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LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE SASAKIAN MANIFOLD WITH A NON-METRIC θ-CONNECTION

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Abstract. In this paper, we study two types of 1-lightlike submanifolds, named by lightlike hypersurface and half lightlike submanifold, of an indefinite Sasakian manifold admitting non-metric θ-connections. We prove that there exist no such two types of 1-lightlike submanifolds of an indefinite Sasakian manifold.

1. Introduction

A linear(affine) connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *non-metric θ-connection* if, for any vector fields X, Y and Z on $\bar{M}$, it satisfies

$$(\bar{\nabla}_X \bar{g})(Y, Z) = -\,\theta(Y)\bar{g}(X, Z) - \theta(Z)\bar{g}(X, Y), \tag{1.1}$$

where θ is a 1-form, associated with a non-vanishing smooth vector field ζ by

$$\theta(X) = \bar{g}(X, \zeta).$$

Two special cases are important for both the mathematical study and the applications to physics: (1) A non-metric θ-connection $\bar{\nabla}$ on $\bar{M}$ is called a *semi-symmetric non-metric connection* if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X, Y) = \theta(Y)X - \theta(X)Y.$$

The notion of semi-symmetric non-metric connections on a Riemannian manifold was introduced by Ageshe and Chafle [1] and later studied by many authors. The lightlike version of Riemannian manifolds with semi-symmetric non-metric connections have been studied by some authors [11, 12, 13, 14, 17].

(2) A non-metric θ-connection $\bar{\nabla}$ is called a *quarter-symmetric non-connection* if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X, Y) = \theta(Y)\phi X - \theta(X)\phi Y,$$

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where ϕ is a $(1,1)$-type tensor field. The quarter-symmetric non-metric connection was introduced by S. Golad [8], and then, studied by many authors [2, 3, 16].

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics. The study of such notion was initiated by Duggal and Bejancu [4] and later studied by many authors [6, 7]. Although now we have lightlike version of a large variety of Riemannian submanifolds, the geometry of lightlike submanifolds of indefinite Sasakian manifolds admitting non-metric θ-connections has not been introduced as yet.

In this paper, we study the geometry of two type of lightlike submanifolds, named by lightlike hypersurface and half lightlike submanifold, of an indefinite Sasakian manifold $\bar{M}$ admitting non-metric θ-connection, in which the 1-form θ and its associated vector field ζ, defined by (1.1), is identical with the structure 1-form θ and its associated vector field ζ, respectively, of the indefinite Sasakian structure $(J,\, \zeta,\, \theta,\, \bar{g})$. We prove the following result:

• *There exist no such two types of* 1*-lightlike submanifolds of an indefinite Sasakian manifold* $\bar{M}$ *admitting non-metric* θ*-connections.*

From these results we deduce to the following result:

• *There exist no such two types of* 1*-lightlike submanifolds of an indefinite Sasakian manifold* $\bar{M}$ *admitting either semi-symmetric non-metric connection or quarter-symmetric non-metric connection.*

2. Non-existence Theorem for Lightlike Hypersurfaces

An odd-dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ is said to be an *indefinite Sasakian manifold* ([9]∼[10]) if there exists a structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$, where J is a (1, 1)-type tensor field, ζ is a vector field which is called the *structure vector field* of $\bar{M}$ and θ is a 1-form such that, for any vector fields X and Y on $\bar{M}$,

$$J^2X = -X + \theta(X)\zeta,\;\; \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\theta(X)\theta(Y),\;\; \theta(\zeta) = 1, \tag{2.1}$$

$$(\bar{\nabla}_X J)Y = \bar{g}(X,Y)\zeta - \epsilon\theta(Y)X \tag{2.2}$$

holds, where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike respectively.

In this case, we show that $J\zeta = 0$ and $\theta\circ J = 0$. The structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite Sasakian structure* of $\bar{M}$. From (2.1) and (2.2), we get

$$\bar{\nabla}_X\zeta = -\epsilon JX, \qquad d\theta(X,Y) = \bar{g}(X,\, JY). \tag{2.3}$$

In the entire discussion of this article, we shall assume that the structure vector field ζ of $\bar{M}$ to be unit spacelike, *i.e.*, $\epsilon = 1$, without loss of generality.

Let (M, g) be a lightlike hypersurface, with a screen distribution $S(TM)$, of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. We follow Duggal and Bejancu [4] for notations and structure equations used in this section. For any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution respectively. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.7}$$

for any X, $Y \in \Gamma(TM)$, where the symbols ∇ and ∇^* are the induced linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ and σ are 1-forms on TM.

The induced connection ∇ on M is not metric and satisfies

$$\begin{aligned}(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\ - \theta(Y)g(X, Z) - \theta(Z)g(X, Y),\end{aligned} \tag{2.8}$$

for any X, Y, $Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N), \quad \forall\, X \in \Gamma(TM).$$

From the fact that $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$, and satisfies

$$B(X, \xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{2.9}$$

From this result and (2.4), for all $X \in \Gamma(TM)$ we obtain

$$\bar{\nabla}_X \xi = -A^*_\xi X - \sigma(X)\xi. \tag{2.10}$$

Now we set $a = \theta(N)$ and $b = \theta(\xi)$. For all $X, Y \in \Gamma(TM)$, the above second fundamental forms B and C are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X, Y) - bg(X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.11}$$

$$g(A_N X, PY) = C(X, PY) - ag(X, PY) - \eta(X)\theta(PY), \tag{2.12}$$
$$\bar{g}(A_N X, N) = -a\eta(X), \qquad \sigma(X) = \tau(X) - b\eta(X).$$

Now we quote the following result by Jin [9]:

Lemma 1. *Let M be a lightlike hypersurface of an indefinite almost contact metric manifold $\bar{M}$. Then $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1.*

Theorem 2.1. *There exist no lightlike hypersurfaces of an indefinite Sasakian manifold admitting a non-metric θ-connection.*

Proof. Now we consider two vector fields V and U on $S(TM)$ such that

$$J\xi = -V, \qquad JN = -U. \tag{2.13}$$

For any $X \in \Gamma(TM)$, the action JX of X by J is expressed as

$$JX = FX + u(X)N, \tag{2.14}$$

where FX is the tangential component of JX and u is a 1-form given by

$$u(X) = g(X, V), \quad \forall\, X \in \Gamma(TM). \tag{2.15}$$

Applying $\bar{\nabla}_X$ to $(2.13)_1$ and using (2.2), (2.4), (2.10) and (2.14), we have

$$\nabla_X V = F(A^*_\xi X) - \sigma(X)V + bX,$$
$$B(X, V) = u(A^*_\xi X), \quad \forall\, X \in \Gamma(TM).$$

On the other hand, taking $Y = V$ to (2.11) and using (2.15), we have

$$B(X, V) = u(A^*_\xi X) + bu(X), \quad \forall\, X \in \Gamma(TM).$$

From the last two equations, we obtain $bu(X) = 0$ for all $X \in \Gamma(TM)$. Taking $X = V$ to this result and using (2.1), we get $b = 0$. This implies that ζ is tangent to M. Replacing Y by ζ to (2.4) and using (2.3), we obtain

$$\nabla_X \zeta = -FX, \quad B(X, \zeta) = -u(X), \quad \forall\, X \in \Gamma(TM). \tag{2.16}$$

Applying ∇_X to $g(\zeta, \zeta) = 1$ and using (2.8), we get

$$g(\nabla_X \zeta,\, \zeta) = \theta(X) - aB(X, \zeta), \quad \forall\, X \in \Gamma(TM).$$

Substituting $(2.16)_1$ into the last equation and using (2.15) and $(2.16)_2$, we have $\theta = 0$ on TM. It is a contradiction as $\theta(\zeta) = 1$. Thus there exist no lightlike hypersurfaces of an indefinite Sasakian manifold admitting a non-metric θ-connection. □

Corollary 1. *There exist no lightlike hypersurfaces of an indefinite Sasakian manifold admitting a semi-symmetric non-metric connection or a quarter-symmetric non-metric connection.*

3. Non-existence Theorem for Half Lightlike Submanifolds

Let (M, g) be a half lightlike submanifold, with a screen distribution $S(TM)$ and the radical distribution $Rad(TM)$, of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. We follow Duggal and Jin [5] for notations and structure equations used in this section. For any null section ξ of $Rad(TM)$, there exists a uniquely defined lightlike vector bundle $ltr(TM)$ and a null vector field N of $ltr(TM)$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. In this case, the local Gauss and Weingartan formulas of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{3.1}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{3.2}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N, \tag{3.3}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{3.4}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{3.5}$$

for all X, $Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are linear operators on TM and τ, ρ, ϕ and σ are 1-forms on TM.

Using (1.1) and (3.1), for all X, Y, $Z \in \Gamma(TM)$ we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) - \theta(Y)g(X, Z) - \theta(Z)g(X, Y). \tag{3.6}$$

From the facts $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \bar{g}(\bar{\nabla}_X Y, L)$, we know that B and D are independent of the choice of $S(TM)$ and satisfy

$$B(X, \xi) = 0, \quad D(X, \xi) = -\phi(X), \quad \forall\, X \in \Gamma(TM). \tag{3.7}$$

From this result and (3.1), for all $X \in \Gamma(TM)$ we obtain

$$\bar{\nabla}_X \xi = -A^*_\xi X - \sigma(X)\xi - \phi(X)L. \tag{3.8}$$

We set $b = \theta(\xi), a = \theta(N)$ and $e = \theta(L)$. For any $X, Y \in \Gamma(TM)$, the above three local second fundamental forms are related to their shape operators by

$$g(A^*_\xi X, Y) = B(X, Y) - bg(X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{3.9}$$

$$g(A_L X, Y) = D(X, Y) - eg(X, Y) + \phi(X)\eta(Y), \tag{3.10}$$
$$\bar{g}(A_L X, N) = \rho(X) - e\eta(X),$$

$$g(A_N X, PY) = C(X, PY) - ag(X, PY) - \eta(X)\theta(PY), \tag{3.11}$$
$$\bar{g}(A_N X, N) = -a\eta(X), \qquad \sigma(X) = \tau(X) - b\eta(X).$$

Now we quote the following result by Jin [10]:

Lemma 2. *Let M be a half lightlike submanifold of an indefinite almost contact metric manifold $\bar{M}$. Then the distributions $J(TM^\perp)$, $J(tr(TM))$ and $J(S(TM^\perp))$ are vector subbundles of $S(TM)$, of rank 1.*

Theorem 3.1. *There exist no half lightlike submanifolds of an indefinite Sasakian manifold admitting a non-metric θ-connection.*

Proof. Now we consider three vector fields V, U and W on $S(TM)$ such that

$$J\xi = -V, \quad JN = -U, \quad W = -JL. \tag{3.12}$$

For any $X \in \Gamma(TM)$, the action JX of X by J is expressed as

$$JX = FX + u(X)N + w(X)L, \tag{3.13}$$

where FX is the tangential component of JX and u and w are 1-forms given by

$$u(X) = g(X, V) \quad w(X) = g(X, W), \quad \forall\, X \in \Gamma(TM). \tag{3.14}$$

Applying $\bar{\nabla}_X$ to $(3.12)_1$ and using (2.2), (3.1), (3.8) and (3.13), we have

$$\nabla_X V = F(A^*_\xi X) - \sigma(X)V - \phi(X)W + bX, \tag{3.15}$$

$$B(X, V) = u(A^*_\xi X), \quad D(X, V) = w(A^*_\xi X), \quad \forall\, X \in \Gamma(TM). \tag{3.16}$$

On the other hand, taking $Y = V$ to (3.9) and using $(3.14)_1$, we have

$$B(X, V) = u(A^*_\xi X) + bu(X).$$

From this and $(3.16)_1$, we obtain $bu(X) = 0$ for any $X \in \Gamma(TM)$. Thus we get $b = 0$. It follow that $B(X, Y) = g(A^*_\xi X, Y)$ and $\tau = \sigma$. Applying $\bar{\nabla}_X$ to $(3.12)_3$ and using (2.2), (3.1), (3.3), (3.12) and (3.13), we have

$$\nabla_X W = F(A_L X) + \phi(X)U + eX, \tag{3.17}$$

$$B(X, W) = u(A_L X), \quad D(X, W) = w(A_L X). \tag{3.18}$$

On the other hand, taking $Y = W$ to (3.10), we have

$$D(X, W) = w(A_L X) + ew(X).$$

From this and $(3.18)_2$, we obtain $ew(X) = 0$ for any $X \in \Gamma(TM)$. Thus we get $e = 0$. As $b = e = 0$, the structure vector field ζ is tangent to M.

Applying $\bar{\nabla}_X$ to (3.13) and using (3.1) $\sim$ (3.3), (3.12) and (3.13), we have

$$\begin{aligned}
(\nabla_X F)Y &= u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W \\
&\quad + g(X,Y)\zeta - \theta(Y)X, \\
(\nabla_X u)Y &= -\,u(Y)\tau(X) - w(Y)\phi(X) - B(X, FY), \\
(\nabla_X w)(Y) &= -\,u(Y)\rho(X) - D(X, FY).
\end{aligned}$$

On the other hand, applying ∇_X to $u(Y) = g(Y, V)$ and $w(Y) = g(Y, W)$ by turns and using (3.6), (3.15), $(3.16)_1$, (3.17), $(3.18)_1$ and $\theta \circ J = 0$, we have

$$\begin{aligned}
(\nabla_X u)(Y) &= -\,u(Y)\tau(X) - w(Y)\phi(X) - B(X, FY) - \theta(Y)u(X), \\
(\nabla_X w)(Y) &= -\,u(Y)\rho(X) - D(X, FY) - \theta(Y)w(X).
\end{aligned}$$

From the last four equations, for all X, $Y \in \Gamma(TM)$ we obtain

$$\theta(Y)u(X) = 0, \qquad \theta(Y)w(X) = 0.$$

Taking $X = U$ and $Y = \zeta$ to the first equation, or taking $X = W$ and $Y = \zeta$ to the second equation, we have $1 = 0$. It is a contradiction. Thus there exist no half lightlike submanifolds of an indefinite Sasakian manifold admitting a non-metric θ-connection. □

Corollary 2. *There exist no half lightlike submanifolds of an indefinite Sasakian manifold admitting a semi-symmetric non-metric connection or a quarter-symmetric non-metric connection.*

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