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# CHAOS AND LYAPUNOV EXPONENT

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ABSTRACT. In this paper, we try to approach chaos with numerical method. After investigating nonlinear dynamics (chaos) theory, we introduce Lyapunov exponent as chaos's index. To look into the existence of chaos in 2-dimensional difference equation, we compute Lyapunov exponent and examine the various behaviors of solutions by bifurcation map.

## 1. Introduction

Chaos theory is very important to understand nonlinear equations which represent complicate phenomena of nature. Lorenz [4] represented clearly that the solutions of simple 3-variable nonlinear differential equations exhibited irregular and complicate behaviors depending on initial values. While studying the dynamics of meteorological changes, he found "the sensitivity to initial conditions in chaos" or "butterfly effect" which means chaos.

In deterministic dynamic system, if the initial conditions can be measured with accuracy, we will be able to predict the output exactly. But it is impossible to determine the initial conditions of system accurately. A slight alteration in the initial conditions can result in enormous differences in the output as the amplification of error. We call this the sensitivity to initial condition. By this feature, the behaviors in the system become unpredictable, therefore the system is changed into irregular system. It follows that although one might be able to predict the states of future with reasonable accuracy in the short term, long-term predictions are futile.

Chaos is the phenomenon that one can not predict the future state of the system for sensitivity in intial condition though it is deterministic system. Chaotic system exhibits very complex motion that is not random, but there are no predictable patterns to it. It has the properties such as irregularity of time series,

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non-periodic solution, orbital unstability, finite attractor, impossibility of long-term predictions, etc.

Chaotic motion is so common in natural and scientific phenomena that many scientists say that it should be considered to be the rule, rather than the exception, in the study of natural phenomena. Chaotic motion has been observed in such diverse areas as fluid dynamics, ecology, optics, the dynamics of the heart and the brain, astrophysics, buckling beams, oceanography, and nonlinear electrical circuits. For this reason it is important to gain an understanding of chaotic motion. In chaotic dynamics one can analyze objects subject to an unpredictable, but not random, behavior.

In recent years the topic of chaotic dynamics has become increasingly popular. Applications of chaotic dynamics will extend to disciplines as diverse as weather prediction, orbits of satellites, chemical reactions, and stock market prices. Interested reader might consult Devaney [1, pp. 52–63] for more detailed explanation.

In this paper, we introduce Lyapunov exponent and period-3 points as numerical index of the existence of chaos and then actually look into the existence of chaos in 2-dimensional difference equation.

## 2. Lyapunov Exponent

A difference equation describes functional relation between states according to discrete time. We consider the first-order difference equation:

$$x_{k+1} = f(x_k), \qquad k = 0, 1, \cdots, \tag{2.1}$$

where $x_k \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $f^n$ denote the composition of the function $f$ with itself $n$ times. We call the sequence $\{f^k(x_0)\}_{k=0}^{\infty}$ of iterates of $x_0$ the orbit of $x_0$. Sometimes we will write $x_k$ for $f^k(x_0)$. In that case, $\{x_k\}_{k=0}^{\infty}$ constitutes the orbit of $x_0$ and (2.1) is deterministic dynamical systems for if initial value $x_0$ is determined, all $x_k$ is determined.

But actually, it is impossible to measure initial conditions exactly. Therefore, it follows that amplification of error occurs. It leads to the different result and converts deterministic dynamical system into chaotic system which is unpredictable. We call this phenomenon sensitivity to initial conditions. We define the following notations according to Gulick [2].

**Definition 2.1** (Sensitive dependence on initial conditions). Let $J$ be an interval, and suppose that $f : J \to J$. Then $f$ has *sensitive dependence on initial conditions at* $x$ if there is an $\epsilon > 0$ such that for $\delta > 0$, there is a $y$ in $J$ and a positive integer $n$ such that

$$|x - y| < \delta \text{ and } |F^n(x) - F^n(y)| > \epsilon.$$

If $f$ has sensitive dependence on initial conditions at each $x$ in $J$, we say that $f$ has *sensitive dependence on initial conditions on* $J$, or that $f$ has *sensitive dependence.*

This property is unique for chaotic system. Although the concept of sensitive dependence on initial conditions is easy to visualize, actually determining that a function has sensitive dependence is usually not so simple. In order to understand and utilize nonlinear dynamic system, it is important that we are able to predict in advance whether chaos will occur or not. Accordingly, we investigate Lyapunov exponent and period-3 points as index of the existence of chaos.

Let $J$ be a bounded interval, and consider a function $f : J \to J$ having a continuous derivative. We assume that for each $x$ in the interior of $J$ and each small enough $\epsilon > 0$ there is a number $\lambda(x)$ such that for each positive integer $n$,

$$|f^n(x + \epsilon) - f^n(x)| \approx [e^{\lambda}(x)]^n \epsilon. \tag{2.2}$$

This implies that

$$e^{n\lambda(x)} \approx \left| \frac{f^n(x + \epsilon) - f^n(x)}{\epsilon} \right| \tag{2.3}$$

so that

$$e^{n\lambda(x)} = \lim_{\epsilon \to 0} \left| \frac{f^n(x + \epsilon) - f^n(x)}{\epsilon} \right| = |(f^n)'(x)|. \tag{2.4}$$

If $(f^n)'(x) \neq 0$, then by taking logarithms and dividing by $n$ in (2.4), we obtain

$$\lambda(x) = \frac{1}{n} \ln |(f^n)'(x)|. \tag{2.5}$$

This leads us to make the following definition.

**Definition 2.2** (Lyapunov exponent). Let $J$ be a bounded interval, and $f : J \to J$ continuously differentiable on $J$. Fix $x$ in $J$, and let $\lambda(x)$ be defined by

$$\lambda(x) = \lim_{n - \infty} \frac{1}{n} \ln |(f^n)(x)| \tag{2.6}$$

provided that the limit exists. In that case, $\lambda(x)$ is the *Lyapunov exponent of* $f$ *at* $x$. If $\lambda(x)$ is independent of $x$ wherever $\lambda(x)$ is defined, then the common value of $\lambda(x)$ is denoted by $\lambda$ and is the *Lyapunov exponent* of $f$.

Lyapunov exponent $\lambda$ can be considered to measure unstability of orbit. In other word, we can use it to measure how fast iterates of neighboring points diverge and how predictable their orbits are. If a given function is simple, we can calculate Lyapunov exponent using (2.6). If $\lambda$ is negative, then the iterates of neighboring points remain close together and it means regularity. By contrast, if $\lambda$ is positive, then the iterates of neighboring points separate from one another. Thus the larger $\lambda$ is, the greater the loss of information of iterates. Therefore, we say that a function $f$ is chaotic if $f$ has a positive Lyapunov exponent.

As another method to determine Lyapunov exponent in nonlinear dynamics, Wolf [7] suggests a way using a time series. This method is described in Figure 1.

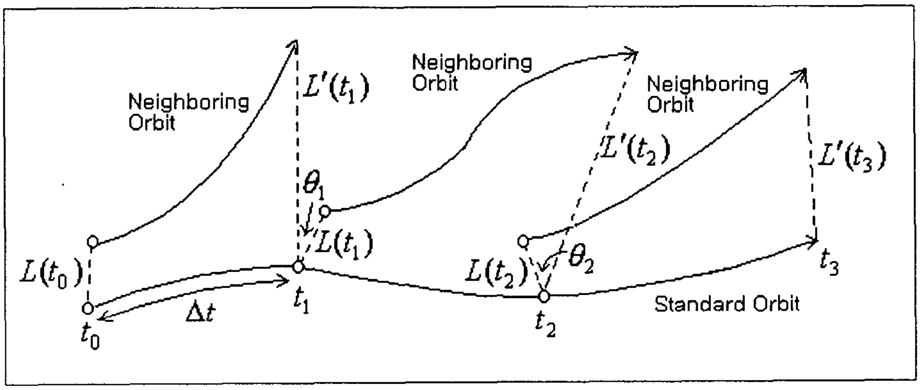


Figure 1. The procedure of determining Lyapunov exponent.

The following five steps are the procedure of computing Lyapunov exponent. First, select the closest point from any point $t_0$ on standard orbit, which is a distance of $L(t_0)$ from $t_0$. Second, let $L'(t_1)$ the distance between a point $t_1$ on standard orbit and neighboring orbit after time *trianglet*. And then calculate the exponential ratio of $L(t_0)$ to $L'(t_1)$. Third, select the closest point which satisfies that $\theta_1$ is minimum at $t_1$ and measure distance $L(t_1)$. Fourth, repeat the second step at $t_2$ after time $\Delta t$ and then calculate the exponential ratio. Fifth, repeat above procedure $M$ times and calculate the average exponential ratio. In time, we have the following definition of Lyapunov exponent $\lambda(x)$.

$$\lambda = \frac{1}{M\Delta t} \sum_{k=1}^{M} \log_2 \frac{L'(t_k)}{L(t_{k-1})} \tag{2.7}$$

where $\Delta t = t_k - t_{k-1}$, and $M$ is repeated times. $L'(t_k)$ and $L(t_{k-1})$ is calculated with Euclidean distance.

One method of displaying the points at which a parameterized family of functions $\{f_\mu\}$ bifurcates is called a *bifurcation map*, and is designed to give information about the behavior of higher iterates of arbitrary members of the domain of $f_\mu$ for all values of the parameter $\mu$. The bifurcation map of $\{f_\mu\}$ is a graph for which the horizontal axis represents values of $\mu$. and the vertical axis represents higher iterates of the variable (normally $x$). For each value of $\mu$, the map includes (in theory) all points of the form $(\mu, f_\mu^n(x))$, for values of $n$ larger than, say, 100 or 200. The reason we only use the higher iterates of $x$ is that the diagram is designed to show eventual behavior of iterates, such as convergence or periodicity or unpredictability.

A basic kind of bifurcations is period-doubling bifurcations at which an attracting period-$n$ cycle becomes repelling and gives birth to an attracting $2n$-cycle. It is a typical pattern where attractor that is not chaotic turns into chaotic attractor. In addition, there are pitchfork bifurcations, flip bifurcations, explosive bifurcations and fold bifurcations.

On the other hand, if there exists a point of period-3 in bifurcation map, we can predict chaos occurs. In bifurcation map where a 3-cycle exists, we can see an uncountable collection of orbits which do not eventually approach any periodic pattern, i.e., chaos.

The following theorem implies that the existence of period-3 points is a sufficient condition for the existence of the chaos in the sense of Li-Yorke [3].

**Theorm 2.1** (Chaos in the sense of Li-Yorke [3]). *Let $J$ be an interval and let $F : J \to J$ be continuous. If there is a point $a \in J$ for which the points $b = F(a)$, $c = F^2(a)$, $d = F^3(a)$ satisfying $d \leq a < b < c$ (or $d \geq a > b > c$) then following properties hold:*

(i) *For every $k = 1, 2, \cdots$ there is a periodic point in $J$ having period $k$.*

(ii) *There is an uncountable set $S \subset J$ (containing no periodic points), which satisfies the following conditions:*

(a) *For every $p, q \in J$ with $p \neq q$,*

$$\lim_{n-\infty} \sup |F^n(p) - F^n(q)| > 0 \text{ and } \lim_{n-\infty} \sup |F^n(p) - F^n(q)| = 0.$$

(b) *For every $p \in S$ and periodic point $q \in J$,*

$$\lim_{n-\infty} \sup |F^n(p) - F^n(q)| > 0.$$

*Proof.* See [3]. □

## 3. Numerical Examples

In this section, we shall now attempt to investigate solution's behaviors where $F : \mathbb{R}^2 \to \mathbb{R}^2$ and check that chaos occurs by computing Lyapunov exponent. We start dividing stable region and unstable region. Next, we investigate solution's behaviors according to parameter in unstable region by describing bifurcation map and then determine Lyapunov exponent to check that chaos occurs. Consider the following two-dimensional difference equation.

$$\begin{cases} x_{k+1} = (ax_k + by_k)(1 - ax_k - by_k) \\ y_{k+1} = x_k \end{cases} \tag{3.1}$$

This problem possesses no special significance, but was selected for investigation since it can be reduced to logistic equation, when $b = 0$, which is one of the most interesting, dynamical systems and is often used to model population dynamics. Since we are primarily interested in only the positive solutions of (3.1), we shall begin by restricting the parameters $a$ and $b$ in the following manner.

Let these parameters lie in the region $R$ of the $(a, b)$-plane described by

$$R = \{(a,b)|\ a \geq 0,\ b \geq 0,\ a + b \leq 4\}.$$

Under these conditions the set $D = \{(x,y)|\ 0 < x \leq \frac{1}{4},\ 0 < y \leq \frac{1}{4}\}$ is invarient under $F$. Let us first examine the qualitative behavior of (3.1) for $(a, b) \in R$.

The local dynamics of difference schemes in a neighborhood of an equilibrium are dependent upon the Jacobian of the function involved. Computing the two fixed points of $F$, we find the trivial one;

$$x_k = y_k = 0$$

and for $a + b > 1$ the positive fixed point;

$$x_k = y_k = \frac{a+b-1}{(a+b)^2}.$$

Also, simple calculation shows that

$$DF(x,y) = \begin{bmatrix} a - 2a(ax+by) & b - 2b(ax-by) \\ 1 & 0 \end{bmatrix}.$$

To compute the eigenvalues $\lambda_1, \lambda_2$ of $F$ at a point $(x, y)$, therefore, we let

$$|DF(x,y) - \lambda I| = 0$$

to obtain

$$\lambda^2 - (a - 2a(ax + by))\lambda - (b - 2b(ax + by)) = 0. \tag{3.2}$$

Evaluating (3.2) at $x = y = 0$, we obtain

$$\lambda^2 - a\lambda - b = 0 \tag{3.3}$$

and we see that for $a + b < 1$, $|\lambda_1|$, $|\lambda_2| < 1$ and thus $(0, 0)$ is stable in the region $R_1 = \{(a, b) |\ a \geq 0,\ b \geq 0,\ a + b \leq 1\}$. However, leaving the region $R_1$ across the line $a + b = 1$, one eigenvalue becomes greater than 1 making $(0, 0)$ unstable.

Now, we consider another fixed point

$$Z = (z, z) = \left(\frac{a + b - 1}{(a + b)^2}, \frac{a + b - 1}{(a + b)^2}\right),$$

whose eigenvalues by (3.3) satisfy

$$\lambda^2 + A\lambda + B = 0 \tag{3.4}$$

where

$$A = \frac{a(a + b - 2)}{(a + b)}, \quad B = \frac{b(a + b - 2)}{(a + b)}. \tag{3.5}$$

Solving (3.4), it is not difficult to check that $Z$ is stable for values of $(a, b)$ close to the line $a + b = 1$. However, moving away from this line, there are two ways in which $Z$ is likely to become unstable;

(i) when both eigenvalues are real and one of them exceeds 1 in norm, while the other remains less, and

(ii) when both eigenvalues, being complex conjugates, have norm greater than 1.

For case (i) we can find the curve in the $(a, b)$-plane along which both eigenvalues are real and one equals 1 in absolute value. Letting the solutions of (3.4) equal $\pm 1$ yields $B \pm A + 1 = 0$. Substituting the values of $A$ and $B$ given by (3.5) implies either $a + b = 1$ or $b^2 - a^2 + 3a - b = 0$. The dynamics across $a + b = 1$ have already been discussed. The latter path, however, separates stability of $Z$ from instability. The behavior across this curve will be discussed below.

For case (ii) note that if the solutions of (3.4) are complex and equal 1 in norm then $B = 1$, and thus the path described by $b^2 + (a - 3)b - a = 0$ also separates stability of $Z$ from instability. Combining this with the result of case (i), we see that $Z$ is locally stable in the sub-region $R_2$ of the region $R = \{(a, b) |\ a \geq 0,\ b \geq 0,\ a + b \leq 4\}$ pictured in Figure 2.

We shall now attempt to look into solution's behaviors according to parameters $a, b$ in unstable regions $R_3$ and $R_4$.

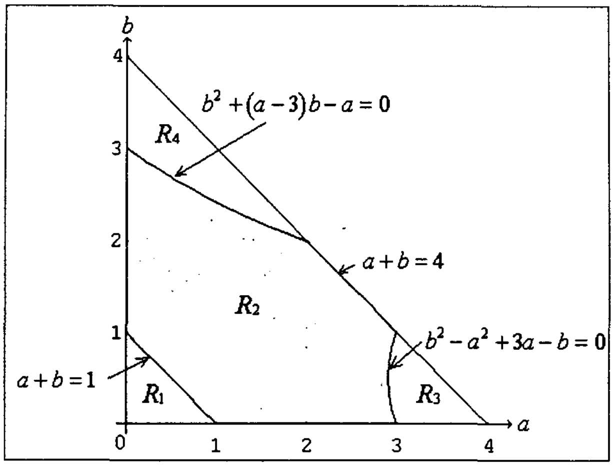


Figure 2. The boundary of stable and unstable region

In order to investigate the dynamics of (3.1) for $(a,b) \in R_3$, we describe bifurcation map and determine Lyapunov exponent to check that chaos occurs. For $(a,b) \in R_3$, we fix $b = 0.25$ and vary parameter values $a$, between $a = 2.9$ and $a = 3.7$ crossing from $R_2$ into $R_3$. Bifurcation map is depicted in Figure 3.

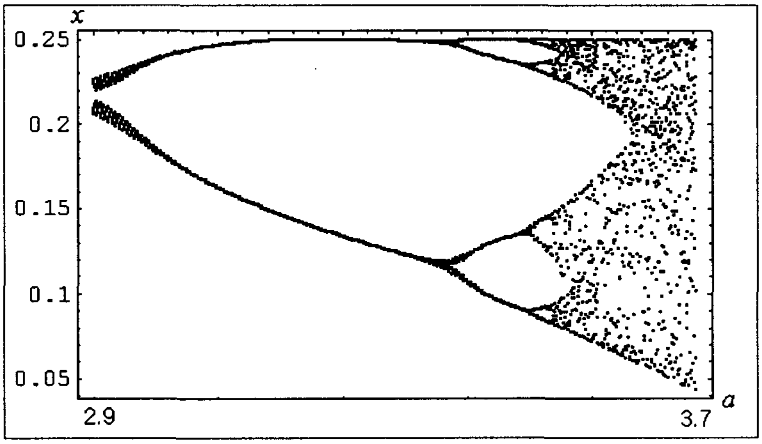


Figure 3. Bifurcation map ($a = 2.9 \sim 3.7$, $b = 0.25$)

In Figure 2 moving further to the right in $R_3$ means that parameter $a$ increase from 2.9 to 3.7 in Figure 3. Moving further to the right in $R_3$, the stable 2-cycle itself becomes unstable and a bifurcation into a stable 4-cycle occurs. Passing in this way through $R_3$, we observe successive bifurcation of $2^k$-cycles. In particular, if $b = 0.25$ and $a = 3.75$, the orbit of solutions exhibits the shape of "strange attractor".

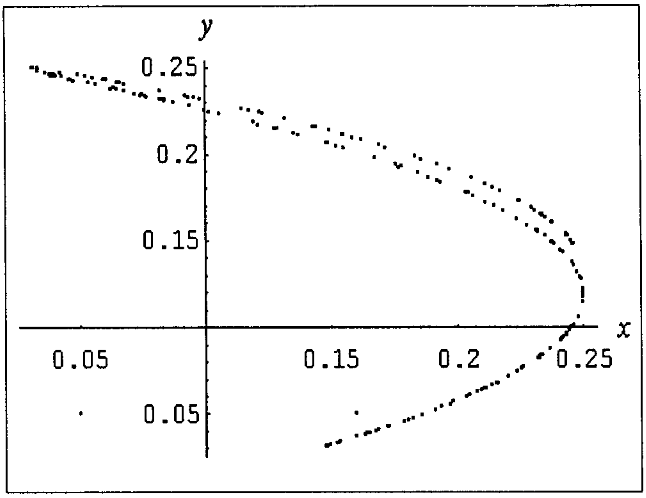


Figure 4. Strange attractor

Next, we compute Lyapunov exponent varying parameter value $a$ with intial condition $x_0 = y_0 = 0.01$. The following table (Table 1) is the program[1] that computes Lyapunov exponent when we fix $b = 0.001$ and increase parameter $a$, from 2.9 to 3.7, by 0.05.

Lyapunov exponent $\lambda$ according to $a$ is described in Figure 5.

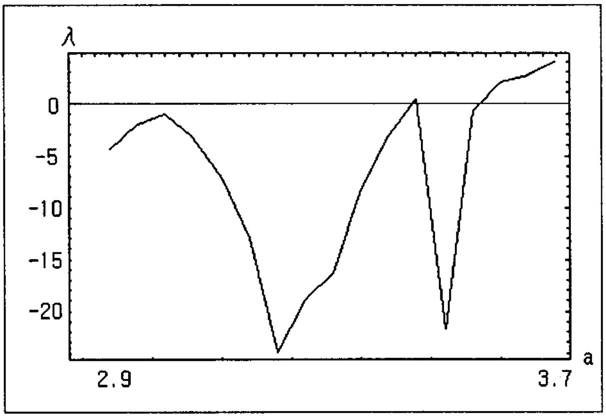


Figure 5. Lyapunov exponent

As parameter values $a$ exceeds 3.55, $\lambda$ becomes positive. So we are able to confirm the existence of chaos in $R_3$.

[1] We have used the Mathematica program for this table and the rest of other figures. Shaw and Tigg [6] was a helpful guiding book for a nonexpert.

Table 1. The program that computes Lyapunov exponent

```
GetLiapunovExp[{x0_,y0_},{a_,b_},eps_,tfrom_,tstep_,opt___] :=
        Module[{x0eps,y0eps,z0eps,lrnz, lrnzeps, pts1,pts2},
                                        x[0] = x0; y[0] = y0;
            lrnz = Table[ {x[i+1],y[i+1]} =
                        {(a x[i] + b y[i]) * (1 - a x[i] - b y[i]), x[i]},
                                        {i,0,tfrom} ];
        {x[0] ,y[0]} = Last[lrnz];
                        lrnz = Table[ {x[i+1],y[i+1]} =
                            {(a x[i] + b y[i]) * (1 - a x[i] - b y[i]), x[i]},
                                    {i,0,tstep} ];


            pts1 = Last[lrnz];
            {x[0] ,y[0]} = {x[0] + eps, y[0] + eps};
            lrnzeps = Table[
                        {x[i+1],y[i+1]} =
                    {(a x[i] + b y[i]) * (1 - a x[i] - by[i]),x[i]},
                                {i,0,tstep} ];
            pts2 = Last[lrnzeps];
            dist1 = Sqrt[2*(eps^2)] /. {0 -> 0.000001, 0.0 -> 0.000001};
    dist2 = Sqrt[ Sum[ (pts1[[i]]-pts2[[i]])^2, {i,1,2} ] ]/.
                {0 -> 0.000001, 0.0 -> 0.000001};
            Return[ Log[dist2/dist1 ] ];
        ];


LiapunovExpList[{x_,y_},a_,b_,eps_,tstep_,imax_] :=
  Module[{ Lep },
            Lep = Table[
        GetLiapunovExp[{x,y},{a,b},eps,i,tstep],
                {i,0,imax,tstep} ];
            Return[Lep];
            ];

LiapunovExpListPlot[{x_,y_},a_,b_,eps_,tstep_,imax_, opt___] :=
  Module[{ Lep },
            Lep = Table[
        GetLiapunovExp[{x,y},{a,b},eps,i,tstep],
                {i,0,imax,tstep} ];
            ListPlot[Lep, opt, PlotJoined -> True];
            Return[Lep];
            ];
```

(continued)

Table 1 (continued)

```
LiapunovExpAvg[{x_,y_},a_,b_,eps_,tstep_,imax_] :=
  Module[{ Lep },
              Lep = Table[
            GetLiapunovExp[{x,y},{a,b},eps,i,tstep],
                  {i,0,imax,tstep} ];
              Return[Apply[Plus,Lep]/Length[Lep]];
              ];
LiapA = Table[ LiapunovExpAvg[{0.01,0.01},a,0.001,0.001,30,100],
                  {a,2.9,3.7,0.05} ];

ListPlot[LiapA, PlotJoined -> True,
          Frame -> True,
          Epilog -> {RGBColor[1,0,0],Line[{{0,0},{100,0}}]} ];
```

From now, we shall look into the dynamics of (3.1) for $(a, b) \in R_4$ pictured in Figure 1. We fix $a = 0.1$ and vary parameter values $b$, between $b = 2.4$ and $b = 3.8$, passing from $R_2$ into $R_4$. The bifurcation map of solutions is described in Figure 6.

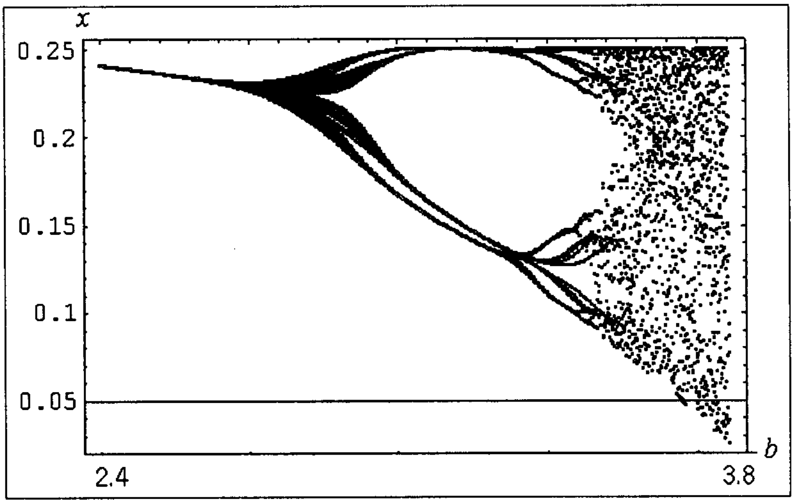


Figure 6. Bifurcation map ($a = 0.1, b = 2.4 \sim 3.8$)

If we look Figure 6 carefully, we can find a point of period-3 and it implies that chaos occurs. As we move deeper into $R_4$, the visual shape of these trajectories changes in the manner plotted in Figures 7~11. At first, the curves and cycles possess well formed circular shapes, but moving further into $R_4$, although still remaining stable, they develop 4-cycle and 8-cycle. Gradually they tend to fill the plane developing $2^k$-cycle as $(a, b)$ moves deeper into $R_4$.

The stability of these curves and cycles vanishes and chaos appears, if $(a, b)$ is moved far enough into $R_4$. The various behaviors of solutions with intial condition $x_0 = y_0 = 0.01$ are described in Table 2.

Table 2. The solution's behaviors according to $a$ and $b$.

| $a$ | $b$ | **Behaviors** | **Graph** |
|---|---|---|---|
| 0.1 | 3.0 | stable continuous curve | Figure 7 |
| 0.1 | 3.25 | 4-cycle | Figure 8 |
| 0.1 | 3.4 | 8-cycle | Figure 9 |
| 0.1 | 3.5 | $2^k$-cycle | Figure 10 |
| 0.1 | 3.9 | chaos | Figure 11 |

Lyapunov exponent computed according to b is described in Figure 12. As we know from Figure 12, Lyapunov exponent $\lambda$ becomes positive as $b$ exceeds 3.56. Therefore we can predict the existence of chaos in unstable region $R_4$.

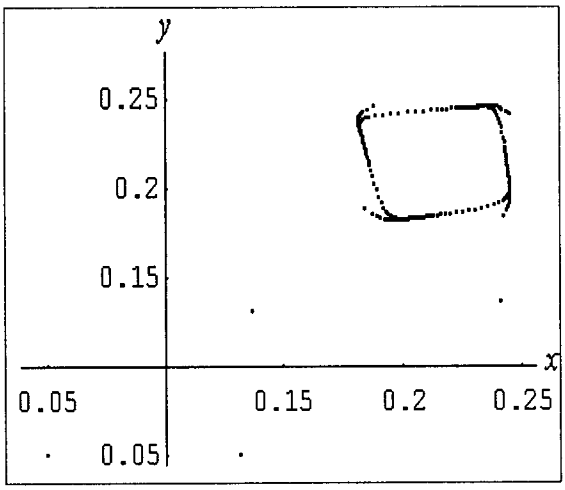


Figure 7. Stable continuous curve

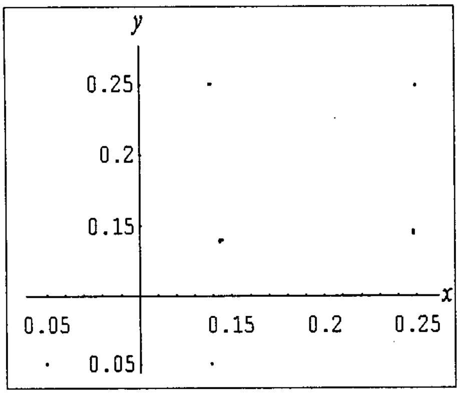


Figure 8. 4-cycle

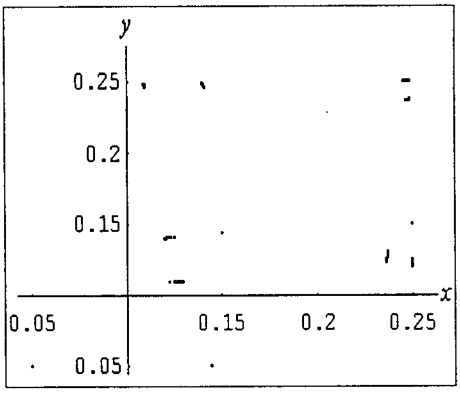


Figure 9. 8-cycle

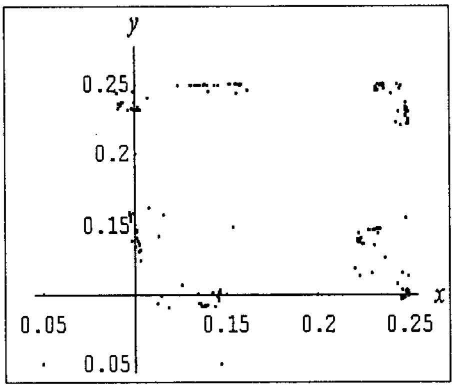


Figure 10. $2^k$-cycle

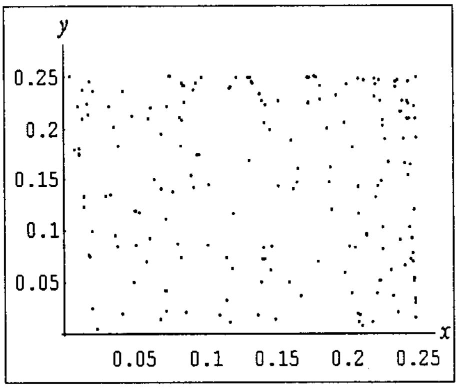


Figure 11. Chaos

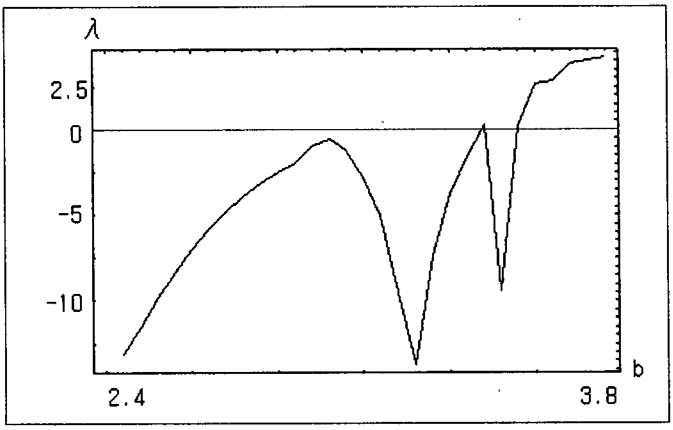


Figure 12. The Lyapunov exponent

## 4. Conclusions

In this paper, we tried to approach chaos using numerical method. Lyapunov exponent are useful in case we can not prove analytically chaos's existence or the problems are given as not specific equation but as many data. Actually, to look into the existence of chaos in 2-dimensional difference equation, we determined Lyapunov exponent and examined behaviors of solutions by bifurcation map. We can see period-3 points in bifurcation map of region where chaos occurs. The existence of period-3 points is a sufficient condition for the existence of chaos. It is a prospective problem for us to investigate the relation between the existence of chaos and the existence of period-3 points in two- or three-dimensional difference equations.

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