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FUNDAMENTAL THEOREM FOR LIGHTLIKE CURVES

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ABSTRACT. The purpose of this paper is to prove the fundamental existence and uniqueness theorems for lightlike curves in a 6-dimensional semi-Euclidean space $\mathbf{R}_q$ of index q, since the general n-dimensional cases are too complicated.

1. INTRODUCTION

Bonnor [2] studied real lightlike curves in a flat four dimensional Minkowski space-time. He constructed a Frenet frame and proved the fundamental existence and uniqueness theorem for this class of lightlike curves. His theory are generalized in a formal way to the lightlike curves in a Lorentz manifold by Bejancu [1]. For the general semi-Riemannian manifold of index greater than one, Duggal & Bejancu [3, Chap. 3, pp. 52–76] have shown (by an example) that Frenet frame appear in Bejancu [1] is not invariant with respect to causal change of any of its generating vector fields and the construction of a Frenet frame for lightlike curve should be done subject to some restrictive conditions on such curves.

Recently we studied the geometry of lightlike curves in a 6-dimensional semi-Riemannian space $\mathbf{M}_q$ of index q (*cf.* Jin [7]). We showed that it is possible to construct three types of Frenet frames suitable for $\mathbf{M}_q$, each invariant under any causal change. This is then followed by constructing general Frenet equations (called *compound Frenet equation*) which include all the possible forms of the three types.

The objective of this paper is to prove the fundamental existence and uniqueness theorems of lightlike curves in a 6-dimensional semi-Euclidean space $\mathbf{R}_q$ of index q, with a variety of Frenet frames of each type.

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2. FRENET EQUATIONS OF LIGHTLIKE CURVES

Let $\mathbf{M}_q$ be a real 6-dimensional semi-Riemannian manifold of constant index q $(1 \leq q \leq 3)$ and C be a smooth lightlike curve in $\mathbf{M}_q$ locally given by

$$x^A = x^A(t), \quad t \in \mathbf{I} \subset \mathbb{R}, \quad A \in \{0, 1, \dots, 5\}$$

for a coordinate neighborhood $\mathcal{U}$ on C. Since C is lightlike curve, the tangent vector field $\frac{d}{dt} = \lambda$ on $\mathcal{U}$ satisfies $g(\lambda, \lambda) = 0$. Denote by TC the tangent bundle of C and $TC^{\perp}$ the TC-perpendicular. Clearly, $TC^{\perp}$ is a vector bundle over C of rank 5 and TC is a vector subbundle of $TC^{\perp}$ of rank 1. This implies that $TC^{\perp}$ is not complementary to TC in $T\mathbf{M}_q|_C$. Thus we must find complementary vector bundle to TC in $T\mathbf{M}_q$ which will play the role of the normal bundle $TC^{\perp}$ consistent with the classical non-degenerate theory. A few researchers (*cf.* Bonnor [2], Duggal & Bejancu [3], Graves [5], Ikawa [6]) have done research on this matter dealing with only specified problems. Bejancu [1] and Duggal & Bejancu [3] developed a general mathematical theory to deal with the lightlike case, which we brief as follows:

Suppose $S(TC^{\perp})$ denotes the complementary vector subbundle to TC in $TC^{\perp}$, *i. e.*, , we have

$$TC^{\perp} = TC \perp S(TC^{\perp})$$

where $\perp$ means the orthogonal direct sum. It follows that $S(TC^{\perp})$ is a non-degenerate 4-dimensional vector subbundle of $T\mathbf{M}_q$. We call $S(TC^{\perp})$ a *screen vector bundle* of C, which being non-degenerate, we have

$$T\mathbf{M}_q|C = S(TC^{\perp}) \perp S(TC^{\perp})^{\perp}, \tag{1}$$

where $S(TC^{\perp})^{\perp}$ is a 2-dimensional complementary orthogonal vector subbundle to $S(TC^{\perp})$ in $T\mathbf{M}_q|_C$. Throughout this paper we denote by $F(C)$ the algebra of smooth functions on C and by $\Gamma(E)$ the $F(C)$ module of smooth sections of a vector bundle E over C. We use the same notation for any other vector bundle.

Theorm 2.1 (Bejancu [1], Duggal & Bejancu [3]). *Let C be a lightlike curve on a semi-Riemannian manifold $\mathbf{M}_q$ and $S(TC^{\perp})$ a screen vector bundle of C. Then there exists a unique vector bundle* $\mathrm{ltr}(C)$ *over C of rank* 1, *such that on each coordinate neighborhood $\mathcal{U} \subset C$ there is a unique section $N \in \Gamma(\mathrm{ltr}(C)|_{\mathcal{U}})$ satisfying*

$$g(\lambda, N) = 1, \quad g(N, N) = g(N, X) = 0, \tag{2}$$

for every $X \in \Gamma(S(TC^{\perp})|\mathcal{U})$.

We call $\mathrm{ltr}(C)$ the *lightlike transversal bundle* of C with respect to $S(TC^{\perp})$. Next consider the vector bundle

$$\mathrm{tr}(C) = \mathrm{ltr}(C) \perp S(TC^{\perp}),$$

which according to (1) and (2) is complementary but not orthogonal to TC in $T\mathbf{M}_q|_C$. More precisely, we have

$$\begin{aligned} T\mathbf{M}_q|C &= TC \oplus \mathrm{tr}(C) \\ &= (TC \oplus \mathrm{ltr}(C)) \perp S(TC^{\perp}). \end{aligned} \tag{3}$$

We call $\mathrm{tr}(C)$ the *transversal vector bundle* of C with respect to $S(TC^{\perp})$. The vector field N in Theorem 2.1 is called the *lightlike transversal vector field* of C with respect to λ. As $\{\lambda, N\}$ is a lightlike basis of $\Gamma((TC \oplus \mathrm{ntr}(C))|_{\mathcal{U}})$ satisfying (2), we obtain

Proposition 2.1 (Bejancu [1], Duggal & Bejancu [3]). *Let C be a lightlike curve on a semi-Riemannian manifold $\mathbf{M}_q$. Then any screen vector bundle of C is semi-Riemannian of index $q-1$.*

Denote ∇ the Levi-Civita connection on $\mathbf{M}_q$. Since any screen vector bundle of C will be semi-Riemannian of index $q-1$ from Proposition 2.1, there are three cases by the causality of the vector fields

$$\begin{aligned} S_1 &= \nabla_\lambda \lambda \setminus \mathrm{Span}\{\lambda, N\}, \\ S_2 &= \nabla_\lambda N \setminus \mathrm{Span}\{\lambda, N, S_1\}, \text{ and} \\ S_3 &= \nabla_\lambda S_1 \setminus \mathrm{Span}\{\lambda, N, S_1, S_2\}, \end{aligned}$$

where S_1 and S_2 are vector fields of $S(TC^{\perp})$. The Frenet equations that all of these vector fields are non-lightlikes is called *Type 1*, one of these vector fields is lightlike is called *Type 2*, two of these vector fields are lightlikes is called *Type 3*. By using (2) and (3) and taking into account that $S(TC^{\perp})$ is a semi-Riemannian vector bundle of rank m, we obtain the following equations (*cf.* Jin [7])

$$\begin{aligned} \nabla_\lambda \lambda &= h\lambda + \kappa_1 W_1 + \tau_1 W_2, \\ \nabla_\lambda N &= -hN + \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3 + \tau_2 W_4, \\ \varepsilon_1 \nabla_\lambda W_1 &= -\kappa_2 \lambda - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3 + \tau_5 W_4, \\ \varepsilon_2 \nabla_\lambda W_2 &= -\kappa_3 \lambda - \tau_1 N - \kappa_4 W_1 + \kappa_6 W_3 + \kappa_7 W_4, \\ \varepsilon_3 \nabla_\lambda W_3 &= -\tau_3 \lambda - \kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \text{ and} \\ \varepsilon_4 \nabla_\lambda W_4 &= -\tau_2 \lambda - \tau_5 W_1 - \kappa_7 W_2 - \kappa_8 W_3, \end{aligned} \tag{4}$$

where ε_i is the signature of W_i. In this case, we call

$$F = \{\lambda, N, W_1, W_2, W_3, W_4\} \tag{5}$$

compound Frenet frame on $\mathbf{M}_q$ along C with respect to a give screen vector bundle $S(TC^{\perp})$ and the equations (4) its *compound Frenet equations* of the lightlike curve. The functions $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ and $\{\tau_1, \tau_2, \tau_3, \tau_5\}$ are called the *curvature functions* and *torsion functions* of C with respect to the frame F.

Remark. We find that the Frenet equations (4) is Type 1 if $\tau_1 = \tau_2 = \tau_3 = \tau_5 = 0$, Type 2 if $\tau_2 = 0$ and Type 3 if $\tau_i \neq 0$ for all i. $\mathbf{M}_3$ have Frenet equations of all three types, $\mathbf{M}_2$ have Frenet equations of two types, named by Type 1 and Type 2, up to the signatures of $W_i's$ and $\mathbf{M}_1$ have Frenet equations of only one type, named by Type 1, up to the signatures of $W_i's$.

We consider the following differential equation

$$\frac{d^2t}{dt^{*2}} - h^* \frac{dt}{dt^*} = 0$$

whose general solution comes from

$$t = a \int_{t_0^*}^{t*} \exp\left(\int_{s_0}^{s} h^*(t^*)dt^*\right) ds + b \quad a, b \in \mathbb{R}.$$

From the first equation of (4), any of these solutions, with $a \neq 0$, might be taken as special parameter on C, such that $h = 0$. Denote one such solution by $p = \frac{t-b}{a}$, where t is the general parameter as defined in above equation. We call p a *distinguished parameter* of C, in terms for which $h = 0$. It is important to note that when t is replaced by p in the compound Frenet equations (4), the first two equations become

$$\begin{aligned} \nabla_\mu \mu &= \kappa_1 W_1 + \tau_1 W_2 \\ \nabla_\mu N &= \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3 + \tau_2 W_4 \end{aligned} \tag{6}$$

and the other equations remain unchanged, where $\mu = \frac{d}{dp}$.

In case $\kappa_1 = 0$, then, since $\tau_1 = 0$ or $\tau_1 = -\kappa_1$, the first equation of (6) takes the following familiar form

$$\frac{d^2x^A}{dp^2} + \sum_{A,B} \Gamma^A_{BC} \frac{dx^B}{dp}\frac{dx^C}{dp} = 0, \quad A, B, C \in \{0, 1, \ldots, 5\}$$

where Γ^A_{BC} are the Christoffel symbols of the second kind induced by ∇. Hence C is a null geodesic of M. The converse follows easily. Thus we have

Theorm 2.2. *Let C be a lightlike curve of a semi-Riemannian manifold M_q. Then C is a lightlike geodesic of M if and only if the first curvature κ_1 vanishes identically on C.*

3. Fundamental theorem for lightlike curves

Let $\mathbf{R}_q\,(1 \leq q \leq 3)$ be the 6-dimensional semi-Euclidean space of index q with the semi-Euclidean metric

$$g(x,y) = -\left(\sum_{i=0}^{q-1} x^i y^i\right) + \left(\sum_{\alpha=q}^{5} x^\alpha y^\alpha\right).$$

Suppose C is a lightlike curve in $\mathbf{R}_q$ locally given by the equations

$$x^A = x^A(t), \qquad t \in \mathbf{I} \subset \mathbb{R}, \qquad A \in \{0, 1, \dots, 5\}.$$

First, we define in $\mathbf{R}_q$ the natural orthonormal basis

$$\begin{aligned} L &= \left(\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right), & L^* &= \left(-\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right), \\ E_1 &= (0,1,0,0,0,0), & E_2 &= (0,0,1,0,0,0), \\ E_3 &= (0,0,0,1,0,0), & E_4 &= (0,0,0,0,1,0), \end{aligned} \tag{7}$$

where $\{L, L^*\}$ are lightlike vectors such that $g(L, L^*) = 1$. In case $q = 1$, E_1, E_2, E_3, E_4 are spacelike vector fields. In case $q = 2$, E_1 is a timelike vector field and E_2, E_3, E_4 are spacelike vector fields. In case $q = 3$, E_1 and E_2 are timelike vector fields and E_3 and E_4 are spacelike vector fields. It is easy to see that

$$L^A L^{*B} + L^B L^{*A} + \sum_{i=1}^{4} \varepsilon_i E_i^A E_i^B = h^{AB}, \tag{8}$$

for any $A, B \in \{0, \dots, 5\}$, where we put

$$h^{AB} = \begin{cases} -1, & A = B \in \{0, 1, \dots, q-1\} \\ 1, & A = B \in \{q, q+1, \dots, 5\} \\ 0, & A \neq B. \end{cases}$$

We are now in a position to state the fundamental existence and uniqueness theorem for lightlike curves of semi-Euclidean space $\mathbf{R}_q$.

Theorm 3.1. *Let $\kappa_1, \kappa_2, \dots, \kappa_8 : [-\varepsilon, \varepsilon] \to \mathbb{R}$ be everywhere continuous functions, $x_0 = (x_0^i)$ be a fixed point of $\mathbf{R}_q$ and let $\{L, L^*, E_1, E_2, E_3, E_4\}$ be the quasi-orthonormal basis in (7). Then there exists a unique lightlike curve $C : [-\varepsilon, \varepsilon] \to \mathbf{R}_q$*

given by the equations $x^i = x^i(p)$, *where* p *is a distinguished parameter on* C, *such that* $C(0) = x_0$ *whose curvature functions are* $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ *and whose Frenet frames of Type 1* $\{\mu, N, W_1, W_2, W_3, W_4\}$ *satisfies*

$$\mu(0) = L, \quad N(0) = L^*, \quad W_\alpha(0) = E_\alpha, \quad \alpha \in \{1, 2, 3, 4\}.$$

Proof. Note that $\nabla_\mu X$ is just X' for any vector field X defined on $\mathcal{U}$. Using the equations (4) and (6) we consider the system of differential equation

$$\begin{aligned} \mu' &= \kappa_1 W_1, \\ N' &= \kappa_2 W_1 + \kappa_3 W_2, \\ \varepsilon_1 W_1' &= -\kappa_2 \mu - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3, \\ \varepsilon_2 W_2' &= -\kappa_3 \mu - \kappa_4 W_4 + \kappa_6 W_3 + \kappa_7 W_4, \\ \varepsilon_3 W_3' &= -\kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \text{ and} \\ \varepsilon_4 W_4' &= -\kappa_7 W_2 - \kappa_8 W_3. \end{aligned} \tag{9}$$

Then there exists a unique solution $\{\mu, N, W_1, W_2, W_3, W_4\}$ satisfying the initial conditions $\mu(0) = L, N(0) = L^*, W_\alpha(0) = E_\alpha, \alpha \in \{1, 2, 3, 4\}$. Now we claim that $\{\mu(p), N(p), W_1(p), W_2(p), W_3(p), W_4(p)\}$ is a quasi-orthonormal basis such that $\{\mu, N\}$ and $\{W_q, W_{q+1}, \ldots, W_4\}$ are lightlike and spacelike vectors respectively and the other $W_i's$ are timelikes for $p \in [-\varepsilon, \varepsilon]$. To this end, by direct calculations using (9), we obtain

$$\frac{d}{dp}\left\{\mu^A N^B + \mu^B N^A + \sum_{i=1}^{4} \varepsilon_i W_i^A W_i^B\right\} = 0. \tag{10}$$

As for $p = 0$ we have (8), from (10) it follows that

$$\mu^A N^B + \mu^B N^A + \sum_{i=1}^{4} \varepsilon_i W_i^A W_i^B = h^{AB}. \tag{11}$$

Further on, construct the field of frames

$$W_0 = \frac{1}{\sqrt{2}}(\mu - N), \qquad W_5 = \frac{1}{\sqrt{2}}(\mu + N), \tag{12}$$

where W_0 is a timelike vector field and W_5 is a spacelike one. Then (10) becomes

$$\sum_{i=0}^{5} \varepsilon_i W_i^A W_i^B = h^{AB}. \tag{13}$$

We define for each $p \in [-\varepsilon, \varepsilon]$ the matrix $D(p) = \left(d^{AB}(p)\right)$ such that

$$d^{ab} = W_b^a, \; d^{ai} = \sqrt{-1} W_i^a, \; d^{hb} = -\sqrt{-1} W_b^h, \; d^{hi} = W_i^h,$$

for any $a, b \in \{0, 1, \ldots, q-1\}, h, i \in \{q, q+1, \ldots, 5\}$. By using (12) it is easy to check that $D(p)D(p)^t = I_6$, which implies that $\{W_1, W_2, W_3, W_4\}$ is an orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. Then, from (11), we conclude that $\{\mu, N, W_1, W_2, W_3, W_4\}$ is a quasi-orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. The lightlike curve is obtained by integrating the system

$$\frac{dx^i}{dp} = \mu(t), \qquad x^i(0) = x_0^i.$$

Taking into account of (10) we see $F = \{\mu, N, W_1, W_2, W_3, W_4\}$ is a Frenet frame of Type 1 for C with curvature functions $\{\kappa_1, \kappa_1, \ldots, \kappa_8\}$. This completes the proof of theorem. □

Next, we define in $\mathbf{R}_q (2 \leq q \leq 3)$ the quasi-orthonormal basis

$$\begin{aligned} L_1 &= \left(\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right), & L_1^* &= \left(-\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right), \\ L_2 &= \left(0, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0\right), & L_2^* &= \left(0, -\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0\right), \\ E_2 &= (0, 0, 1, 0, 0,), & E_3 &= (0, 0, 0, 1, 0, 0), \end{aligned} \tag{14}$$

where $\{L_1, L_2, L_1^*, L_2^*\}$ are lightlike vector fields such that

$$g(L_i, L_j^*) = \delta_{ij}, \quad g(L_i, L_j) = 0, \quad g(L_i^*, L_j^*) = 0$$

and $\{E_2, E_3\}$ are orthonormal spacelike vector fields for $q = 2$ or orthonormal timelike and spacelike vector fields respectively for $q = 3$. In this case also we find

$$\sum_{i=1}^{2} (L_i^A L_i^{*B} + L_i^B L_i^{*A}) + \sum_{\alpha=2}^{3} \varepsilon_\alpha E_\alpha^A E_\alpha^B = h^{AB}, \tag{15}$$

for any $A, B \in \{0, 1, \ldots, 5\}$. Let $E_1 = \frac{L_2 - L_2^*}{\sqrt{2}}$ and $E_4 = \frac{L_2 + L_2^*}{\sqrt{2}}$, then $\{E_1, E_4\}$ are orthonormal timelike and spacelike vector fields respectively which is orthogonal to $\{E_2, E_3\}$. The equation (15) becomes

$$L_1^A L_1^{*B} + L_1^B L_1^{*A} + \sum_{\alpha=1}^{4} \varepsilon_\alpha E_\alpha^A E_\alpha^B = h^{AB}. \tag{16}$$

Theorm 3.2. *Let $\kappa_1, \kappa_2, \ldots, \kappa_8; \tau_1, \tau_3, \tau_5 : [-\varepsilon, \varepsilon] \to \mathbb{R}$ be everywhere continuous functions, $x_0 = (x_0^i)$ be a fixed point of $\mathbf{R}_q$ and let $\{L_i, L_i^*, E_\alpha\}, i \in \{1, 2\}, \alpha \in \{2, 3\}$ be the quasi-orthonormal basis in (14). Then there exists a unique lightlike curve $C : [-\varepsilon, \varepsilon] \to \mathbf{R}_q$ such that $x^i = x^i(p), C(0) = x_0$ and $\{\kappa_1, \ldots, \kappa_8\}$ are the curvature* *functions and* $\{\tau_1, \tau_3, \tau_5\}$ *the torsion functions with respect to a Frenet frame of Type 2* $F = \{\mu, N, W_1, W_2, \dots, W_4\}$ *that satisfies*

$$\mu = L_1, \quad N(0) = L_1^*, \quad W_\alpha(0) = E_\alpha, \quad \alpha \in \{1, 2, 3, 4\}.$$

Proof. We consider the system of differential equations

$$\begin{aligned}
\mu' &= \kappa_1 W_1 + \tau_1 W_2, \\
N' &= \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3, \\
\varepsilon_1 W_1' &= -\kappa_2 \mu - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3 + \tau_5 W_4, \\
\varepsilon_2 W_2' &= -\kappa_3 \mu - \kappa_4 W_4 + \kappa_6 W_3 + \kappa_7 W_4, \\
\varepsilon_3 W_3' &= -\kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \text{ and} \\
\varepsilon_4 W_4' &= -\kappa_7 W_2 - \kappa_8 W_3.
\end{aligned} \tag{17}$$

Then, there exists a unique solution $\{\mu(p), N(p), W_1(p), W_2(p), \dots, W_4(p)\}$ satisfying the initial conditions $\mu(0) = L_1, N(0) = L_1^*, W_\alpha(0) = E_\alpha, \alpha \in \{1, 2, 3, 4\}$ such that

$$\mu^A N^B + \mu^B N^A + \sum_{\alpha=1}^{4} W_\alpha^A W_\alpha^B = h^{AB}. \tag{18}$$

Let $N_2 = \frac{W_1 + W_4}{\sqrt{2}}$ and $N_2^* = \frac{W_1 - W_4}{\sqrt{2}}$, then $\{N_2, N_2^*\}$ are lightlike vector fields such that $g(N_2, N_2^*) = 1$ and $N_2(0) = L_2; N_2^*(0) = L_2^*$. Thus $\{\mu(p) = N_1(p), N(p) = N_1^*(p), N_2(p), N_2^*(p)\}$ are lightlike vector fields such that

$$g(N_i(p), N_j^*(p)) = \delta_{ij}, \quad g(N_i(p), N_j(p)) = 0, \quad g(N_i^*(p), N_j^*(p)) = 0$$

and $\{W_2, W_3\}$ are spacelike vector fields for $q = 2$ or timelike and spacelike vector fields respectively for $q = 3$ for $p \in [-\varepsilon, \varepsilon]$. To this end, construct the field of frames

$$W_0 = \frac{1}{\sqrt{2}}(\mu - N), \qquad W_5 = \frac{1}{\sqrt{2}}(\mu + N), \tag{19}$$

then W_0 is a timelike and W_5 is a spacelike vector field and (18) becomes

$$\sum_{\alpha=0}^{5} W_\alpha^A W_\alpha^B = h^{AB} \tag{20}$$

and it is easy to check that the matrix $D(p) = (d^{AB}(p))$ satisfies $D(p)D(p)^t = I_6$, which implies also that $\{W_0, W_1, \dots, W_5\}$ is an orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. Then from (19) we conclude that $\{\mu, N, N_2, N_2^*, W_1, W_2, W_3, W_4\}$ is a quasi-orthonormal basis for any $p \in [-\varepsilon, \varepsilon]$. Thus there is a lightlike C such that $C(0) = x_0$

and $F = \{\mu, N, W_1, W_2, W_3, W_4\}$ is a Frenet frame of Type 2 for C with curvature functions $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ and torsion functions $\{\tau_1, \tau_3, \tau_5\}$.

In the last case, we define in $\mathbf{R}_3$ the quasi-orthonormal basis

$$\begin{aligned} L_1 &= \left(\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right), \quad L_1^* = \left(-\frac{1}{\sqrt{2}}, 0, 0, 0, 0, \frac{1}{\sqrt{2}}\right), \\ L_2 &= \left(0, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0\right), \quad L_2^* = \left(0, -\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0\right), \\ L_3 &= \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \quad L_3^* = \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \end{aligned} \tag{21}$$

where $\{L_1, L_2, L_3, L_1^*, L_2^*, L_3^*\}$ are lightlike vector fields such that

$$g(L_i, L_j^*) = \delta_{ij}, \quad g(L_i, L_j) = 0, \quad g(L_i^*, L_j^*) = 0.$$

In this case also we find

$$\sum_{i=1}^{3} (L_i^A L_i^{*B} + L_i^B L_i^{*A}) = h^{AB}, \tag{22}$$

for any $A, B \in \{0, 1, \ldots, 5\}$. Let

$$E_1 = \frac{L_2 - L_2^*}{\sqrt{2}}, \; E_4 = \frac{L_2 + L_2^*}{\sqrt{2}} \quad \text{and} \quad E_2 = \frac{L_3 - L_3^*}{\sqrt{2}}, \; E_3 = \frac{L_3 + L_3^*}{\sqrt{2}},$$

then $\{E_1, E_2\}$ and $\{E_3, E_4\}$ are mutually orthogonal orthonormal sets of timelike and spacelike vector fields respectively. The equation (22) becomes

$$L_1^A L_1^{*B} + L_1^B L_1^{*A} + \sum_{\alpha=1}^{4} \varepsilon_\alpha E_\alpha^A E_\alpha^B = h^{AB}. \qquad \square$$

Theorm 3.3. *Let $\kappa_1, \kappa_2, \ldots, \kappa_8; \tau_1, \tau_2, \tau_3, \tau_5 : [-\varepsilon, \varepsilon] \to \mathbb{R}$ be everywhere continuous functions, $x_0 = (x_0^i)$ be a fixed point of $\mathbf{R}_3$ and let $\{L_i, L_i^*, E_\alpha\}, i \in \{1, 2\}, \alpha \in \{2, 3\}$ be the quasi-orthonormal basis in (21). Then there exists a unique lightlike curve $C : [-\varepsilon, \varepsilon] \to \mathbf{R}_3$ such that $x^i = x^i(p), C(0) = x_0$ and $\{\kappa_1, \kappa_2, \ldots, \kappa_8\}$ are the curvature functions and $\{\tau_1, \tau_2, \tau_3, \tau_5\}$ the torsion functions with respect to a Frenet frame $F = \{\mu, N, W_1, W_2, W_3, W_4\}$ of Type 3 that satisfies*

$$\mu = L_1, \quad N(0) = L_1^*, \quad W_\alpha(0) = E_\alpha, \quad \alpha \in \{1, 2, 3, 4\}.$$

Proof. We consider the system of differential equations

$$\begin{aligned}
\mu' &= \kappa_1 W_1 + \tau_1 W_2,\\
N' &= \kappa_2 W_1 + \kappa_3 W_2 + \tau_3 W_3 + \tau_2 W_4,\\
\varepsilon_1 W_1' &= -\kappa_2 \mu - \kappa_1 N + \kappa_4 W_2 + \kappa_5 W_3 + \tau_5 W_4,\\
\varepsilon_2 W_2' &= -\kappa_3 \mu - \kappa_4 W_4 + \kappa_6 W_3 + \kappa_7 W_4,\\
\varepsilon_3 W_3' &= -\kappa_5 W_1 - \kappa_6 W_2 + \kappa_8 W_4, \text{ and}\\
\varepsilon_4 W_4' &= -\kappa_7 W_2 - \kappa_8 W_3.
\end{aligned}$$

Then, there exists a unique solution $\{\mu(p), N(p), W_1(p), W_2(p), W_3(p), W_4(p)\}$ satisfying the initial conditions $\mu(0) = L_1, N(0) = L_1^*, W_\alpha(0) = E_\alpha, \alpha \in \{1,2,3,4\}$ such that

$$\mu^A N^B + \mu^B N^A + \sum_{\alpha=1}^{4} W_\alpha^A W_\alpha^B = h^{AB}.$$

By a procedure same as for Theorems 3.1 and 3.2, we can prove the theorem.

4. Concluding remark

In this paper we proved the fundamental existence and uniqueness theorems of a lightlike curve in a 6-dimensional semi-Riemannian manifold of index q, with a variety of Frenet frames of Type 1, Type 2 and Type 3. This is only a step further of the earlier works of Duggal & Bejancu [3] on lightlike curves of Lorentzian manifolds. However, the general case of lightlike curves in semi-Riemannian manifolds of arbitrary dimension is still an open problem. We guess that the general cases are too complicated. But We hope that the publication of this paper will help in solving the general case.

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