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ON LINEAR PERTURBATIONS AND ABSOLUTE ROOT BOUND FUNCTIONALS

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ABSTRACT. We will show that any linear perturbation of polynomials that introduces bounded perturbations into the roots of polynomial is some linear combination of the derivatives of a polynomial. And we will derive an absolute root bound functional which is in some sense better than the other known absolute root bound functionals.

1. INTRODUCTION AND NOTATIONS

Let $p(z)$ be a polynomial in the complex variable z. The *first divided difference* of $p(z)$ is denoted by $p[z_0, z_1]$ and defined by the relation

$$p[z_0, z_1] \;=\; \frac{p(z_1) - p(z_0)}{z_1 - z_0}.$$

The *n-th divided difference* is defined by induction in terms of the $(n-1)$-th divided difference by the formula:

$$p[z_0, z_1, \ldots, z_n] = \frac{p[z_0, \ldots, z_{n-2}, z_n] - p[z_0, \ldots, z_{n-2}, z_{n-1}]}{z_n - z_{n-1}}.$$

In order to derive a new formula for the divided differences which is useful in studying perturbation of roots, we need the following lemma.

Lemma 1.1.

$$p[z_0, z_1, \ldots, z_n] \;=\; \frac{1}{2\pi i}\int_\Gamma \frac{p(z)}{(z - z_0)(z - z_1)\cdots(z - z_n)}\, dz,$$

where the points $z_0, z_1, \ldots, z_n$ lie inside the contour Γ.

From Lemma 1.1, interchange of any two of the arguments does not alter the value of the divided difference. Therefore $p[z_0, z_1, \ldots, z_n]$ is a symmetric function

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(*i. e.*, invariant under all permutations of the variables $z_0, \dots, z_n$ even if some of them coincide). For $n+1$ coincident arguments z_0 (*i. e.*, $z_1 = z_2 = \cdots = z_n = z_0$) we obtain the equality

$$p[z_0, \dots, z_0] = \frac{1}{n!} p^{(n)}(z_0).$$

By Cauchy's integral formula, we have the following estimate;

$$\big|p[z_0, \dots, z_n]\big| \leq \frac{1}{n!} \sup_{z \in \mathcal{D}} |p^{(n)}(z)|,$$

where $\mathcal{D}$ is any convex region in the complex plane $\mathbb{C}$, containing $z_0, \dots, z_n$.

If $p(z)$ is a polynomial of degree n, then by Newton's interpolation formula, $p(z)$ can be reconstructed uniquely from the values of the divided differences at $z_0, \dots, z_n$, as follows:

$$p(z) = p[z_0] + p[z_0, z_1](z - z_0) + p[z_0, z_1, z_2](z - z_0)(z - z_1) + \cdots \\ + p[z_0, z_1, \dots, z_n](z - z_0)(z - z_1) \cdots (z - z_{n-1}).$$

Notations. Let $p(z)$ be a polynomial with degree n. We denote the set of roots of $p(z)$ by the finite sequence $Q_n = \{q_1, \dots, q_n\}$ because some of roots may coincide, the letters $\alpha, \beta, \gamma, \dots$ will denote subsequences of Q_n and denote by $\alpha', \beta', \gamma', \dots$ complements of these subsequences in Q_n. For $\alpha \subseteq Q_n$ we denote by $p[\alpha]$ for the divided difference of $p(z)$. We set, for any $\alpha \subseteq Q_n$,

$$(z - q)^\alpha = \begin{cases} \prod_{q_j \in \alpha} (z - q_j) & \text{for } \alpha \neq \varnothing \\ 1 & \text{for } \alpha = \varnothing. \end{cases}$$

Let $r(z)$ be a polynomial of degree $\leq n-1$ and $\tilde{q_1}, \dots, \tilde{q_n}$ the roots of

$$\tilde{p}(z) = p(z) + r(z) = (z - \tilde{q}_1) \cdots (z - \tilde{q}_n).$$

We also set, for any $\alpha \subseteq Q_n$,

$$(q - \tilde{q})^\alpha = \begin{cases} \prod_{q_j \in \alpha} (q_j - \tilde{q}_j) & \text{for } \alpha \neq \varnothing \\ 1 & \text{for } \alpha = \varnothing. \end{cases}$$

If $Q_n = \{q_1, \dots, q_n\}$ is a fixed sequence of complex numbers. Then for any subsequence $\{q_j, q_k, q_\ell, \dots\} \subseteq Q_n$, we shall always set $j < k < \ell < \cdots (\leq n)$ throughout this paper. If β is a finite sequence of complex numbers, we denote $\#(\beta)$ the number of terms (components) in β.

We will now define $(q_\alpha - \tilde{q})^\nu$ as follows:

Let $\beta \subseteq Q_n$ and $\#(\beta) = m \leq n$. For any subsequence $\alpha \subseteq \beta$, set $\alpha = \{q_{\alpha_1}, q_{\alpha_2}, \ldots\}$, $\beta' = \{q_{c_1}, q_{c_2}, \ldots\} \subseteq Q_n$ (remember β' is the complement of β in Q_n). Choose

$$\nu = \{q_{c_{j_1}}, q_{c_{j_2}}, \ldots, q_{c_{j_{\#(\nu)}}}\} \subseteq \beta'$$

so that $\#(\nu) = n + 1 - m - \#(\alpha)$, then we define

$$(q_\alpha - \tilde{q})^\nu = (q_{\alpha_{i_1}} - \tilde{q}_{c_{j_1}})(q_{\alpha_{i_2}} - \tilde{q}_{c_{j_2}}) \cdots (q_{\alpha_{i_{\#(\nu)}}} - \tilde{q}_{c_{j_{\#(\nu)}}})$$

so that $i_1 = j_1$, $i_2 = j_2 - 1$, $\ldots$, $i_{\#(\nu)} = j_{\#(\nu)} - \#(\nu) + 1$.

Now for our work we need the new formula for the divided differences.

Theorem 1.2. *Let $Q_n = \{q_1, \ldots, q_n\}$. Suppose that*

$$p(z) = (z - q_1)(z - q_2) \cdots (z - q_n)$$

and

$$p(z) + r(z) = (z - \tilde{q}_1) \cdots (z - \tilde{q}_n), \qquad \deg(r(z)) \leq n - 1.$$

Then

(i) *For any subset $\beta \subseteq Q_n$ such that $\#(\beta) = m \leq n$,*

$$r[\beta] = \frac{1}{2\pi i} \int_\Gamma \frac{(z - \tilde{q}_1) \cdots (z - \tilde{q}_n)}{(z - q)^\beta} \, dz = \sum_{\varnothing \neq \alpha \subseteq \beta} (q - \tilde{q})^\alpha \sum_{\substack{\nu \subseteq \beta' \\ \#(\nu) = n + 1 - m - \#(\alpha)}} (q_\alpha - \tilde{q})^\nu.$$

(ii) *For a given $\rho \geq 0$, if $|q_j - \tilde{q}_j| \leq \rho$ for all j, then for any non-empty subset $\beta \subseteq Q_n$,*

$$|r[\beta]| \leq \sum_{\varnothing \neq \alpha \subseteq \beta} \rho^{\#(\alpha)} \sum_{\substack{\nu \subseteq \beta' \\ \#(\nu) = n + 1 - m - \#(\alpha)}} (q_\alpha - \tilde{q})^\nu_\rho. \tag{1}$$

where $(q_\alpha - q)^\nu_\rho = (|q_{\alpha_{i_1}} - q_{c_{j_1}}| + \rho) \cdots (|q_{\alpha_{i_{\#(\nu)}}} - q_{c_{j_{\#(\nu)}}}| + \rho)$.

Conversely if (1) holds for all non-empty subset $\beta \subseteq Q_n$, then there exists $C(n)$ depending only on n that the roots $\tilde{q}_1, \ldots, \tilde{q}_n$ of $p(z) + r(z)$ can be indexed in such way that $|q_j - \tilde{q}_j| \leq C(n)\rho$, $j = 1, \ldots, n$.

Proof. See Park [11] for details. □

Let us write down the formula $r[\beta]$ for $n = 4$ for an example.

Example 1.3. For $n = 4$, let $p(z) = (z - q_1)(z - q_2)(z - q_3)(z - q_4)$ and $\deg(r(z)) \leq 3$. Then we have the following forms.

$$r[q_j] = (q_j - \tilde{q}_1)(q_j - \tilde{q}_2)(q_j - \tilde{q}_3)(q_j - \tilde{q}_4)$$

for $1 \le j \le 4$;

$$r[q_j, q_k] = (q_j - \tilde{q}_j)(q_j - \tilde{q}_\ell)(q_j - \tilde{q}_m) + (q_k - \tilde{q}_k)(q_j - \tilde{q}_\ell)(q_k - \tilde{q}_m) + (q_j - \tilde{q}_j)(q_k - \tilde{q}_k)(q_j - \tilde{q}_\ell)$$

for $1 \le j < k \le 4$, $\ell \ne m \in \{1,2,3,4\} \setminus \{j,k\}$;

$$r[q_j, q_k, q_\ell] = (q_j - \tilde{q}_j)(q_j - \tilde{q}_m) + (q_k - \tilde{q}_k)(q_k - \tilde{q}_m) + (q_\ell - \tilde{q}_\ell)(q_\ell - \tilde{q}_m) + (q_j - \tilde{q}_j)(q_k - \tilde{q}_k) + (q_j - \tilde{q}_j)(q_\ell - \tilde{q}_\ell) + (q_k - \tilde{q}_k)(q_\ell - \tilde{q}_\ell)$$

for $1 \le j < k < \ell \le 4$, $m \in \{1,2,3,4\} \setminus \{j,k,\ell\}$;

$$r[q_1, q_2, q_3, q_4] = (q_1 - \tilde{q}_1) + (q_2 - \tilde{q}_2) + (q_3 - \tilde{q}_3) + (q_4 - \tilde{q}_4).$$

From the fact that, for $\beta \subseteq Q_n$ such that $\#(\beta) = m$, $r[\beta]$ has $\binom{n}{n+1-m}$ terms, we obtain the following results.

Corollary 1.4. *Let $p(z) = (z - z_0)^n, r(z)$ be a polynomial of degree $\le n-1$ and $\tilde{p}(z) = p(z) + r(z) = (z - \tilde{q}_1)\cdots(z - \tilde{q}_n)$. If $|z_0 - \tilde{q}_j| \le \rho$ for all j, then for $\beta = \{z_0, \ldots, z_0\}$ with $\#(\beta) = m \le n$, we have*

$$|r[\beta]| \le \binom{n}{n+1-m} \rho^{n+1-m} \tag{2}$$

Conversely if (2) holds for all $\#(\beta) \ne 0$, then we have

$$|z_0 - \tilde{q}_j| \le \frac{1}{\sqrt[n]{2} - 1}\rho \qquad (j = 1, 2, \ldots, n).$$

2. Linear perturbations

In this section we will show that all linear perturbations of polynomials that introduce bounded perturbations into the roots of $p(z)$ are some linear combinations of the derivatives of $p(z)$. Let us denote by H_n the linear space of all polynomials of degree $\le n$ with complex coefficients. The following estimates are well known results, see Taylor [12] for polynomials of several variables.

Lemma 2.1. Let $p(z)$ be a polynomial of degree n and let $d(z) = \operatorname{dist}(z, V)$, the distance from $z \in \mathbb{C}$ to $V = \{z \in \mathbb{C} : p(z) = 0\}$. Then there exist constants $C_1(n)$ and $C_2(n)$ depending only on n such that

$$C_1(n) \le d(z) \sum_{k=1}^{n} \left| \frac{p^{(k)}(z)}{p(z)} \right|^{1/k} \le C_2(n). \tag{3}$$

Proof. We have $p(z+\zeta) - p(z) = \sum_{k=1}^{n} \frac{\zeta^k}{k!} p^{(k)}(z)$. Hence

$$\frac{p(z+\zeta)}{p(z)} - 1 = \sum_{k=1}^{n} \frac{\zeta^k}{k!} \frac{p^{(k)}(z)}{p(z)}. \tag{4}$$

Pick $C_1(n)$ such that $\sum_{k=1}^{n} \frac{C_1(n)^k}{k!} < 1$. If $|\zeta| < C_1(n) \min_k \left| \frac{p^{(k)}(z)}{p(z)} \right|^{-1/k}$ with fixed z. Then (4) implies $p(z+\zeta) \neq 0$. Hence

$$d(z) \geq C_1(n) \min_k \left| \frac{p^{(k)}(z)}{p(z)} \right|^{-1/k}$$

which proves the first part of (3).

For the second part, let $|\zeta| \leq d(z)$. Then if $g(t) = p(z + t\zeta)$ $(t \in \mathbb{R})$, we have

$$\left| \frac{p(z+\zeta)}{p(z)} \right| = \left| \frac{g(1)}{g(0)} \right| = \prod_k \left| \frac{1-t_k}{t_k} \right| \leq 2^n$$

where t_k are the roots of $g(t)$ such that $t_k \geq 1$.

Thus $|p(z+\zeta)| \leq 2^n |p(z)|$ if $|\zeta| \leq d(z)$. By Cauchy's integral formula, we get

$$|p^{(k)}(z)| \leq \frac{\tilde{C}_2(n) 2^n |p(z)|}{d(z)^k}$$

which completes the proof. □

Theorem 2.2. *Let $T : H_n \to H_{n-1}$ be a linear operator and $\deg(p(z)) = n$. If the roots $\tilde{q}_1, \ldots, \tilde{q}_n$ of $p + Tp$ are indexed so that $|q_j - \tilde{q}_j| \leq \rho$, $j = 1, \ldots, n$, then*

$$T = \sum_{k=1}^{n} \alpha_k (d/dz)^k, \quad |\alpha_k| \leq \frac{\rho^k}{k!} \ (k = 1, \ldots, n). \tag{5}$$

Conversely, if T has the above form (5), then for any $p(z)$ with $\deg(p(z)) = n$ the roots $\tilde{q}_1, \ldots, \tilde{q}_n$ of $p + Tp$ are indexed so that

$$|q_j - \tilde{q}_j| \leq C(n)\rho, \quad j = 1, \ldots, n, \tag{6}$$

where $C(n)$ depends only on n.

Proof. It was known (*cf.* Tulovsky [14]) that

$$T = \sum_{k=1}^{n} \alpha_k (d/dz)^k, \quad |\alpha_k| \leq C\rho^k \ (k = 1, \ldots, n)$$

for some constant C, and moreover the coefficients α_k do not depend on polynomials $p(z)$ with $\deg(p(z)) = n$.

In order to get more precise estimates for $|\alpha_k|$, let us choose $p(z) = z^n$. Then we get

$$z^n + T(z^n) = (z - \tilde{q}_1) \cdots (z - \tilde{q}_n) = z^n + \sum_{k=1}^{n} \alpha_k (d/dz)^k (z^n).$$

From this equality we will get the following relation

$$\alpha_k n(n-1) \cdots (n-k+1) = \sum_{i_1 < \cdots < i_k} (-1)^k \tilde{q}_{i_1} \cdots \tilde{q}_{i_k}$$

where $\{i_1, i_2, \ldots, i_k\}$ is a subset of $\{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_k$.

Now taking into account that $|\tilde{q}_j| \le \rho$, we obtain

$$|\alpha_k| \, n(n-1) \cdots (n-k+1) \le \frac{n!}{k!(n-k)!} \rho^k$$

which gives

$$|\alpha_k| \le \frac{\rho^k}{(k)!}, \; k = 1, \ldots, n.$$

To prove the converse, let $d = \min_j |z - q_j|$. Then there exist constants $C_1(n)$ and $C_2(n)$ depending only on n such that

$$C_1(n) \le d \sum_{k=1}^{n} \left| \frac{p^{(k)}(z)}{p(z)} \right|^{1/k} \le C_2(n). \tag{7}$$

As the proof of (ii) in Theorem 1.2 (see Park [11, pp. 73–79]), we will use Rouche's theorem and

$$G = \bigcup_{j=1}^{n} B(q_j, k\rho)$$

with boundary Γ to find $C(n)$ depending only on n. If $z \in \Gamma$ then, by (7) and $|\alpha_j| \le \frac{\rho^j}{(jk)!}$, we get

$$|Tp(z)| \le \sum_{j=1}^{n} |\alpha_j| \, |p^{(j)}(z)| \le |p(z)| \sum_{j=1}^{n} C_2^j(n) k^{-j} < |p(z)|.$$

From Rouche's Theorem, we can see (6) holds for $C(n)$ depending only on n. □

For any $p(z)$ with degree n, the set $\{(d/dz)^k p : k = n-m, n-m+1, \ldots, n\}$ forms a basis for H_m when $m < n$. Therefore we can obtain the following result.

Corollary 2.3. *Let $T : H_n \to H_m$ be a linear operator with $m < n$. If for any $p(z)$ with $\deg(p(z)) = n$, the roots $\tilde{q}_1, \ldots, \tilde{q}_n$ of $p + Tp$ are indexed so that $|q_j - \tilde{q}_j| \le \rho$,*

$j = 1, \dots, n$, *then*

$$T = \sum_{j=n-m}^{n} \alpha_k (d/dz)^j, \quad |\alpha_k| \le \frac{\rho^j}{j!}, \ j = n-m, \dots, n.$$

3. Absolute root bound functionals

We will now show how the new formula for the divided differences leads to an absolute root bound functional that in some sense gives better estimates than the classical absolute root bound functionals. First of all, we will give some definitions and well known results in this area.

Definition 3.1. For $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ with roots $q_1, \dots, q_n$, we define $U(p)$ by

$$U(p) = \max_j |q_j|.$$

Ξ will denote the class of monic complex polynomials of degree n. A *root bound* for $p(z)$ will be a real number m such that $m \ge U(p)$. A *root-bound* (**rb**) *functional* on Ξ will be a real functional $M : \Xi \to \mathbb{R}$ such that

$$M(p) \ge U(p) \quad \text{for all } p(z) \in \Xi.$$

A **rb** functional M on Ξ such that $M(p) = M(\tilde{p})$ whenever

$$\begin{cases} p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0 \\ \tilde{p}(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0 \end{cases}$$

with $|c_j| = |b_j|$ $(0 \le j \le n-1)$ is called an *absolute* **rb** functional on Ξ.

Definition 3.2. For $p(z) \in \Xi$ and $r > 0$, we will denote the polynomial defined by $p^r(z) = r^n p(z/r)$. A **rb** functional $M : \Xi \to R$ is called *homogeneous* if $M(p^r(z)) = rM(p(z))$ for $p(z) \in \Xi$ and $r > 0$.

Definition 3.3. M is called *normal* if M is a continuous **rb** functional and $M(p^r(z))$ is an increasing function of $r > 0$ for which $M(p^r(z)) > \inf(M(\Xi))$.

$$\text{Mof}_M(t) = \frac{t}{\inf\{U(p) : p(z) \in \Xi, M(p) = t\}}$$

is called the *maximum over estimation factor.* Note that if M is a homogeneous **rb** functional on Ξ, then $\text{Mof}_M(t)$ is independent of t. In this case we will write Mof_M.

Remark 3.4. Now, some results on this field will be presented without proof. For the proofs and references, see van der Sluis [15].

(a) If $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$. Then, by Cauchy's Theorem (*cf.* Park [9]), the unique positive root z_0 of

$$z^n - |b_{n-1}|z^{n-1} - \cdots - |b_1|z - |b_0| = 0$$

is an absolute **rb** functional. For any $p(z)(\neq z^n) \in \Xi$, we denote the corresponding z_0 as $B(p) = z_0$ and also define $B(z^n) = 0$, then B is the best absolute **rb** functional of all absolute **rb** functionals. While B is optimal, the positive root z_0 of the equation $z^n - |b_{n-1}|z^{n-1} - \cdots - |b_1|z - |b_0| = 0$ cannot be easily calculated.

(b) Let

$$S(p) = 2\max\left\{|b_{n-1}|, \sqrt{|b_{n-2}|}, \ldots, \sqrt[n-1]{|b_1|}, \sqrt[n]{\frac{|b_0|}{2}}\right\}.$$

Van der Sluis [15] showed that for the absolute **rb** functional S, $S(p) \leq 2B(p)$ for all $p(z) \in \Xi$ and hence S is nearly optimal among all absolute **rb** functionals which are well-known from the literature.

Lemma 3.5. Let $B : \Xi \to \mathbb{R}$ be the absolute **rb** functional defined in (a) of Remark 3.4. Then the followings are hold.

(i) For any normal absolute **rb** functional M, we obtain $\mathrm{Mof}_M(t) \geq \mathrm{Mof}_B(t)$ for the best absolute **rb** functional $B(p)$.
(ii) $\mathrm{Mof}_B = \frac{1}{\sqrt[n]{2}-1} \approx \frac{n}{ln2} \approx 1.4n, B(p)$ is homogeneous normal.
(iii) $\mathrm{Mof}_S = 2n, S(p)$ is homogeneous normal.

Lemma 3.6. For any absolute **rb** functional M on Ξ and any

$$p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0 \neq z^n,$$

there exists a sequence of non-negative numbers $\alpha_1, \ldots, \alpha_n$ with $\sum_{j=1}^n \alpha_j \leq 1$ such that

$$M(p) = \max{}' \left\{\frac{|b_{n-j}|}{\alpha_j}\right\}^{1/j},$$

where the prime indicates that elements with $b_{n-j} = 0$ are skipped for determination of the maximum.

Conversely, for each n-tuple (α_j) of non-negative numbers with $\sum_{j=1}^n \alpha_j \leq 1$,

$$\max{}' \left\{\frac{|b_{n-j}|}{\alpha_j}\right\}^{1/j}$$

gives an absolute **rb** functional on Ξ.

Let $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ with roots $q_1, \ldots, q_n$. From Theorem 1.2, if $|b_{n-j}| \le \binom{n}{j}\rho^j$ for all j, then we get the estimate $|q_j| \le \frac{\rho}{\sqrt[n]{2}-1}$, for all j. Note that $z^m p(z) = z^m(z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0)$ has the same roots as $p(z)$ except 0. So,

$$|b_{n-j}| \le \binom{n+m}{j}\rho^j.$$

and

$$\sqrt[j]{\frac{j!\,|b_{n-j}|}{(n+m)(n+m-1)\cdots(n+m-j+1)}} \le \rho.$$

Set $t = n + m$. Then

$$\frac{\rho}{\sqrt[t]{2}-1} \ge \frac{\sqrt[j]{j!\,|b_{n-j}|}}{(\sqrt[t]{2}-1)\sqrt[j]{t(t-1)\cdots(t-j+1)}}.$$

By using L'hôpital's rule, we conclude that all roots of $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ lie in

$$|z| \le \frac{1}{\ln 2}\max_{1\le j\le n}\sqrt[j]{j!\,|b_{n-j}|}.$$

Now the above inequality, Remark 3.4 and Lemmas 3.5–3.6 suggest a simpler absolute **rb** functional. Namely, take α_k as follows; $\alpha_1 = \ln 2$, $\alpha_2 = \frac{(\ln 2)^2}{2}$, $\alpha_j = \left(\frac{\ln 2}{2}\right)^j$ for $j \ge 3$. Then we can see that $\sum_{k=1}^{\infty}\alpha_k < 1$. Therefore we have the following result.

Theorem 3.7. *For $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$, if $I : \Xi \to R$ is defined by*

$$I(p) = \frac{1}{\ln 2}\max\left\{|b_{n-1}|,\ \sqrt{2|b_{n-2}|},\ 2\sqrt[3]{|b_{n-3}|}, \ldots,\ 2\sqrt[n]{|b_0|}\right\},$$

then I is an absolute **rb** *functional.*

Now we are going to show that our absolute **rb** functional I gives much better estimate than the nearly optimal S in the sense of the maximum over estimation factor. Applying Lemmas 3.5, we obtain the following results.

Theorem 3.8. *For $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$, if $I : \Xi \to R$ is defined by*

$$I(p) = \frac{1}{\ln 2}\max\left\{|b_{n-1}|, \sqrt{2|b_{n-2}|}, 2\sqrt[3]{|b_{n-3}|}, \ldots, 2\sqrt[n]{|b_0|}\right\},$$

then we have the following properties.

1) *$I(p)$ is homogeneous normal.*
2) $\mathrm{Mof}_I = \frac{n}{\ln 2} \approx \mathrm{Mof}_B$ *for $n \le 11$.*

3) $\mathrm{Mof}_I = \frac{2}{\ln 2}\sqrt[3]{\binom{n}{3}} \approx 1.59n$ *for* $n \leq 12$.

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