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AN APPLICATION OF THE STRING AVERAGING METHOD TO ONE-SIDED BEST SIMULTANEOUS APPROXIMATION

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ABSTRACT. For $t = 1, 2, \ldots, \ell$, let $I_t = (i_1^t, i_2^t, \ldots, i_{\ell(t)}^t)$ be an ordered $\ell(t)$-tuple of numbers in $\{1, 2, \ldots, \ell\}$ and let T_t be chosen from a finite composition of orthogonal projections

$$R_{i_1^t}, R_{i_2^t}, \ldots, R_{i_{\ell(t)}^t}$$

acting on the normed linear space $C_1(X)$ to closed convex subset $S(f_{i_j^t})$ respectively. In this paper, we study the convergence of the sequence

$$x_i = \sum_{t=1}^{\ell} w_t T_t(x_{i-1}), \quad i = 1, 2, \ldots,$$

where $\sum_{i=1}^{\ell} w_i = 1$ and $w_i > 0$ for $i = 1, \ldots, \ell$.

0. INTRODUCTION

This paper is concerned with one-sided best simultaneous approximation on

$$C(X) = \{f \mid f \text{ is a real valued continuous function on } X\},$$

where X is a compact subset of $\mathbb{R}^N$. We denote that $C_1(X) = (C(X), \|\cdot\|_1)$ with the $L_1(X, \mu)$-norm where μ is a finite positive admissible measure defined on X. Then $C_1(X)$ is not a Banach space and it is a dense linear subspace of $L_1(X, \mu)$.

We define a norm on the space of all ℓ-tuples of functions in $C(X)$ as follows: For any ℓ elements $f_1, \ldots, f_\ell$ in $C(X)$, let $F = (f_1, \ldots, f_\ell)$ and

$$\|F\| = \|(f_1, \ldots, f_\ell)\| = \max_{(w_1, \ldots, w_\ell) \in A} \left\| \sum_{i=1}^{\ell} w_i f_i \right\|_1$$

where $A = \{(w_1, \ldots, w_\ell) \mid \sum_{i=1}^{\ell} w_i = 1 \text{ and } w_i > 0 \text{ for } i = 1, \ldots, \ell\}$.

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Now suppose that an ℓ-tuple $F = (f_1, \ldots, f_\ell)$ of functions in $C(X)$ is given and S is a finitely generated subspace of $C(X)$. We define the sets, for $i = 1, \ldots, \ell$,

$$S(f_i) := \{s \in S \mid s(x) \leq f_i(x) \text{ for } x \in X\}$$

and the set

$$S(F) : = \bigcap_{i=1}^{\ell} S(f_i) : = \bigcap_{i=1}^{\ell} \{s \in S \mid s(x) \leq f_i(x) \text{ for } x \in X\}$$

in normed linear space $C_1(X)$. By the definitions of $S(f_i)$ and $S(F)$ above, it is trivial to show that $S(F)$ is non-empty for every ℓ-tuple F of functions in $C(X)$ if S contains a strictly positive function. Throughout this paper we shall restrict to those F for which $S(F)$ is non-empty.

In this paper, we use a new algorithmic scheme for finding one-sided best approximation of an ℓ-tuple F from a set $S(F)$. The new iterative algorithm (the string averaging method of Censor, Elfving & Herman [1] in approximation theory) has been used for solving the convex feasibility problem of finding a point x in the nonempty intersection $C = \bigcap_{i=1}^{m} C_i$ of finitely many closed and convex sets C_i in the Euclidean space $\mathbb{R}^N$.

1. One-sided best approximation

If $f^* \in S(F)$, we define

$$r(F, f^*) : = \sup_{g \in F} \|f^* - g\|.$$

The *Chebyshev radius* of F is defined by

$$\text{rad}_{S(F)}(F) : = \inf_{f^* \in S(F)} r(F, f^*).$$

An element $f^* \in S(F)$ satisfying $r(F, f^*) = \text{rad}_{S(F)}(F)$ is called a *one-sided best simultaneous L_1-approximation* of F (*cf.* Park & Rhee [7]). If $\ell = 1$ (*i. e.*, $F = \{f\}$ is singleton), it is called a *one-sided best L_1-approximation* (*cf.* Pinkus [8] and Park & Rhee [7]).

Definition 1. Let V be a nonempty subset of a normed linear space Y and $\mathbf{F}$ be a given family of closed and bounded subsets of Y.

A *minimizing sequence* $\{f_n\}$ of F in V is a sequence $\{f_n\}$ such that $r(F, f_n) \to \text{rad}_V(F)$ has a subsequence which is convergent in V.

The set V is said to be *cent-compact relative to* $\mathbf{F}$ if, for each $F \in \mathbf{F}$, each minimizing sequence $\{f_n\}$ in V.

We say that the set V satisfies the *center property* for $\mathbf{F}$ if, for each $F \in \mathbf{F}$, there exists a one-sided best simultaneous L_1-approximation of F.

Remark 1. For each ℓ-tuple $F = (f_1, \ldots, f_\ell)$ of functions in $C_1(X)$, a minimizing sequence in $S(F)$ is bounded, so the sequence has a subsequence which is convergent in $S(F)$. Thus we can say that $S(F)$ is *cent-compact relative to* F.

Mhaskar & Pai [6] has showed that if a closed set V in a normed space Y is *cent-compact relative to* Y, then V satisfies *center property* for each family of closed and bounded sets.

Consider a case when $\mathbf{F} = \{F\}$ is singleton and F is an ℓ-tuple of functions in $C_1(X)$. Then $S(F)$ is center-compact relative to $\mathbf{F}$. Thus we have the following proposition.

Proposition 2. *Let F is any ℓ-tuple of functions in of $C_1(X)$. Then $S(F)$ satisfies center property for $\mathbf{F} = \{F\}$.*

2. AN ALGORITHM FOR ORTHOGONAL PROJECTIONS

Projection algorithmic schemes for the convex feasibility problem and for the best approximation problem are, in general, either sequential or simultaneous or block-iterative. In the following, we explain these terms in the framework of the algorithmic scheme proposed in this paper.

For each $t = 1, 2, \ldots, \ell$, let the *string* I_t be an ordered $\ell(t)$-tuple of numbers in $\{1, 2, \ldots, \ell\}$ of the form

$$I_t = (i_1^t, i_2^t, \ldots, i_{\ell(t)}^t).$$

We will assume that for any t, the components i_j^t $(j = 1, 2, \ldots, \ell(t))$ of I_t are distinct from each other and every element of $\{1, 2, \ldots, \ell\}$ appears in at least one of the strings $I_t (t = 1, 2, \ldots, \ell)$.

Algorithmic Scheme. Let $w_t > 0$ (for $t = 1, 2, \ldots, \ell$), $\sum_{t=1}^{\ell} w_t = 1$ and $R_{i_j^t}$ be *the metric projection onto* $S(f_{i_j^t})$. We use iterative steps:

Given $F_0 = (f_1, \ldots, f_\ell)$,

(1) calculate, for all $t = 1, 2, \ldots, \ell$,

$$T_t f_t = R_{i_1^t} R_{i_2^t} \cdots R_{i_{\ell(t)}^t} f_t,$$

(2) and then calculate $\quad F_1 = \sum_{t=1}^{\ell} w_t T_t f_t;$

given the current F_k $(k \geq 2)$, iterate

(3) calculate, for all $t = 1, 2, \ldots, \ell$,

$$T_t F_k = R_{i_1^t} R_{i_2^t} \cdots R_{i_{\ell(t)}^t} F_k,$$

(4) and then calculate $\quad F_{k+1} = \sum_t w_t T_t F_k.$

We demonstrate our algorithmic scheme. For simplicity, we take $\ell = 1$. Then we get a one-sided L_1-approximation (*cf.* Park & Rhee [7]).

In this framework, we get a simultaneous algorithm by the choice

$$I_t = (t), \quad t = 1, 2, \ldots, \ell,$$

and a sequential algorithm by the choice

$$I_t = (1, 2, \ldots, \ell).$$

For some case, the sequence $\{F_k\}$, generated by the algorithmic scheme, converges to a point $F^* \in S(F_0)$, which is a one-sided best simultaneous L_1-approximation of F_0 in $S(F_0)$.

Now we turn to questions of when the sequence $\{F_k\}$ converges to some element in $S(F_0)$ and when the sequence converges to a one-sided best simultaneous L_1-approximation to $F_0 = (f_1, \ldots, f_\ell)$.

The subspace S is called a *one-sided simultaneous L_1-unicity space* if for each $F = (f_1, \ldots, f_\ell)$, there is a unique one-sided best simultaneous L_1-approximation (*cf.* Park & Rhee [7]).

The next example satisfies that the sequence $\{F_k\}$ converges to a one-sided best simultaneous approximation of $F = (f_1, f_2)$.

Example 3. Let $F_0 = (\sin x, \cos x), X = [0, \pi]$ and $S = \mathbb{R}$ (*i. e.*, S consists of all constant functions). Then S is a one-sided simultaneous L_1-unicity space for $C_1(X)$ and

$$S(\sin x) = (-\infty, 0], \; S(\cos x) = (-\infty, -1].$$

So $S(F_0) = (-\infty, -1]$.

Let $I_t = (t)$; w_1, w_2 are positive and $w_1 + w_2 = 1$. By Algorithm Scheme, we have

$$\begin{aligned} F_1 &= -w_2, \\ F_2 &= w_1(F_1) - w_2 = w_1(-w_2) - w_2 = -w_2(1 + w_1), \\ F_3 &= w_1(F_2) - w_2 = w_1(-w_2(1 + w_1)) - w_2 = -w_2(1 + w_1 + w_1^2) \end{aligned}$$

and therefore we have

$$F_n = w_1(F_{n-1}) - w_2 = -w_2(1 + w_1 + w_1^2 + \cdots + w_1^{n-1})$$

for all $n \geq 1$. Then the sequence F_n converges to $F^* = -1 \in S(F_0)$ and F^* is a one-sided best simultaneous approximation of F_0 from $S(F_0)$.

3. Proofs of convergence

We now turn to prove our results concerning the convergence of $\{F_k\}$, generated by algorithmic scheme.

The framework in the previous section does not introduce relaxation parameters into the algorithm. However, according to Censor & Reich [3], we will do so for the special case of orthogonal projections.

We define, for $i = 1, 2, \ldots, \ell$, the algorithmic operators

$$R_i x = x + \alpha_i(P_{S(f_i)}x - x) \text{ for } x \in \mathbb{R}^\ell$$

where α_i $(0 < \alpha_i < 2)$ are fixed for each set $S(f_i)$. The algorithmic operators R_i are called *relaxed orthogonal projections.*

Definition 2. An operator T is said to be *strictly nonexpansive*, if

$$\|T(x) - T(y)\| < \|x - y\| \text{ or } T(x) - T(y) = x - y$$

for all x and y.

Definition 3. A continuous operator T is said to be *paracontracting*, if for any x and any fixed point y (*i. e.*, $y = T(y)$),

$$\|T(x) - y\| < \|x - y\| \text{ or } T(x) = x.$$

Remark 4. Obviously, a strictly nonexpansive operator is paracontracting. But the inverse implication does not hold (*cf.* Elsner, Koltracht & Neumann [4, Example 1]).

Next theorem shows that if ℓ-tuple $F_0 = (f_1, \dots, f_\ell)$ consist of constant functions then the sequence $\{F_k\}$ converges to some element in $S(F_0)$.

Theorem 5 (Censor, Elfving & Herman [1]). *Let F_0 be an ℓ-tuple of elements in $\mathbb{R}$ and S be a closed convex set in $\mathbb{R}^\ell$. Then the sequence $\{F_k\}$, generated by the algorithmic scheme for the relaxed orthogonal projection, converges to a point $F^* \in S(F_0)$.*

Corollary 6. *Let F_0 be an ℓ-tuple of elements in $\mathbb{R}$ and $S = \mathbb{R}$. Then the sequence $\{F_k\}$, in the above theorem, converges to a point $F^* \in S(F_0)$.*

By the definition, F^* is a one-sided best simultaneous approximation for $F_0 = (f_1, \dots, f_\ell)$ if and only if F^* satisfies that $F^* \in S(F_0)$ attains the supremum in $\sup_{f \in S(F_0)} \int_X f\, d\mu$. Obviously, if $\sup_{f \in S(F_0)} \int_X f\, d\mu = 0$ for all $f \in S(F_0)$, then S is not a one-sided simultaneous L_1-unicity space for $C_1(X)$. Equivalently, if

$$\dim S = 1, \quad \sup_{f \in S(F_0)} \int_X f\, d\mu \neq 0 \quad \text{for some } f \in S$$

and $S(F_0) \neq \varnothing$ for any $F_0 = (f_1, \dots, f_\ell)$, then S is a one-sided simultaneous L_1-unicity space for $C_1(X)$.

Proposition 7. *Suppose that S is the set of all constant functions and let $\ell \geq 2$ and $I_t = (t)$. Then for any ℓ-tuple $F_0 = (f_1, \dots, f_\ell)$ in $C_1(X)$, the sequence $\{F_k\}$, constructed as in the algorithmic scheme with respect to relaxed projection operators, converges to a one-sided best simultaneous approximation of F_0 from $S(F_0)$.*

Proof. Since S is a one-sided simultaneous L_1-unicity space and $S(F_0) = (-\infty, \alpha)$ where

$$\alpha = \min_{i \in \{1,2,\dots,\ell\}} \{\min_{x \in X}(f_i(x))\},$$

we have

$$F_1 = \sum_{i=1}^{\ell} w_i \cdot \min_{x \in X}(f_i(x)).$$

If $F_1 \in S(F_0)$, then the proof is complete. If $F_1 \notin S(F_0)$, (*i. e.*, there exists i_0 such that $F_1 \notin S(f_{i_0})$), then $F_2 \leq F_1$. Inductively, if $F_k \notin S(F_0)$ then $F_{k+1} \leq F_k$. So F_k is decreasing and convergent to a element F^* in $S(F_0)$ and F^* attains the supremum in $\int_X f\, d\mu$ for all $f \in S(F_0)$. Hence, F^* is a one-sided best simultaneous approximation of F_0 from $S(F_0)$. □

4. Continuity of the projection $P_{S(\cdot)}(\cdot)$

For each ℓ-tuples F of $C(X)$, $S(F)$ is a closed and convex subset of $C_1(X)$. The metric projection $P : F \to P_{S(F)}(F)$ is a set valued map the approximating set depends on some ℓ-tuples F.

Now we consider a continuity of the projection $P_{S(\cdot)}(\cdot)$. By Remark 1, $S(F)$ is a cent-compact and convex subset of S. Thus we can show that if the map $F \to S(F)$ is Hausdorff continuous at F_0, then the projection $P_{S(\cdot)}(\cdot)$ is upper semicontinuous at F_0.

Mabizela [5] proved that for each F, $S(F)$ is a m-dimensional subspace of $C_1(X)$ with basis $\{e_1^F, \dots, e_m^F\}$ with some conditions. Additionally if F_n converges to F_0 with $\|e_i^{F_n} - e_i^{F_0}\| \to 0$, then the map $S(\cdot)$ is Hausdorff continuous on F_0.

Theorem 8. *The metric projection $P_{S(\cdot)}(\cdot)$ is upper semicontinuous.*

Proof. It suffices to show that the set

$$P_{S(\cdot)}^{-1}(E) : = \ \{F \mid P_{S(F)}(F) \cap E \neq \varnothing\}$$

is closed in $C_1(X)$ for any closed set E. Let $\{F_n\} \subset P_{S(F)}^{-1}(E)$ be a sequence such that $\{F_n\}$ converge to F_0 with respect to Hausdorff metric, denote that $H(F_n, F_0) \to 0$ as $n \to \infty$. We can pick $u_n \in P_{S(F_n)}(F_n) \cap E$ for all $n \in \mathbb{N}$. Then $r(F_n, u_n) = \mathrm{rad}_{S(F_n)}(F_n)$ and

$$\begin{aligned} |r(F_0, u_n) - \mathrm{rad}_{S(F_0)}(F_0)| &\leq |r(F_0, u_n) - r(F_n, u_n)| + |\,\mathrm{rad}_{S(F_n)}(F_n) - \mathrm{rad}_{S(F_0)}(F_0)| \\ &\leq 2H(F_n, F_0). \end{aligned}$$

Thus, $r(F_0, u_n)$ approach to $\mathrm{rad}_{S(F_0)}(F_0)$. Hence $\{u_n\}$ is bounded and so there is a subsequence $\{u_k\}$ which is converge to $u_0 \in S$. Since the function $u \to r(F_0, u)$ is lower semicontinuous, we obtain

$$r(F_0, u_0) \leq \liminf r(F_0, u_k) = \mathrm{rad}_{S(F_0)}(F_0).$$

Thus $u_0 \in P_{S(F_0)}(F_0) \cap E$, that is, $P_{S(\cdot)}^{-1}(E)$ is closed. □

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