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CENTRAL LIMIT THEOREM ON CHEBYSHEV POLYNOMIALS

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Abstract. Let T_l be a transformation on the interval $[-1, 1]$ defined by Chebyshev polynomial of degree l ($l \geq 2$), i.e., $T_l(\cos\theta) = \cos(l\theta)$. In this paper, we consider T_l as a measure preserving transformation on $[-1, 1]$ with an invariant measure $\frac{1}{\pi\sqrt{1-x^2}}\,dx$. We show that If $f(x)$ is a nonconstant step function with finite k-discontinuity points with $k < l - 1$, then it satisfies the Central Limit Theorem. We also give an explicit method how to check whether it satisfies the Central Limit Theorem or not in the cases of general step functions with finite discontinuity points.

1. Introduction

Let (X, μ) be a probability measure space. A measurable transformation $T : X \to X$ is said to be *measure preserving* if $\mu(T^{-1}E) = \mu(E)$ for every measurable subset E. A measure preserving transformation T on X is called *ergodic* if $f(Tx) = f(x)$ holds only for constant functions and it is called weakly mixing if the constant function is the only eigenfunction with respect to T [3, 5].

Let $\mathbf{1}_E$ be the characteristic function of a set E and consider the behavior of the sequence $\sum_{k=0}^{n-1} \mathbf{1}_E(T^k x)$ which equals the number of times that the points $T^k x$ visit E. The Birkhoff Ergodic Theorem applied to the ergodic transformation $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift on $\prod_{k=0}^{\infty}\{0, 1\}$ gives the Laws of the Large Numbers.

Let T be a transformation which is piecewise expanding on the unit interval $X = [0, 1)$ and $g(x) \equiv \frac{1}{|T'(x)|}$ be a function of bounded variation, where $T'(x)$ is the appropriate one-sided derivative at the discontinuities. Then it is well-known that there exists an absolutely continuous invariant measure with respect to the Lebesgue measure. Furthermore if T is weakly mixing with respect to the T-invariant absolutely continuous measure, $f(x)$ is a bounded variation function and the functional

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equation

$$f = g \circ T - g + c$$

does not have any solution $g(x)$ for any constant $c \in \mathbb{R}$, then we can apply the Central Limit Theorem to the function $f(x)$ [2].

For each natural number $l (l \geq 2)$, let T_l be the transformation on the interval $[-1, 1]$ defined by Chebyshev polynomial of degree l. In this paper, we consider T_l as a measure preserving transformation on $[-1, 1]$ with an invariant measure $\frac{1}{\pi\sqrt{1-x^2}}\, dx$. We show that if $f(x)$ is a step function with finite k-discontinuity points($k < l - 1$) then it satisfies the Central Limit Theorem. We also give a explicit method how to check whether it satisfies the Central Limit Theorem or not in the cases of general step functions with finite discontinuity points. It is known that the entropy of $([-1, 1], \frac{1}{\pi\sqrt{1-x^2}}\, dx, T_l)$ is $\log l$ for each $l \geq 2$[1].

2. Properties of Chebyshev Polynomials

Let T_l be the Chebyshev polynomial of degree l $(l \geq 2)$. Recall that T_l is defined by

$$T_l(\cos x) = \cos(lx)$$

on $[-1, 1]$. Chebyshev polynomials are orthogonal in the Hilbert space

$$H = L^2([-1, 1], \rho(x)\, dx)$$

where

$$\rho(x) = \frac{1}{\pi\sqrt{1 - x^2}}.$$

Let $T : (X, \mu) \to (X, \mu)$ and $\Lambda : (Y, \nu) \to (Y, \nu)$ be measure preserving. Two measure preserving transformations are said to be measure theoretically isomorphic if there exists an isomorphism $\psi : (X, \mu) \to (Y, \nu)$ such that $\psi \circ T = \Lambda \circ \psi$, in other words, the following diagram commutes:

$$\begin{array}{ccc} (X, \mu) & \xrightarrow{T} & (X, \mu) \\ \psi \big\downarrow & & \psi \big\downarrow \\ (Y, \nu) & \xrightarrow{\Lambda} & (Y, \nu) \end{array}$$

From now on, let ν be the Lebesgue measure on $[0, 1]$ and μ be an absolutely continuous measure on $[-1, 1]$ with the density function $\rho(x)$. i.e., the measure μ is defined by

$$\mu(E) = \int_E \rho(x)\, dx.$$

Definition 1. For each $l \in \mathbb{N}$, let Λ_l be a map on $[0,1]$ defined by

$$\Lambda_l(x) = \begin{cases} lx - k, & \text{if } k \text{ is even,} \\ k + 1 - lx, & \text{if } k \text{ is odd} \end{cases}$$

for $\frac{k}{l} \leq x \leq \frac{k+1}{l}$, $k = 0, 1, \cdots, l-1$.

It is well-known that Λ_l preserves the Lebesgue measure ν and it is weakly mixing.

Lemma 1. *Let T_l be the l-th Chebyshev polynomial of order $l \geq 2$. Then T_l preserves the measure μ on $([-1,1])$ and is measure theoretically isomorphic to the transformation Λ_l on $([0,1], \nu)$ by a topological homeomorphism $\psi(x) = \frac{1}{\pi}\arccos(x)$.*

Proof. Let $\phi(y)$ be the inverse function of $\psi(x)$, i.e., $\phi(y) = \cos(\pi y)$ from $[0,1]$ to $[-1,1]$. It is obvious that $\phi \circ \Lambda_l = T_l \circ \phi$ holds. Hence $\psi \circ T_l = \Lambda_l \circ \psi$. So it is enough to show that ϕ is a measure theoretical isomorphism. Note that the inverse image of $[\phi(y), 1]$ under ϕ is $[0, y]$, which has Lebesgue measure equal to y. For ϕ to be an measure theoretical isomorphism, it must satisfy

$$\mu([\phi(y), 1]) = y$$

for all $0 \leq y \leq 1$. Thus $\mu([x,1]) = \psi(x)$ and $\mu([0,x]) = 1 - \psi(x)$ for all $-1 \leq x \leq 1$, because μ is a probability measure on $[-1,1]$. Since

$$\frac{d}{dx}(\mu([0,x]) = \frac{d}{dx}(1 - \psi(x)) = \frac{1}{\pi\sqrt{1-x^2}} = \rho(x),$$

ϕ is a isomorphism and the following diagram commutes.

$$\begin{array}{ccc} ([-1,1], \mu) & \xrightarrow{T_l} & ([-1,1], \mu) \\ \psi \big\downarrow & & \psi \big\downarrow \\ ([0,1], \nu) & \xrightarrow{\Lambda_l} & ([0,1], \nu) \end{array}$$

Hence T_l is a measure preserving transformation on $([-1,1], \mu)$ and weakly mixing. $\square$

3. The Central Limit Theorem

The following lemma gives a sufficient condition for a special class of transformations on which the Central Limit Theorem holds [2]. In Lemma 2, μ is an arbitrary absolutely continuous measure.

Lemma 2. *Let T be a piecewise continuously differentiable and expanding transformation on an interval $[a,b]$, i.e., there exists a partition*

$$a = a_0 < a_1 < \cdots < a_{k-1} < a_k = b$$

such that T is continuously differentiable on each $[a_{i-1}, a_i]$ $(1 \le i \le k)$ and $|T'(x)| > B$ for some constant $B > 1$(At the endpoints of an interval we consider directional derivatives). Assume that $\frac{1}{|T'(x)|}$ is a function of bounded variation. Suppose that T is weakly mixing with respect to an invariant probability measure μ. Let $f(x)$ be a function of bounded variation such that the equation

$$f(x) = g(Tx) - g(x) + c,$$

where c is constant, has no solution $g(x)$ of bounded variation. Then

$$\sigma^2 = \lim_{n\to\infty} \int_0^1 \left(\frac{S_n f - n\mu(f)}{\sqrt{n}} \right)^2 d\mu > 0$$

and, for every α,

$$\lim_{n\to\infty} \mu\left\{ x : \frac{S_n f(x) - n\mu(f)}{\sigma\sqrt{n}} \le \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp\left(-t^2/2\right) dt,$$

where

$$S_n f(x) = \sum_{j=0}^{n-1} f(T^j(x))$$

and

$$\mu(f) = \int_0^1 f(x)\, d\mu(x).$$

Since T_l is measure theoretically isomorphic to Λ_l by a topological homeomorphism, we may assume that the transformation T_l on $([-1,1],\mu)$ satisfies all the conditions of Lemma 2.

Proposition 1. *For the measure preserving transformation T_l on $[-1,1]$ defined by l-th Chebyshev polynoimal, if an $\mathbb{R}$-valued function $f(x)$ is a step function with finite discontinuity points and $f(x) = g(T_l x) - g(x) + c$ with a constant c, then $g(x)$ is also a step function with finite discontinuity points.*

Proof. Recall that the measure preserving transformation T_l on $([-1,1],\mu)$ and the measure preseving transformation Λ_l on $([0,1],\nu)$ are measure theoretically isomorphic via the topological homeomorphism $\psi(x) = \frac{1}{\pi}\arccos(x)$ by Lemma 1. As in Lemma 1, let $\phi(y) = \cos(\pi y)$ be the inverse function of $\psi(x)$. Note that $f(x)$ is a step function with finite discontinuity points if and only if $f(\phi(y))$ is a step function with finite discontinuity points. Furthermore the functional equation

$$f(x) = g(T_l x) - g(x) + c$$

has a solution if and only if the functional equation

$$f(\phi(y)) = g(T_l(\phi(y))) - g(\phi(y)) + c$$

has a solution. Let v be the variation of $f(x)$, $F(y) = \exp(-2\pi i \frac{1}{v} f(\phi(y))) \times \exp(2\pi i \frac{1}{v} c)$ and $G(y) = \exp(2\pi i \frac{1}{v} g(\phi(y)))$. Note that the number of discontinuity points of $f(x)$ is equal to the number of discontinuity points of $F(y)$ and if the functional equation

$$f(\phi(y)) = g(T_l(\phi(y))) - g(\phi(y)) + c$$

has a solution then the functional equation

$$F(y)G(\Lambda_l y) = G(y)$$

has a solution. So it is enough to show that $G(y)$ is also a step function with finite discontinuity points, because if $g(\phi(y))$ is a bounded variation function and $G(y) = \exp(2\pi i \frac{1}{v} g(\phi(y)))$ is a step function with finite discontinuity points, then $g(\phi(y))$ also has to be a step function with finite discontinuity points. For the notational simplicity, we will prove the proposition in the case $l = 2$.

Let $\mathcal{P}$ be a partition of $[0,1]$ defined by $\mathcal{P} = \{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}$, and $\mathcal{P}_N = \bigvee_{k=0}^{N-1} T_2^{-k}\mathcal{P}$. Let $D = \{z \,|\, F(y) \text{ is discontinuous at } y = z\}$, m be the cardinality of discontinuity D and D_ϵ be the ϵ-neighborhood of D, i.e., $D_\epsilon = \{\bigcup_{z\in D}(z-\epsilon, z+\epsilon)\}$. Then there exists ϵ_0 such that for all $0 < \epsilon < \epsilon_0$, $\nu(D_\epsilon) = 2m\epsilon$. Now choose an integer N such that $(\frac{1}{2})^N < \epsilon_0$ and $m \cdot (\frac{1}{2})^{N-1} < \frac{1}{2}$.

If $I \in \mathcal{P}_N$ and if $I \cap D \neq \phi$, then $I \subset D_\epsilon$ for $\epsilon = (\frac{1}{2})^N$. Hence the totality of $I \in \mathcal{P}_N$ with $I \cap D \neq \phi$ measures at most $2m \cdot (\frac{1}{2})^N$. By the similar argument, the totality of $I \in \mathcal{P}_{N+j}$, $j \geq 0$ such that $I \cap D \neq \phi$ measures at most $2m \cdot (\frac{1}{2})^{N+j}$.

Fix $L > 0$ and consider the collection of $I \in \mathcal{P}_{N+L}$ having the property that $T^j I \cap D \neq \phi$ for some $0 \leq j \leq L-1$. Since $T^j I \in \mathcal{P}_{N+L-j}$ for these j, and Λ_2 is Lebesgue measure preserving, these intervals have the total Lebesgue measure at most

$$\sum_{j=1}^{L-1} 2m \cdot \left(\frac{1}{2}\right)^{N+L-j} \leq m \cdot \left(\frac{1}{2}\right)^{N-1} \leq \frac{1}{2}.$$

Let $Q(N,L)$ be the sub-collection of $\mathcal{P}_{N+L}$ such that $T^j I \cap D = \phi$ for all $0 \le j \le L-1$. Then for each $I \in Q(N,L)$, $F(y)F(\Lambda_2 y)\cdots F(\Lambda_2^{L-1}y)$ is constant, say $\lambda_{I,L}$ with $|\lambda_{I,L}| = 1$. Since $G(y) = F(y)G(\Lambda_2 y)$, $G(y) = F(y)F(\Lambda_2 y)\cdots F(\Lambda_2^{L-1}y)G(\Lambda_2^L y)$. Hence $G(y) = \lambda_{I,L} \cdot G(\Lambda_2^L y)$ holds almost everywhere on I. Letting $\Lambda_2^L I = J \in \mathcal{P}_N$, the map $\Lambda_2^L : I \to J$ is bijective and it is easily shown that

$$\left|\frac{1}{\nu(I)}\int_I G(y)\,d\nu(y)\right| = \left|\frac{1}{\nu(J)}\int_J G(z)\,d\nu(z)\right|. \tag{1}$$

Since $Q(N,L)$ measures at least $\frac{1}{2}$, the set of y which is interior to some $I \in Q(N,L)$ for an infinitely number of L must also measures at least $\frac{1}{2}$. Fixing such an y, we have that (1) holds. We may assume that y is also a Lebesgue point of G. Since $\mathcal{P}_N$ is finite, it can be assumed J is always the same on the right side of (1). By the Lebesgue density theorem[4], we can assume that the left side of (1) tends to $G(y)$. Hence

$$\left|\frac{1}{\nu(J)}\int_J G(z)\,d\nu(z)\right| = |G(y)| = 1.$$

Since $|G(z)| = 1$ for all $z \in [0,1]$, $G(z)$ has to be constant on J. Since $F(y)$ is a step function with finite discontinuity and $\Lambda_2^N J = [0,1]$, $G(y)$ is also a step function with finite discontinuity. Hence the conclusion follows. □

Theorem 1. *Let T_l be a measure preserving transformation on* $([-1,1], \frac{1}{\pi\sqrt{1-x^2}}\,dx)$ *defined by Chebyshev polynomial of degree l ($l \ge 2$). If $f(x)$ is a nonconstant step function with finite k-discontinuity points with $k < l-1$ then it satisfies the Central Limit Theorem, i.e.,*

$$\sigma^2 = \lim_{n\to\infty}\int_0^1 \left(\frac{S_n f - n\mu(f)}{\sqrt{n}}\right)^2 d\mu > 0$$

and, for every α,

$$\lim_{n\to\infty}\mu\left\{x : \frac{S_n f(x) - n\mu(f)}{\sigma\sqrt{n}} \le \alpha\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}\exp\left(-t^2/2\right)dt,$$

where $S_n f(x) = \sum_{j=0}^{n-1} f(T^j(x))$, $d\mu = \frac{1}{\pi\sqrt{1-x^2}}\,dx$ and $\mu(f) = \int_0^1 f(x)\,d\mu(x)$.

Proof. It is enough to show that the functional equation

$$f(x) = g(T_l x) - g(x) + c$$

has no solution. Suppose it is not, by Proposition 1, $g(x)$ is also a step function with finite discontinuity points. Hence $g(x)$ can be expressed as

$$g(x) = \sum_{j=0}^{m-1} c_j \, \mathbf{1}_{[a_j, a_{j+1}]}(x)$$

where $-1 = a_0 < a_1 < \cdots < a_m = 1$. Since $g(x)$ has $m-1$ discontinuity points, $g(T_l x)$ has at least $l(m-1)$ discontinuity points and $g(T_l x) - g(x) + k$ has at least $(l-1)(m-1)$ discontinuity points. Since $f(x)$ has k discontinuity points, we have

$$0 \le m - 1 \le \frac{k}{l-1}.$$

So if $k < l-1$ then m has to be 1 and $g(x)$ has to be a constant function. It is a contradiction to the assumption that $f(x)$ is not a constant function. $\square$

Theorem 2. *Let T_l be a measure preserving transformation on* $([-1,1], \frac{1}{\pi\sqrt{1-x^2}}\, dx)$ *defined by Chebyshev polynomial of degree l ($l \ge 2$). If $f(x)$ is a nonconstant step function with finite discontinuity points and $f(x)$ is constant on the interval* $[-1, \cos(\frac{l-1}{l}\pi)]$*, then it satisfies the Central Limit Theorem.*

Proof. Letting $J = [-1, \cos(\frac{l-1}{l}\pi)]$, we have $T_l(J) = [-1,1]$. Suppose there exists an function $g(x)$ which satisfies the functional equation,

$$f(x) = g(T_l x) - g(x) + c.$$

By Proposition 1, there exists x_1 such that $g(x)$ is constant on $[-1, T_l(x_1)] \supset [-1, x_1]$. If we take any $x \in [-1, x_1]$, then both x and $T_l(x)$ are in $[-1, x_1]$ and $g(T_l(x)) = g(x)$. Since $f(x) = g(T_l x) - g(x) + c$, we have $f(x) = c$ for all $x \in J$. Therefore $g(T_l(x)) = g(x)$ for all $x \in J$, and $g(x) = g(-1)$ for all $x \in [-1, T_l(x_1)]$. If $T_l([-1, x_1]) = [-1,1]$, then $g(x)$ has to be a constant function and $f(x)$ also has to be constant. it completes the proof. Otherwise, letting $x_2 = T_l(x_1)$, we have $g(x)$ is a constant on $T_l([-1, x_2))$ by exactly the same argument by using x_2 in the place of x_1. Iterating this argument if we need it, we get $g(x)$ is constant and the conclusion follows. $\square$

In the following Proposition, we give an explicit method how to check whether it satisfies the Central Limit Theorem or not in the cases of general step functions with finite discontinuity points. For the simplicity, we consider the case $l = 2$ and $f(x)$ is a step function with 1 or 2 discontinuity points.

Proposition 2. *Let T_2 be a measure preserving transformation on* $([-1,1], \frac{1}{\pi\sqrt{1-x^2}}\, dx)$ *defined by Chebyshev polynomial of degree 2. If $f(x)$ is a step function with finite k-discontinuity points with $k \le 2$, then it satisfies the Central Limit Theorem with respect to T_2 except for the functions of the form*

$$f(x) = b \cdot \mathbf{1}_{[-1,-\frac{1}{2}]}(x) - b \cdot \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x) + c$$

with some constants b, c.

Proof. As in the proof of Proposition 1, let $\phi(y) = \cos(\pi y)$. Then the functional equation

$$f(x) = g(T_l x) - g(x) + c$$

has a solution if and only if the functional equation

$$f(\phi(y)) = g(T_l(\phi(y)) - g(\phi(y)) + c$$

has a solution. Furthermore $F(y) = f(\phi(y))$ has the same discontinuity points as $f(x)$.

Case 1) Suppose that $f(x)$ has 1-discontinuity point, i.e., $F(y) = f(\phi(y))$ has the form of

$$F(y) = b_0 \cdot \mathbf{1}_{[0,d]}(y) + b_1 \cdot \mathbf{1}_{[d,1]}(y)$$

and $G(y) = g(\phi(y))$ is the solution of the functional equation $F(y) = G(\Lambda_2 y) - G(y) + c$, then $G(y)$ can be expressed as

$$G(y) = \sum_{j=0}^{m-1} c_j \, \mathbf{1}_{[a_j, a_{j+1}]}(y)$$

where $0 = a_0 < a_1 < \cdots < a_m = 1$. By exactly the same argument as in the proof of Theorem 1, we have $m = 1$ or 2. When $m = 1$, $G(y)$ has to be constant, and $f(x)$ also has to be constant. It is a contradiction. When $m = 2$, then $G(y)$ has the form of $G(y) = c_0 \mathbf{1}_{[0,a]}(y) + c_1 \mathbf{1}_{[a,1]}(y)$ with some constants c_0, c_1 and constant $0 < a < 1$. Since both $G(y)$ and $G(\Lambda_2 y)$ have the same value on the interval $[0, a/2]$, we have $b_0 = c$. Integrating the functional equation

$$F(y) = G(\Lambda_2 y) - G(y) + c,$$

we get a equation $dc + (1 - d)b_1 = c$. Hence $b_1 = c$ and $f(x)$ is constant. It is a contradiction to the assumption of $f(x)$. Thus if $f(x)$ has 1-discontinuity point, then it satisfies the Central Limit Theorem.

Case 2) Suppose that $f(x)$ has 2-discontinuity points. Then $F(y) = f(\phi(y))$ has the form of

$$F(y) = b_1 \cdot \mathbf{1}_{[0,d_1]}(y) + b_2 \cdot \mathbf{1}_{[d_1,d_2]}(y) + b_3 \cdot \mathbf{1}_{[d_2,1]}(y).$$

As in the case 1, letting $G(y)$ be a solution of the functional equation $F(y) = G(\Lambda_2 y) - G(y) + c$, we have $G(y) = \sum_{j=0}^{m-1} c_j \, \mathbf{1}_{[a_j, a_{j+1}]}(y)$ with $m = 2$ or 3. When

$m = 2$, the discontinuity points of $G(\Lambda_2 y) - G(y) + c$ are $\{\frac{a_1}{2}, a_1, 1 - \frac{a_1}{2}\}$. Hence we have $a_1 = 1 - \frac{a_1}{2}$ and $a_1 = \frac{2}{3}$. Thus

$$G(y) = c_0 \cdot \mathbf{1}_{[0,\frac{2}{3}]}(y) + c_1 \cdot \mathbf{1}_{[\frac{2}{3},1]}(y)$$

and $F(y)$ has to be in the form of

$$F(y) = (c_1 - c_0) \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}(y) + (c_0 - c_1) \cdot \mathbf{1}_{[\frac{2}{3},1]}(y) + c,$$

and $f(x) = b \cdot \mathbf{1}_{[-1,-\frac{1}{2}]}(x) - b \cdot \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x) + c$ with some constants b, c.

When $m = 3$, by the similar argument as in the case $m = 2$, we have $a_1 = \frac{1}{3}$, $a_2 = \frac{2}{3}$ and

$$G(y) = c_0 \cdot \mathbf{1}_{[0,\frac{1}{3}]}(y) + c_1 \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}(y) + c_2 \cdot \mathbf{1}_{[\frac{2}{3},1]}(y).$$

Thus $G(\Lambda_2 y) - G(y)$ has the form of

$$\begin{aligned} G(\Lambda_2 y) - G(y) = (c_1 - c_0) \cdot \mathbf{1}_{[\frac{1}{6},\frac{1}{3}]}(y) + (c_2 - c_1) \cdot \mathbf{1}_{[\frac{1}{3},\frac{2}{3}]}(y) \\ + (c_1 - c_2) \cdot \mathbf{1}_{[\frac{2}{3},\frac{5}{6}]}(y) + (c_0 - c_2) \cdot \mathbf{1}_{[\frac{5}{6},1]}(y). \end{aligned}$$

Hence for $G(\Lambda_2 y) - G(y)$ having 2-discontinuity points, we have $c_2 - c_1 = 0$. It contradicts the assumption that $G(y)$ has 2-discontinuity points. $\square$

Remark 1. By exactly the same argument as in the proof of the case in Proposition 2, if $f(x)$ has only 1-discontinuity point, then it satisfies the Central Limit Theorem with respect to any Chebyshev polynomials of degree $l \geq 2$.

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