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MOTION IN A HANGING CABLE WITH VARIOUS DIFFERENT PERIODIC FORCING

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Abstract. We investigate long-term motions of the cable when cable has different types of periodic forcing term. Various different types of solutions are presented by using the 2nd order Runge-Kutta method under various initial conditions. There appeared to be small- and large-amplitude solutions which have different nodal structure.

1. Introduction

Cable vibration problems are of considerable antiquity. Starting with Taylor in 1713 and continuing with the work of D'Alembert, Euler, and Daniel Bernoulli during the first half of the eighteenth century, did its mathematical theory become firmly established. In 1744, Euler derived the correct equations for the large vibrations of a string in a plane. He regarded the equations as the limit of those for a finite collection of beads joined by massless springs as the number of beads approach infinity while their total mass remains fixed [2]. The motion of the system of beads is described by a finite system of ordinary differential equations [2].

It is known that hanging cables sometimes oscillate with large amplitudes under moderate wind force. Such motions, known as galloping in the literature, can cause collapse of the supporting towers.

Galloping has been an observed problem for many years [4, 6, 8, 12]. It is the large amplitude up-and-down motion that can break lines and cause catastrophic failure towers of their supporting according to the Electric Power Research Institute. Considerable effort has been expended in trying to solve the problem of galloping in long spans of power transmission lines [11].

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If by chance a gust of wind or some other disturbance causes a tremor in the cable, these small forces actually will transmit energy to the cable [13]. Should their frequencies correspond to, or very nearly to, a frequency of resonant vibration for the cable, the cable gradually will start vibrating in a number of loops such that the frequency of vibration of the cable coincides with the frequency of the eddy currents [13]. Thus, even though the magnitude of the forcing term is small, the phenomenon of linear resonance is supposedly enough to explain the large oscillation [13] and break lines [11]. However, this explanation fails to reconcile the fact that galloping occurs under a wide range of forcing frequencies due to wind.

It is known that the equation

$$y'' + \delta y' + by^+ - ay^- = c + \lambda \sin \mu t \tag{1.1}$$

has related large amplitude solutions even for small forcing terms which have a natural frequency away from the natural "equilibrium frequency" of $\sqrt{b}$ ([5]). We develop a model of the hanging cable which exhibits this phenomena.

We modeled a cable as a multiple particle oscillator and performed extensive numerical experiments.

We shall show that under a variety of periodic forcing, multiple stable periodic solutions exist. Some of these solutions possess large amplitudes and can be induced by proper initial conditions. Thus it might be explain why wind gust can cause a cable to gallop even after the wind die down. We shall investigate the cable motion when the supporting towers have small displacements. We find that longitudinal motions may occur even when they have only vertical forcing. Longitudinal motions, one-nodal, and two-nodal symmetric solutions are evident in our numerical results.

2. Governing Equation

Let a cable be as a series of equally contributed point masses. Let point masses be connected by nonlinear springs having the same unstretched lengths. In the loaded hanging cable, the restoring force between two particles in a cable is such that it strongly resists extension, but does not resist compression in large motions. Thus, the simplest function to model the restoring force of the cable between two particles would be a nonlinear term u^+, which is a constant times u, the extension, if u is positive, but zero if u is negative, corresponding to compression. The vertical periodic forces commonly influence the restoring forces. Let damping force be supposed to work in an opposite direction of the motion having a magnitude proportionate to

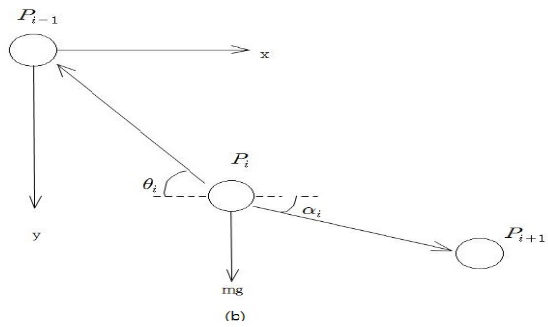

Figure 1. Diagram for equally distributed particles.

the instantaneous velocity.

Let a cable be hung between two fixed particles whose distance are L and height are same. The line joining two fixed particles which located at $x = 0$ and $x = L$ will be considered as the x-axis. Let $(u_i(t), v_i(t))$ be the instantaneous position of the i-th particle at time t. It has the positive directions for x and y and is presented in Figure 1.

Then we have the following equations which derived by Newton's second law and Hooke's law.

$$\begin{aligned} \rho l d^2 u_i/dt^2 &= -k(\overline{P_{i-1}P_i} - l)^+ \cos\theta_i + k(\overline{P_iP_{i+1}} - l)^+ \cos\alpha_i - cldu_i/dt \\ \rho l d^2 v_i/dt^2 &= -k(\overline{P_{i-1}P_i} - l)^+ \sin\theta_i + k(\overline{P_iP_{i+1}} - l)^+ \sin\alpha_i - cldv_i/dt \\ &\quad +\rho lg + lf(\tilde{u}_i, t) \end{aligned} \tag{2.1}$$

where $\tilde{u}_i = i\Delta x/L, \Delta x = L/(N+1)$, and N is the number of particles discretizing the cable.

$\overline{P_{i-1}P_i}$ is the distance between two particles P_{i-1} and P_i at time t. $\overline{P_iP_{i+1}}$ $(1 \leq i \leq N)$ is the same to $\overline{P_{i-1}P_i}$. Here, $P_0(u_0, v_0) = P_0(0, 0)$ and $P_{N+1}(u_{N+1}, v_{N+1}) = P_{N+1}(L, 0)$ are the two fixed supports.

The letter ρ denotes the mass per unit length of unstretched cable and the letter l denotes the unstretched length of the spring between two point masses. The letter c denotes the damping coefficient per unit unstretched length and the letter g denotes the accelation due to gravity. We denote the spring constant as k and $k = \frac{EA}{l}$, where E is Young's modulus and A is the cross section area. When we model a cable, l can be reduced by half in doubling the number of particles. So k can be increased by two. Hence, The letter θ_i and α_i denote the angles which the cable forms with the x-axis, that are shown in figure 1.

When we substitute the geometric relations between lengths and angles into equations (2.1) and divide by mass, we have the following equations,

$$\begin{aligned}
\frac{d^2u_i}{dt^2} &= -\frac{k}{\rho l}(\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}-l)^+\frac{(u_i-u_{i-1})}{\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}} \\
&+\frac{k}{\rho l}(\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}-l)^+\frac{(u_{i+1}-u_i)}{\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}} \\
&-\frac{c}{\rho}du_i/dt, \\
\frac{d^2v_i}{dt^2} &= -\frac{k}{\rho l}(\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}-l)^+\frac{(v_i-v_{i-1})}{\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}} \\
&+\frac{k}{\rho l}(\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}-l)^+\frac{(v_{i+1}-v_i)}{\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}} \\
&-\frac{c}{\rho}dv_i/dt+g+\frac{1}{\rho}f(\tilde{u}_i,t),
\end{aligned}$$

where $u_0(t)=0, u_{N+1}(t)=L, v_0(t)=0, v_{N+1}(t)=0, 1\le i\le N$. We take the forcing term $f(\tilde{u}_i,t)$ as $\lambda\sin 3\pi\tilde{u}_i\sin\mu t$, where $\tilde{u}_i=i\Delta x/L$ and $\Delta x=L/(N+1)$.

Let $\vec{u}=(u_1,v_1,\cdots,u_N,v_N)^T, \vec{S}=(S_1,T_1,\cdots,S_N,T_N)^T$. Then the given nonautonomous system becomes

$$\frac{d^2\vec{u}}{dt^2}=\vec{S}(t,\vec{u}),$$

where

$$\begin{aligned}
S_i &\quad (u_{i-1},v_{i-1},u_i,v_i,u_{i+1},v_{i+1},t) \\
&= -\frac{k}{\rho l}(\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}-l)^+\frac{(u_i-u_{i-1})}{\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}} \\
&\quad +\frac{k}{\rho l}(\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}-l)^+\frac{(u_{i+1}-u_i)}{\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}} \\
&\quad -\frac{c}{\rho}du_i/dt, \\
T_i &\quad (u_{i-1},v_{i-1},u_i,v_i,u_{i+1},v_{i+1},t) \\
&= -\frac{k}{\rho l}(\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}-l)^+\frac{(u_i-u_{i-1})}{(\sqrt{(u_i-u_{i-1})^2+(v_i-v_{i-1})^2}}
\end{aligned}$$

$$+\frac{k}{\rho l}(\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}-l)^{+}\frac{(v_{i+1}-v_i)}{\sqrt{(u_{i+1}-u_i)^2+(v_{i+1}-v_i)^2}}$$
$$-\frac{c}{\rho}dv_i/dt+g+\frac{1}{\rho}f(\tilde{u}_i,t).$$

Here, $f(\tilde{u}_i,t)=\lambda\sin 3\pi\tilde{u}_i\sin\mu t$ where $\tilde{u}_i=i\Delta x/L$ and $\Delta x=L/(N+1)$.

We will solve the system of differential equations numerically. In order to study the system numerically, we will search for stable periodic solutions by solving the initial value problem for various initial conditions and allowing the solution to run for large time.

We stress that we have little concern about the transient solutions to the initial value problems, solely their ultimate long-term behavior. To solve the system, we use the 2nd-order Runge-Kutta method using higher precision.

3. Main Results

Rather than use the original approach to the vibrating cable adopted by Euler, we solve the system for several numbers of particles by computation. Since cable is a continuous structure, we need more particles to approximate the cable. To approximate the cable, we solve the system by computation using various different numbers of particles. Then we discover that it achieves the convergence, which means the motion remains in the same even as the number of particles is increased. In fact, we get the convergence for the solutions as the number of particles is increased to 63, 127. We illustrate the 63-particle case for the galloping cable because the pictures for the larger number of particles are identical with figures presented.

We let the interval of length be 1, unstretched length of cable be 1.2, the total mass be 5, and $f(\tilde{u}_i,t)=\lambda\sin 3\pi\tilde{u}_i\sin\mu t$, where $\tilde{u}_i=i\Delta x/L$ and $\Delta x=L/(N+1)$. Based on the previous experience in [5], we expect to find interesting results near linear resonance. We consider the long-term solutions of the 63-particle case.

The interesting solutions are investigated when sufficient time has elapsed for the transient behaviors to have disappeared. We observe the shapes of cables in the last 8 periods. In order to find whether the solutions are periodic or not, we observe the Poincaré section. If they are periodic, we find out their corresponding periods. Each of the plots show the last five equally spaced profiles of motions in the final period. We shall compare cable motion in two-noded forcing

$$(f(\tilde{u}_i,t)=\lambda\sin 3\pi\tilde{u}_i\sin\mu t)$$

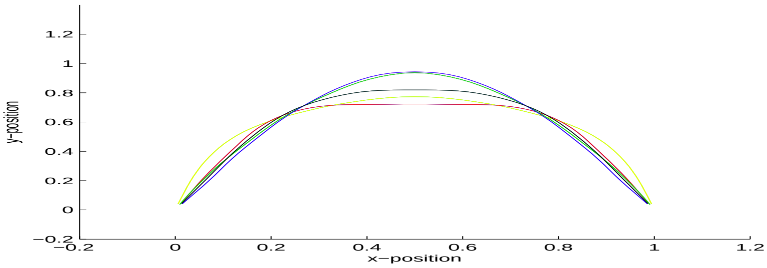

Figure 2. The small-amplitude solution for $\mu = 4.8$ and $\lambda = 5.2$.

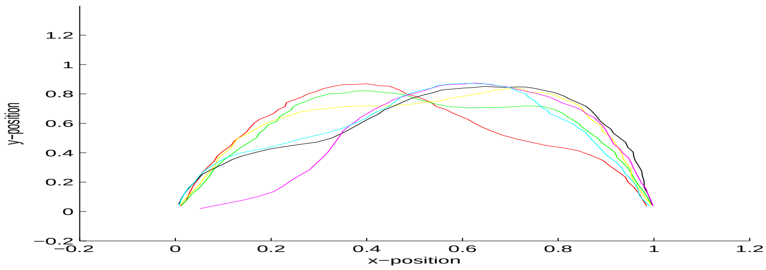

Figure 3. The small-amplitude solution for $\mu = 8.6$ and $\lambda = 2.4$.

with that in no-noded forcing

$$(f(\tilde{u}_i, t) = \lambda \sin \pi \tilde{u}_i \sin \mu t).$$

3.1. Shapes of cables

3.1.1. *Small-amplitude solutions* In the case of no-noded forcing, as long as the initial values and λ are small, we have typical small-amplitude oscillations over wide range of frequencies [10].

In the case of two noded forcing, small-amplitude oscillations appear around the frequencies which observed in no-noded forcing. The small-amplitude oscillations have two types of movements. One is two-nodal motion and the other is one-nodal motion.

(1) Two-nodal motion Around $\mu = 5.0$, the small-amplitude solution has the maximum displacement in the middle as we predicted from the forcing. Figure 2 shows a two-nodal motion which is symmetric about the mid point at $\mu = 4.8$ and $\lambda = 5.2$.

(2) One-nodal motion

Around $\mu = 8.0$, the small-amplitude solution barely moves in the middle of the cable. Figure 3 shows a one-nodal motion which has a longitudinal component at $\mu = 8.6$ and $\lambda = 2.4$.

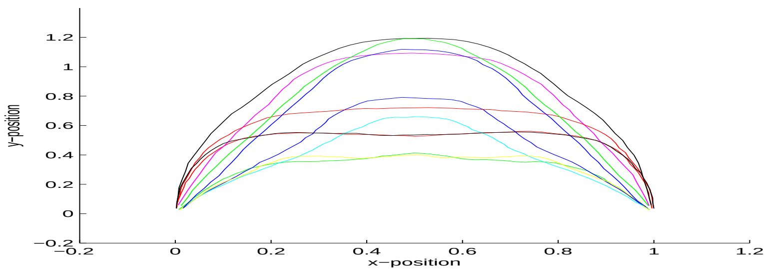

Figure 4. The large-amplitude solution for $\mu = 5.2$ and $\lambda = 0.4$.

3.1.2. *Large-amplitude solutions* Differently from the case of small initial condition, in the case of using large initial condition, small amplitude of forcing does not guarantee that the long term solution of small amplitude solution would converge to that solution.

In the case of no-noded forcing, three types of nonlinear behavior were found [10]. Solutions found are subharmonic solution which is a solution of double the period of the forcing term (Figure 4), large-amplitude solution which is of near periodic and no-noded, and longitudinal solution (similar to Figure 5).

In the case of two-noded forcing, the motions of cables with the large initial values are no-nodal motion, one-nodal motion, and two-nodal motion.

(1) No-nodal motion

The large-amplitude solutions around $\mu = 5.0$ are almost same as solutions with no-nodal forcing. It is presented in figure 5. It is slightly unsymmetric and no-nodal motion which is close to periodic with the same period as the forcing term. This large-amplitude no-nodal motions occur at $\mu = 4.8$, existing from $\lambda = 4.4$ to $\lambda = 6.0$, at $\mu = 5.0$, existing from $\lambda = 3.2$ to $\lambda = 6.0$, and at $\mu = 5.2$, existing from $\lambda = 0.8$ to $\lambda = 2.2$. Hence, large-amplitude solutions of this type exist without regard to the nodal types of forcing term.

Similarly to the case of no-noded forcing, as the frequency of the forcing term becomes small about the linear resonance, the amplitude of the solution with large initial displacement tends to increase and the motion of it becomes more disorganized.

(2) One-nodal motion

At higher frequency, we find a different type of oscillation. We observe a longitudinal motion in the first few periods even though the initial conditions are purely vertical, but after long time this changes to up and down motion. It is observed at $\mu = 8.6$ and

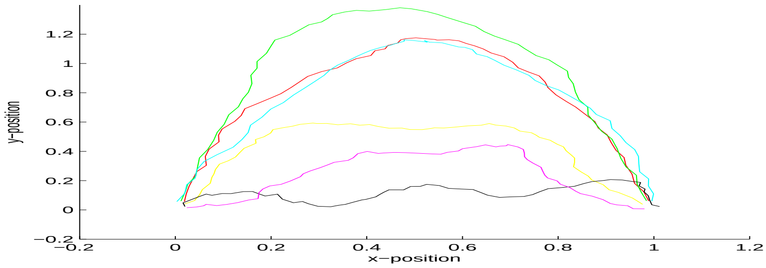

Figure 5. The large-amplitude solution for $\mu = 4.8$ and $\lambda = 5.2$.

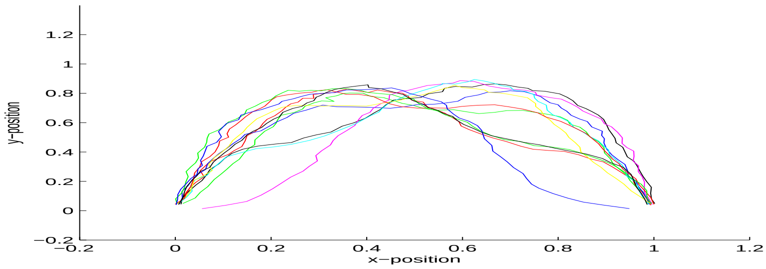

Figure 6. One-nodal solution for $\mu = 8.6$ and $\lambda = 2.8$.

$\lambda = 2.8$ (Figure 6). It shows a one-nodal motion which has a pronounced longitudinal component. One-nodal motion is periodic with double period of forcing.

(3) Two-nodal motion

Around $\mu = 8.6$, two-nodal motion is found. It is the solution got by large initial condition. It looks symmetric and fuzzily. This is observed at $\mu = 8.6$, from $\lambda = 3.4$ to $\lambda = 5.6$.

At $\mu = 8.6$ and $\lambda = 2.8$, the solution got by large initial condition is one-nodal (Figure 6) and the solution got by small initial condition is two-nodal (Figure 7). But at $\mu = 8.6$ and from $\lambda = 3.4$, the opposite phenomenon is appeared. The solution got by large initial condition is two-nodal and one got by small condition is one-nodal. Two-nodal motion appeared to be the double period as the forcing term.

The large-amplitude solutions, which are periodic with double period of the forcing, or almost periodic with a period of the forcing, reflect the nodal type of the forcing.

3.1.3. *Multiple periodic solutions* We have the small- and large-amplitude solutions at $\mu = 4.8$, existing from $\lambda = 4.4$ to $\lambda = 6.0$, at $\mu = 5.0$, existing from $\lambda = 3.2$ to $\lambda = 6.0$, and at $\mu = 5.2$, existing from $\lambda = 0.8$ to $\lambda = 2.2$. Figure 2 and 5 show the

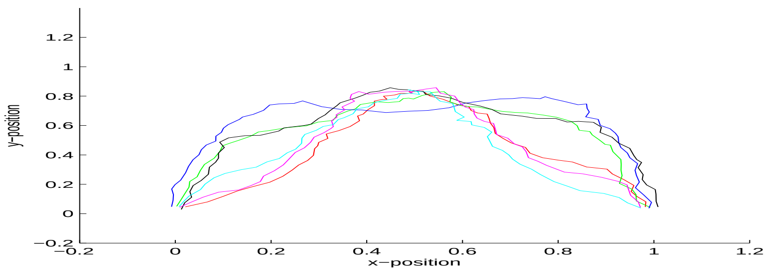

Figure 7. Two-nodal solution for $\mu = 8.6$ and $\lambda = 2.8$.

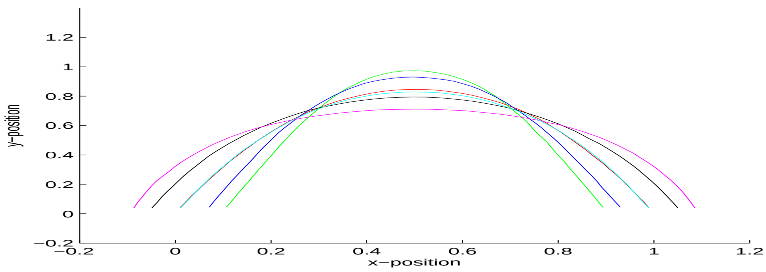

Figure 8. The small-amplitude solution for $\mu = 4.8$, and $\lambda = 0.1$ when two supports with forces out of phase move.

small- and large-amplitude solutions at $\lambda = 4.8$, $\mu = 5.2$, respectively. Figure 6 and 7 show multiple solutions at $\lambda = 8.6$, $\mu = 2.8$.

While in the case of no-noded forcing, small- and large-amplitude solution have one type of node, which is no-nodal, small- and large-amplitude solution have some types of node in the case of two-noded forcing.

Even when there exist small-amplitude solutions for the given forcing, large initial conditions could result in lasting large-amplitude solutions. This means that the solution after large time is keen to initial conditions.

3.2. Other type of forcing term In the case of the suspension cable, it is kept in an equilibrium position of extension by the suspended weight of the road-bed, and periodic forces would be exerted either by aerodynamic effects of the wind on the road-bed or by the oscillations of the towers. The oscillation of the towers could be induced by motions of the side-spans, or wind effects on the towers, or wind effects on the cable system [12].

We investigate the effect of forcing caused by vibrations on the motions of the cable in two supports. We change the forcing just on two supports P_0 and P_{N+1}

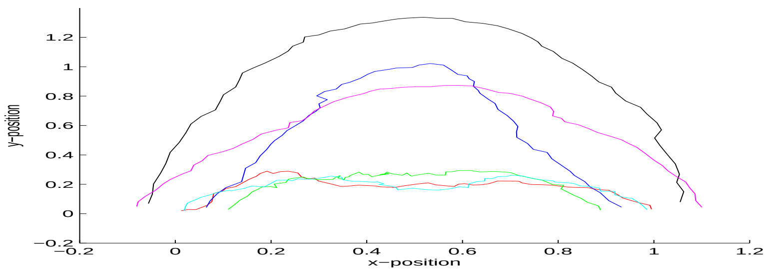

Figure 9. The large-amplitude solution for $\mu = 4.8$, and $\lambda = 0.1$ when two supports with forces out of phase move.

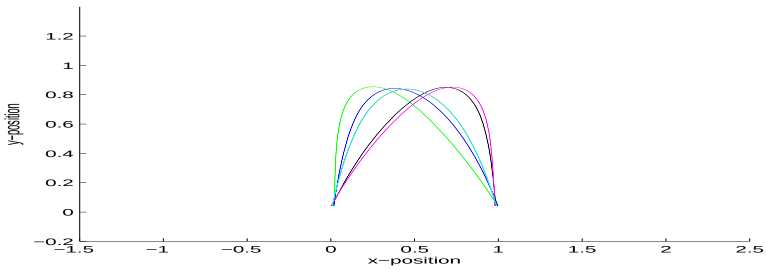

Figure 10. The small-amplitude solution for $\mu = 4.2$ and $\lambda = 0.02$ when two supports with forces in phase move.

which are the left endsupport and the right endsupport with the forces out of phase. This might be the force exerted by vibrating towers with towers out of phase. This experiment consists of employing the forcing $f(t) = \lambda \sin \mu t$ on $P_0(0,0)$ which is the left endsupport and $f(t) = -\lambda \sin \mu t$ on $P_{N+1}(1,0)$ which is the right endsupport.

The small- and large-amplitude solutions are found. The small-amplitude solution is symmetric about the mid point and periodic with the same period as the forcing term. This is presented in figure 8. The large-amplitude solutions are vertical motions which are likely analogous to the large-amplitude solutions which found in consequence of a vertical forcing (Figure 9). Hence, there exist large and small-amplitude solutions at $\mu = 4.8$ and $\lambda = 0.1$. Until the amplitude of the forcing term induces the magnitude of the solution to arrive in a certain critical value, the linear picture is found.

If we change the forcing just on two supports P_0 and P_{N+1} with the forces in phase, the cable motion is quite different from the case where the supporting towers are out of phase. Small or large motions are observed, even though we give forcing with much smaller displacements in the supports (Figure 10, 11, 12). The motions

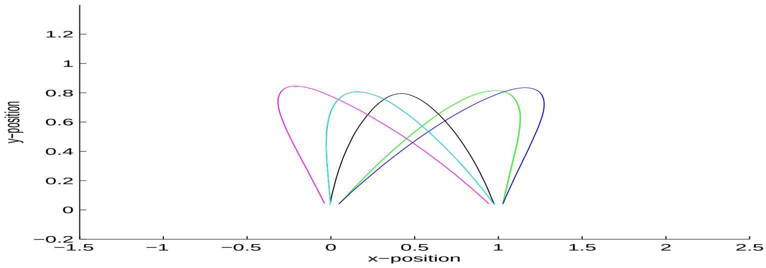

Figure 11. The large-amplitude solution for $\mu = 4.2$ and $\lambda = 0.02$ when two supports with forces in phase move.

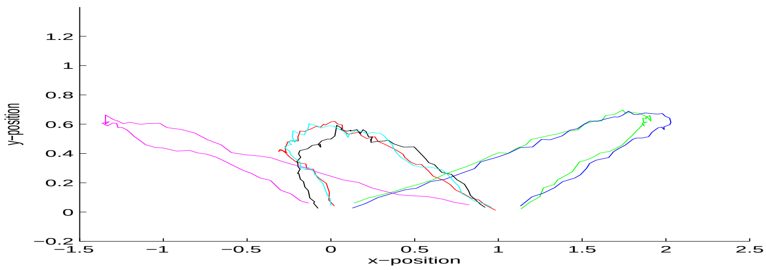

Figure 12. The longitudinal solution for $\mu = 4.6$ and $\lambda = 0.08$ when two supports with forces in phase move.

are longitudinal and the amplitude of solutions can be large.

This may well be the explanation of why cables of suspension bridge develop longitudinal motions, since the towers of the suspension bridges have been known to oscillate in both the in-phase and out-of-phase mode, possibly due to motions in the side spans [1].

4. Conclusion

In the third section, we mainly discussed about the effects of other types of forcing term ($f(\tilde{u}_i, t) = \lambda \sin 3\pi\tilde{u}_i \sin \mu t$), comparing with the result of the no-noded forcing ($f(\tilde{u}_i, t) = \lambda \sin \pi\tilde{u}_i \sin \mu t$). These result from careful and exhaustive computational experiments. The phenomenon found is that the solution of the system after long time depends on the initial conditions over a large scope of μ and λ. If the initial conditions are close to equilibrium, the solutions are small motions which are two-noded or one-noded in character. However, if initial conditions are large displacement, the long term solution can either stay in a large-amplitude solution, which is of one-noded, two-noded, or no-noded, or can converge to the small-amplitude solution.

The large-amplitude solutions may have two different types from the view of period. One is near-periodic solution which is in a state of disorder, the other is periodic solution which is double period of the forcing term. What we found was that there appeared to be a symmetry-breaking, in which initial conditions and purely vertical forcing terms could give rise to a pronounced longitudinal motions in the cable model for some time.

To investigate the effect of motion induced by vibrations in the supporting towers, we move the supporting towers, out of phase and in phase in a periodic way. We find that longitudinal motions may occur even when they have only vertical forcing. It is proven that this is a lasting problem in the cable of suspension bridge. In fact, after the Bronx-Whitestone bridge was constructed, it was actually modified to stop this behavior.

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