*)$ be a complete residuated lattice with the law of double negation defined by

$$x\odot y=(x+y-1)\vee 0,\ \ x\to y=(1-x+y)\wedge 1,\ x^*=1-x.$$

Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$ be sets and $f : X \to Y$ as follows:

$$f(a) = x, f(b) = y, f(c) = z.$$

(1) Define $R_X \in L^{X\times X}, R_Y \in L^{Y\times Y}$ as follows

$$R_X = \begin{pmatrix} 0.5 & 0.9 & 0.1 \\ 0.7 & 0.8 & 0.5 \\ 0.9 & 0.6 & 0.7 \end{pmatrix}, R_Y = \begin{pmatrix} 0.6 & 0.9 & 0.7 \\ 0.8 & 1 & 0.5 \\ 0.9 & 0.7 & 0.8 \end{pmatrix}.$$

Then $R_X(a, b) \le R_Y(f(a), f(b))$ for all $a, b \in X$.

$$R^r_X = \begin{pmatrix} 1 & 0.9 & 0.1 \\ 0.7 & 1 & 0.5 \\ 0.9 & 0.6 & 1 \end{pmatrix}, R^r_Y = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.8 & 1 & 0.5 \\ 0.9 & 0.7 & 1 \end{pmatrix},$$

For $n \ge 2$, $(R^r_X)^2 = (R^r_X)^n$ and $(R^r_Y)^2 = (R^r_Y)^n$ as follows:

$$(R^r_X)^2 = \begin{pmatrix} 1 & 0.9 & 0.4 \\ 0.7 & 1 & 0.5 \\ 0.9 & 0.8 & 1 \end{pmatrix}, (R^r_Y)^2 = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.8 & 1 & 0.5 \\ 0.9 & 0.8 & 1 \end{pmatrix}.$$

Then

$$\overline{R}_X = \bigvee_{n\in N} (R^r_X)^n = (R^r_X)^2,$$

$$\overline{R}_Y = \bigvee_{n\in N} (R^r_Y)^n = (R^r_Y)^2.$$

Moreover,

$$R_{\tau_{R_X}}(a, b) = \bigwedge_{B\in\tau_{R_X}} (B(a) \to B(b)) = (R^r_X)^2(a, b)$$

$$R_{\tau_{R_Y}}(x, y) = \bigwedge_{B\in\tau_{R_Y}} (B(x) \to B(y)) = (R^r_Y)^2(x, y).$$

Then $R_{\tau_{R_X}}(a, b) \le R_{\tau_{R_X}}(f(a), f(b))$ for all $a, b \in X$.

(2)

$$\begin{aligned} \tau_{R_X} &= \{\textstyle\bigvee_{x\in X}(a_x \odot \overline{R}_X(x, -)\} \\ &= \{(a_1 \odot \overline{R}_X(a, -)) \vee (a_2 \odot \overline{R}_X(b, -)) \vee (a_3 \odot \overline{R}_X(c, -))\} \end{aligned}$$

$$\begin{aligned} \tau^*_{R_X} &= \tau_{R^{-1}_X} = \{\textstyle\bigwedge_{x\in X}(\overline{R}_X(x, -) \to a_x)\} \\ &= \{(\overline{R}_X(a, -) \to a_1) \wedge (\overline{R}_X(b, -) \to a_2) \wedge (\overline{R}_X(c, -) \to a_3)\} \end{aligned}$$

where $a_i \in L$ and

$$\overline{R}_X(a, -) = (1, 0.9, 0.4),\ \overline{R}_X(b, -) = (0.7, 1, 0.5),\ \overline{R}_X(c, -) = (0.9, 0.6, 1).$$

For $A = (0.5 \odot \overline{R}_X(a, -)) \vee (0.9 \odot \overline{R}_X(b, -)) \vee (0.8 \odot \overline{R}_X(b, -)) = (0.7, 0.9, 0.8) = \bigvee_{x\in X}(A(x) \odot \overline{R}_X(x, -) \in \tau_{R_X}$.

For $B = (\overline{R}_X(a,-) \to 0.5) \wedge (0.9 \odot \overline{R}_X(b,-) \to 0.9) \wedge (\overline{R}_X(b,-) \to 0.8) = (0.5, 0.6, 0.8) = \bigwedge_{x\in X}(\overline{R}_X(x,-) \to B(x)) \in \tau^*_{R_X}$.

$$\begin{aligned}\tau_{R_X} &= \{\bigwedge_{y\in X}(\overline{R}_X(-,y) \to b_y)\}\\ &= \{(\overline{R}_X(-,a) \to b_1) \wedge (\overline{R}_X(-,b) \to b_2) \wedge (\overline{R}_X(-,c) \to b_3)\}\end{aligned}$$

$$\begin{aligned}\tau^*_{R_X} &= \tau_{R_X^{-1}} = \{\bigvee_{y\in X}(\overline{R}_X(-,y) \odot b_y)\}\\ &= \{(\overline{R}_X(-,a) \odot b_1) \vee (\overline{R}_X(-,b) \odot b_2) \vee (\overline{R}_X(-,c) \odot b_3)\}\end{aligned}$$

where $b_i \in L$ and

$$\overline{R}_X(-,a) = (1, 0.7, 0.9),\ \overline{R}_X(-,b) = (0.9, 1, 0.8),\ \overline{R}_X(-,c) = (0.4, 0.5, 1).$$

For $A = (\overline{R}_X(-,a) \to 0.3) \wedge (\overline{R}_X(-,b) \to 0.5) \wedge (\overline{R}_X(-,c) \to 0.2) = (0.3, 0.6, 0.4) \wedge (0.6, 0.5, 0.7) \wedge (0.8, 0.7, 0, 2) = (0.3, 0.5, 0, 2) = \bigwedge_{x\in X}(\overline{R}_X(-,x) \to A(x)) \in \tau_{R_X}$.

For $B = (\overline{R}_X(-,a) \odot 0.3) \vee (\overline{R}_X(-,b) \odot 0.5) \vee (\overline{R}_X(-,c) \odot 0.2) = (0.3, 0, 0.2) \vee (0.4, 0.5, 0.3) \vee (0, 0, 0, 2) = (0.4, 0.5, 0, 3) = \bigvee_{x\in X}(\overline{R}_X(-,x) \odot A(x)) \in \tau^*_{R_X}$.

(3)

$$\begin{aligned}\tau_{R_Y} &= \{\bigvee_{x\in Y}(a_x \odot \overline{R}_Y(x,-)\}\\ &= \{(a_1 \odot \overline{R}_Y(x,-)) \vee (a_2 \odot \overline{R}_Y(y,-)) \vee (a_3 \odot \overline{R}_Y(z,-))\}\end{aligned}$$

$$\begin{aligned}\tau^*_{R_Y} &= \tau_{R_Y^{-1}} = \{\bigwedge_{x\in Y}(\overline{R}_Y(x,-) \to a_x)\}\\ &= \{(\overline{R}_Y(x,-) \to a_1) \wedge (\overline{R}_Y(y,-) \to a_2) \wedge (\overline{R}_Y(z,-) \to a_3)\}\end{aligned}$$

where $a_i \in L$ and

$$\overline{R}_Y(x,-) = (1, 0.9, 0.7),\ \overline{R}_Y(y,-) = (0.8, 1, 0.5),\ \overline{R}_Y(z,-) = (0.9, 0.8, 1).$$

$$\begin{aligned}\tau_{R_Y} &= \{\bigwedge_{y\in Y}(\overline{R}_Y(-,y) \to b_y)\}\\ &= \{(\overline{R}_Y(-,x) \to b_1) \wedge (\overline{R}_Y(-,y) \to b_2) \wedge (\overline{R}_Y(-,z) \to b_3)\}\end{aligned}$$

$$\begin{aligned}\tau^*_{R_Y} &= \tau_{R_Y^{-1}} = \{\bigvee_{y\in Y}(\overline{R}_Y(-,y) \odot b_y)\}\\ &= \{(\overline{R}_Y(-,x) \odot b_1) \vee (\overline{R}_Y(-,y) \odot b_2) \vee (\overline{R}_Y(-,z) \odot b_3)\}\end{aligned}$$

where $b_i \in L$ and

$$\overline{R}_Y(-,x) = (1, 0.8, 0.9),\ \overline{R}_Y(-,y) = (0.9, 1, 0.8),\ \overline{R}_Y(-,z) = (0.7, 0.5, 1).$$

(4) For $A = (0.2, 0.8, 0.6) \in L^X$,

$$\begin{aligned}C_{R_X}(A) &= (0.5, 0.8, 0.6),\ C_{R_Y}(f(A)) = (0.6, 0.8, 0.6)\\ I_{R_X}(A) &= (0.2, 0.5, 0.3),\ C_{R_Y}(f(A)) = (0.2, 0.4, 0.3)\end{aligned}$$

$$\begin{aligned}C_{R_X^{-1}}(A) &= (0.7, 0.8, 0.6),\ C_{R_Y^{-1}}(f(A)) = (0.7, 0.8, 0.6)\\ I_{R_X^{-1}}(A) &= (0.2, 0.3, 0.6),\ I_{R_Y^{-1}}(f(A)) = (0.2, 0.3, 0.5).\end{aligned}$$

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