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THE EXISTENCE OF THE RISK-EFFICIENT OPTIONS

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Abstract. We prove the existence of the risk-efficient options proposed by Xu [7]. The proof is given by both indirect and direct ways. Schied [6] showed the existence of the optimal solution of equation (2.1). The one is to use the Schied's result. The other one is to find the sequences converging to the risk-efficient option.

1. Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a complete filtered probability space. Let $S=(S_t)_{0\le t\le T}$ be an adapted positive process which is a semimartingale. It is assumed that the riskless interest rate is zero for simplicity and

$$\mathcal{M} = \{Q \,|\, Q \sim P,\ S \text{ is a local martingale under } Q\} \neq \emptyset$$

to avoid the arbitrage opportunities [4].

Definition 1.1. A *self-financing* strategy (x, ξ) is defined as an initial capital $x \geq 0$ and a predictable process ξ_t such that the value process (value of the current holdings)

$$X_t = x + \int_0^t \xi_u dS_u, \quad t \in [0, T]$$

is P-a.s. well-defined.

The set of admissible self-financing portfolios $\mathcal{X}(x)$ with initial capital x is defined as

$$\mathcal{X}(x) = \left\{ X \,\middle|\, X_t = x + \int_0^t \xi_u \, dS_u \geq c,\ c \text{ is a constant},\ t \in [0, T] \right\}.$$

Let L^0 be the set of all measurable functions in the given probability spaces.

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Definition 1.2. A *coherent measure of risk* $\rho : L^0 \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying the following properties for $X, Y \in L^0$

(1) $\rho(X+Y) \le \rho(X) + \rho(Y)$ (subadditivity),
(2) $\rho(\lambda X) = \lambda \rho(X)$ for $\lambda \ge 0$ (positive homogeneity),
(3) $\rho(X) \ge \rho(Y)$ if $X \le Y$ (monotonicity) ,
(4) $\rho(Y+m) = \rho(Y) - m$ for $m \in \mathbb{R}$ (translation invariance).

The conditions of subadditivity (1) and positive homogeneity (2) in Definition 1.2 can be relaxed to a weaker quantity, i.e., convexity

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda \rho(X) + (1-\lambda)\rho(Y) \quad \text{for any } \lambda \in [0,1]. \tag{1.1}$$

Convexity means that diversification does not increase the risk. Also refer to the papers [1, 3] for coherent or convex risk measures.

Definition 1.3. A map $\rho : L^0 \to \mathbb{R}$ is called a *convex risk measure* if it satisfies the properties of convexity (1.1), monotonicity (3) and translation invariance (4).

Definition 1.4. The *minimal risk* $\rho^x(\cdot)$ with initial capital x is defined as the risk

$$\rho^x(L) = \inf_{X \in \mathcal{X}(x)} \rho(L - X_T) \tag{1.2}$$

where the liability L is a random variable bounded below by a constant at time T, $X_T = x + \int_0^T \xi_u \, dS_u$ and $\rho(L - X_T)$ is a final risk.

Assumption 1.5. The convex risk measure ρ satisfies the *Fatou property*

$$\rho(X) \le \lim_{n\to\infty} \inf \rho(X_n) \text{ if } X_n \to X \quad a.s. \text{ as } n \to \infty. \tag{1.3}$$

Assumption 1.6. $\rho : L^0 \to \mathbb{R}$ satisfies $\rho(X) = \rho(Y)$ whenever $X = Y$ $P-$a.s. and for the positive payoff function H, the bounded conditions

$$\rho(L+H) < \infty \text{ and } -\infty < \rho^0(0). \tag{1.4}$$

Lemma 1.7 ([7])**.** *The minimal risk defined as (1.2) is a convex risk measure. Moreover, the translation invariance property satisfies the following relations*

$$\rho^{x_1}(X - x_2) = \rho^{x_1+x_2}(X) = \rho^{x_1}(X) - x_2 \quad \textit{for any } x_1, x_2 \in \mathbb{R}^+. \tag{1.5}$$

Lemma 1.8 ([7])**.** *Let L be the initial liability bounded below by a constant and H be the positive payoff function. Then for any fixed number x*

$$-\infty < \rho(L-H) \le \rho(L) \le \rho(L+H) < \infty \textit{ and} \tag{1.6}$$

$$-\infty < \rho^x(L-H) \le \rho^x(L) \le \rho^x(L+H) < \infty. \tag{1.7}$$

The risk-efficient options are defined as the options having the same selling price, which minimize the risk. That is, the risk-efficient options are the H that minimizes $\rho^{x_0+\alpha}(L+H)$ with the constraint $p(H)=\alpha$, where $p(H)$ is the selling price of the option H, L is the initial liability, x_0 is the initial capital, and $\rho^{x_0+\alpha}(L+H)$ is the minimal risk obtained by optimal hedging with capital $x_0+\alpha$ as defined in (1.2). Here ρ is a risk measure. Xu [7] defined such risk-efficient options and asked a question of their existence. The option seller could get the same minimal risk even though he or she choose any one of available risk-efficient options. Every contingent claim is replicable, i.e., perfectly hedged in a complete market. We should consider risk-efficient options in an incomplete market.

This paper is structured as follows. We prove the existence of risk-efficient options by using Schied's result in Section 2. We prove it by finding the sequences converging to the risk-efficient option in Section 3.

2. Indirect Proof

In this section, we assume that ρ is convex risk measure satisfying Fatou property and H is $\mathcal{F}_T-$measurable contingent claim which is bounded. Xu [7] treated option H which is positive.

Schied [6] supposes an agent wishes to raise the capital $v(\geq 0)$ by selling a contingent claim and tries to find a contingent claim such that the risk of the terminal liability is minimal among all claims satisfying the issuer's capital constraints, i.e.,

$$\min_{\substack{0\leq H\leq K\\ E[\varphi H]\geq v}} \rho(-H), \tag{2.1}$$

where the price density φ is a $P-$a.s. strictly positive random variable with $E[\varphi]=1$. The problem is called the *Neyman-Pearson problem* for the risk measure ρ.

Lemma 2.1 ([6]). *Assume that the conditions of convexity (1.1), monotonicity in Definition 1.2 and Fatou property (1.3) hold. Then there exists a solution to the Neyman-Pearson problem (2.1).*

Lemma 2.2 ([6]). *Any solution H^* of the Neyman-Pearson problem (2.1) with capital constraint $v\in[0,K]$ satisfies $E[\varphi H^*]=v$.*

In terms of liabilities $-X$ and $-Y$, the properties of convexity (1.1), monotonicity (3) and translation invariance (4) in Definition 1.2 are respectively expressed as

$$\rho(\lambda(-X)+(1-\lambda)(-Y)) \le \lambda\rho(-X)+(1-\lambda)\rho(-Y) \text{ for } \lambda \in [0,1], \tag{2.2}$$

$$\rho(-X) \le \rho(-Y) \text{ if } X \le Y, \tag{2.3}$$

$$\rho(-X+m) = \rho(-X)+m \text{ for } m \in \mathbb{R}. \tag{2.4}$$

The properties of (2.2), (2.3) and (2.4) can be easily derived by taking $\rho(-X) = \psi(X)$ for a convex risk measure $\psi(X)$.

For the option payoff function H and an initial capital x_0, we show that in Theorem 2.4 there exists a *risk-efficient option* H^* satisfying

$$\inf_{\substack{0\le H\le K\\ E[\varphi H]\ge x}} \rho^{x+x_0}(L+H) = \rho^{x+x_0}(L+H^*),$$

where L is the initial liability uniformly bounded below by c_L, and the price density φ is a $P-$a.s. strictly positive random variable with $E[\varphi]=1$.

In a term of liability $-H$, define η as

$$\eta(-H) := \rho^{x+x_0}(L+H). \tag{2.5}$$

Then η is well defined by Assumption 1.6.

Lemma 2.3. *$\eta(-H)$ is a convex risk measure and law-invariant.*

Proof. First, let's prove the convexity. Let H_1, H_2 and H be $\mathcal{F}_T$-measurable payoff functions and $\lambda \in [0,1]$, $m \in \mathbb{R}$.

$$\begin{aligned}
\eta(\lambda(-H_1)+(1-\lambda)(-H_2)) &= \rho^{x+x_0}(L+\lambda H_1+(1-\lambda)H_2)\\
&= \rho^{x+x_0}(\lambda(L+H_1)+(1-\lambda)(L+H_2))\\
&\le \lambda\rho^{x+x_0}(L+H_1)+(1-\lambda)\rho^{x+x_0}(L+H_2)\\
&= \lambda\eta(-H_1)+(1-\lambda)\eta(-H_2).
\end{aligned}$$

Secondly, let's prove the monotonicity. Let $H_1 \le H_2$. Then

$$\begin{aligned}
\eta(-H_1) = \rho^{x+x_0}(L+H_1) &= \inf_{X\in\mathcal{X}(x+x_0)} \rho(L+H_1-X_T)\\
&\le \inf_{X\in\mathcal{X}(x+x_0)} \rho(L+H_2-X_T) = \eta(-H_2).
\end{aligned}$$

Thirdly, let's prove the translation invariance.

$$\begin{aligned}
\eta(-H+m) &= \rho^{x+x_0}(L-(-H+m)) = \inf_{X\in\mathcal{X}(x+x_0)} \rho(L+H-X_T-m)\\
&= \inf_{X\in\mathcal{X}(x+x_0)} \rho(L+H-X_T)+m = \rho^{x+x_0}(L+H)+m\\
&= \eta(-H)+m.
\end{aligned}$$

So η is a convex risk measure.

Last, let's prove $\eta(-H_1) = \eta(-H_2)$ whenever $H_1 = H_2$ $P-$a.s.. Let $H_1 = H_2$ $P-$a.s.. Then we have $L + H_1 = L + H_2$ $P-$a.s.. Since $\rho(L + H_1) = \rho(L + H_2)$, we get

$$\eta(-H_1) = \rho^{x+x_0}(L + H_1) = \rho^{x+x_0}(L + H_2) = \eta(-H_2).$$

□

Theorem 2.4. *If $x \in (0, K)$, then there exists $H^* \in [0, K]$, $E[\varphi H^*] = x$ such that*

$$\inf_{\substack{0\le H\le K\\ E[\varphi H]\ge x}} \eta(-H) = \eta(-H^*) \Longleftrightarrow \inf_{\substack{0\le H\le K\\ E[\varphi H]\ge x}} \rho^{x+x_0}(L + H) = \rho^{x+x_0}(L + H^*).$$

Proof. $\eta(H)$ is a convex risk measure by Lemma 2.3. By Lemmas 2.1 and 2.2, it is proved. □

Now we give bounded conditions to x for the $E[\varphi H^*] = x$ to be a no-arbitrage price. Xu [7] defined the selling price SP and the buying price BP of the option $H(\geq 0)$ as

$$SP(H) = \min\{x : \rho^{x_0+x}(L + H) \leq \rho^{x_0}(L)\}, \quad (2.6)$$

$$BP(H) = \max\{x : \rho^{x_0-x}(L - H) \leq \rho^{x_0}(L)\} \quad (2.7)$$

respectively.

By the translation invariance relation (1.5), the equations (2.6) and (2.7) become

$$\begin{aligned} SP(H) &= \min\{x : \rho^{x_0}(L + H) - \rho^{x_0}(L) \leq x\} \\ &= \rho^{x_0}(L + H) - \rho^{x_0}(L), \\ BP(H) &= \max\{x : x \leq \rho^{x_0}(L) - \rho^{x_0}(L - H)\} \\ &= \rho^{x_0}(L) - \rho^{x_0}(L - H) \end{aligned}$$

respectively. Since the final risk exposure both $\rho^{x_0+x}(L + H)$ and $\rho^{x_0-x}(L - H)$ do not exceed the initial risk $\rho^{x_0}(L)$, i.e.,

$$\begin{aligned} \rho^{x_0}(L + H) - x = \rho^{x_0+x}(L + H) \leq \rho^{x_0}(L), \\ \rho^{x_0}(L - H) + x = \rho^{x_0-x}(L - H) \leq \rho^{x_0}(L), \end{aligned}$$

we have

$$SP(H) = \rho^{x_0}(L + H) - \rho^{x_0}(L) \leq x \leq \rho^{x_0}(L) - \rho^{x_0}(L - H) = BP(H). \quad (2.8)$$

Thus for the $E[\varphi H^*] = x$ to be a no-arbitrage price of H^*, it should satisfy the inequalities

$$SP(H) \leq E[\varphi H^*] = x \leq BP(H).$$

3. Direct Proof

In this section, we find the sequences converging to the risk-efficient option for the proof of its existence.

Lemma 3.1 (Föllmer and Schied [5]). *Let $(\xi_n)_{n\geq 1}$ be a sequence in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ such that $\sup_n |\xi_n| < +\infty$ P-a.s.. Then there exists a sequence of convex combinations*

$$\eta_n \in conv\{\xi_n, \xi_{n+1}, \ldots\}$$

which converges P-a.s. to some $\eta \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$.

Define

$$\mathcal{X}(x, b) = \{X \mid X \in \mathcal{X}(x) \text{ and } X_T \geq x - b\}.$$

Then we have

$$\mathcal{X}(x) = \bigcup_{b \in \mathbb{R}^+} \mathcal{X}(x, b).$$

Theorem 3.2 ([7]). *Under two assumptions (1.3) and (1.4) and $\mathcal{M} \neq \emptyset$, there exists an optimal admissible hedging portfolio $X^* \in \mathcal{X}(x, b)$ which is the solution of the minimal risk problem*

$$\rho_b^x(L) := \inf_{X \in \mathcal{X}(x,b)} \rho(L - X_T) = \rho(L - X_T^*), \tag{3.1}$$

for any $b \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

Let H be a payoff function of an option, $x \in \mathbb{R}^+$, and let $Q \in \mathcal{M}$ be fixed.

Lemma 3.3. *There exists $\mathcal{F}-$measurable H^* and $X_T^{b,*} \in \mathcal{X}(x, b)$, depending on H^* such that $E^Q[H^*] = x$,*

$$\inf_{E^{\mathbb{Q}}[H]=x} \rho_b^x(L + H) \quad = \quad \rho(L + H^* - X_T^{b,*}) := \rho_b^x(L + H^*).$$

Proof. By Theorem 3.2, for each H there exists $X_T^{b,H} \in \mathcal{X}(x, b)$ such that

$$\rho_b^x(L + H) := \inf_{X \in \mathcal{X}(x,b)} \rho(L + H - X_T) = \rho(L + H - X_T^{b,H}).$$

Choose the sequences H_n and $X_T^n \in \mathcal{X}(x,b)$ satisfying

$$E^Q[H_n] = x,$$
$$\rho(L + H_n - X_T^n) \searrow \inf_{E^Q[H]=x} \rho_b^x(L+H).$$

Then Lemma 3.1 implies that there exist the sequences $\tilde{X}_T^n \in conv\{X_T^n, X_T^{n+1}, \cdots\}$ such that

$$\tilde{X}_T^n \longrightarrow X_T^{b,*} \in \mathcal{X}(x,b) \quad \text{as } n \to \infty.$$

The sequence $\tilde{X}_T^n$ can be expressed as the convex combination

$$\tilde{X}_T^n = \sum_{i=k_1}^{k_m} \lambda_i^n X_T^i, \quad n \le k_1 < \cdots < k_m, \ \sum_{i=k_1}^{k_m} \lambda_i^n = 1, \ \lambda_i^n \ge 0.$$

Set $\tilde{H}_n = \sum_{i=k_1}^{k_m} \lambda_i^n H_i$, in which is the sequence H_i in the chosen pair H_i and $X_T^i \in \mathcal{X}(x,b)$.

It is easy to see

$$E^Q[\tilde{H}_n] = \sum_{i=k_1}^{k_m} \lambda_i^n E^Q[H_i] = x. \tag{3.2}$$

If we apply the Lebesgue Dominated Convergence Theorem to the equation (3.2), then there exists H^* such that $\lim_{n\to\infty} \tilde{H}_n = H^*$ Q-a.s., and $E^Q[H^*] = x$.

So we have

$$\begin{aligned}
\rho(L + \tilde{H}_n - \tilde{X}_T^n) &= \rho\Big(L + \sum_{i=k_1}^{k_m} \lambda_i^n H_i - \sum_{i=k_1}^{k_m} \lambda_i^n X_T^i\Big) \\
&= \rho\Big(\sum_{i=k_1}^{k_m} \lambda_i^n (L + H_i - X_T^i)\Big) \le \sum_{i=k_1}^{k_m} \lambda_i^n \rho(L + H_i - X_T^i) \\
&\le \rho(L + H_n - X_T^n) \sum_{i=k_1}^{k_m} \lambda_i^n = \rho(L + H_n - X_T^n) \\
&\le \sup_{m \ge n} \rho(L + H_m - X_T^m). \qquad (3.3)
\end{aligned}$$

By applying the Fatou property to $\rho(L + \tilde{H}^n - \tilde{X}_T^n)$ and also using the inequality (3.3), we have

$$\begin{aligned}\rho(L+H^* - X_T^{b,*}) &\leq \liminf_{n\to\infty}\rho(L+\tilde{H}_n - \tilde{X}_T^n)\\ &\leq \lim_{n\to\infty}\sup_{m\geq n}\rho(L+H_m - X_T^m)\\ &= \inf_{E^{\mathbb{Q}}[H]=x}\rho_b^x(L+H).\end{aligned}$$

Since $E^Q[H^*]=x$ and $X_T^{b,*}\in\mathcal{X}(x,b)$, we have

$$\rho(L+H^* - X_T^{b,*}) = \inf_{E^Q[H]=x}\rho_b^x(L+H).$$

□

Theorem 3.4. *Let* $p(H)=E^Q[H]$ *be the* pricing rule *of the option* H *for a fixed* $Q\in\mathcal{M}$. *Let* x_0 *be an initial capital. Then there exists a* risk-efficient option H^* *satisfying*

$$\inf_{p(H)=x}\rho^{x_0+x}(L+H)=\rho^{x_0+x}(L+H^*),$$

where L *is the initial liability uniformly bounded below by* c_L.

Proof. Let $Q\in\mathcal{M}$ be fixed. Since $\rho^{x+x_0}(L+H)=\rho^x(L+H)-x_0$, we need only to consider

$$\rho^x(L+H).$$

For $X\in\mathcal{X}(0)$, by Assumption 1.6 and translation invariance property, the following both inequality and equality

$$\begin{aligned}\rho(L+H-X_T) &\geq \rho(c_L+0-X_T)\geq c_L+\rho(-X_T)\\ &\geq c_L+\rho^0(0)>-\infty, \qquad \text{and}\\ \rho^x(L+H) &= \rho^0(L+H)-x\end{aligned}$$

imply that $\rho^x(L+H)$ is well-defined for all $X\in\mathcal{X}(x)$.

By Theorem 3.2, for each H there exists $X_T^{b,H}\in\mathcal{X}(x,b)$ such that

$$\rho_b^x(L+H):=\inf_{X\in\mathcal{X}(x,b)}\rho(L+H-X_T)=\rho(L+H-X_T^{b,H}).$$

Let $\epsilon>0$. Then since

$$\rho_b^x(L+H)\searrow\rho^x(L+H) \text{ as } b\nearrow\infty,$$

there exists a large nonnegative integer $N\in\mathbb{Z}^+$ satisfying

$$b>N\Longrightarrow\rho^x(L+H)+\epsilon>\rho_b^x(L+H). \tag{3.4}$$

The equation (3.4) and Lemma 3.3 imply the following inequality

$$\inf_{E^Q[H]=x} \rho^x(L+H) + \epsilon > \inf_{E^Q[H]=x} \rho_b^x(L+H) = \rho_b^x(L+H^*).$$

So we have

$$\inf_{E^Q[H]=x} \rho^x(L+H) \geq \rho_b^x(L+H^*),$$

and so

$$\inf_{E^Q[H]=x} \rho^x(L+H) \geq \lim_{b \nearrow \infty} \rho_b^x(L+H^*). \tag{3.5}$$

On the other hand, since $\rho^x(L+H) < \rho_b^x(L+H)$ we have the inequality

$$\inf_{E^Q[H]=x} \rho^x(L+H) \leq \inf_{E^Q[H]=x} \rho_b^x(L+H) = \rho_b^x(L+H^*)$$

and by letting b go to infinity we get

$$\inf_{E^Q[H]=x} \rho^x(L+H) \leq \lim_{b \nearrow \infty} \rho_b^x(L+H^*). \tag{3.6}$$

By the inequalities (3.5) and (3.6), we get

$$\inf_{E^Q[H]=x} \rho^x(L+H) = \rho^x(L+H^*).$$

The theorem has been proved. □

For the pricing rule $E^Q[H] = x$ of the option H to be an no-arbitrage price, it should also satisfy

$$SP(H) \leq x \leq BP(H),$$

as we showed the reason in Section 2.

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