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FRACTAL HEDGEHOGS

MAMTA RANI AND VINOD KUMAR

ABSTRACT. The study of fractal hedgehogs is a recent development in the ambit of fractal theory and nonlinear analysis. The intent of this paper is to present a study of fractal hedgehogs along with some of their special constructions. The main result is a new fractal hedgehog theorem. As a consequence, a fractal projective hedgehog theorem of Martinez-Maure is obtained as a special case, and several fractal hedgehogs and similar images are discussed.

1. INTRODUCTION

In 1874, Weierstrass (1815–1897) astounded the mathematical world by publishing an ingenious example of a continuous function, which is nowhere differentiable (see Theorem 1 below). A large number of his mathematical findings became possessions of the mathematical world, not through publication by him, but through notes taken of his lectures. It was in his lecture of 1861 that he first discussed his example of a continuous nondifferentiable function, which was finally published in 1874 (see Eves [3]).

Theorem 1 (Weierstrass Theorem, *cf.* Martinez-Maure [15]). *Let f be a real function of the form*

$$f(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x) \tag{1}$$

where $a \in [0,1]$, b is an odd natural number and $ab > 1 + 3\pi/2$. The function f is continuous everywhere and differential nowhere.

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Note. See Theorem 2 as well. For another example of everywhere continuous and nowhere differentiable function due to van der Waerden, one may refer to Goldberg [4, pp. 256–258], and for a constructive example to Munkers [17, p. 297].

Fractal hedgehogs have been studied by Langevin, Levitt & Rosenberg [6] and Martinez-Maure [7]–[16]. Some fractal hedgehogs look like a Koch curve (*cf.* Peitgen, Jürgen & Saupe [18]). Using the example given in Theorem 1, Martinez-Maure [15] has recently constructed a fractal projective hedgehog (*cf.* Theorem 3) in a plane. In this paper, we generalize Theorem 3 to obtain a fractal hedgehog theorem (*cf.* Theorem 5). Further, we construct various hedgehogs using different parameters of Theorem 5. Surprisingly, one of the constructions is akin to the Indian mythological divine weapon *Sudarśana Cakra* (*cf.* Figure 6) and similar others may be obtained as its variants. Interestingly, one of the invalid constructions looks like *Aśok Cakra.*

2. PRELIMINARIES

Consider a 2π-periodic real function, which is of class C^1 on the real line and defines a curve $C \subset \mathbb{R}^2$ as an envelope of the family of lines. Let S^1 denote the unit sphere of 2-dimensional Euclidean space.

For any $h \in C^1(S^1;\mathbb{R})$, i. e., h is a continuously differentiable function (see Martinez-Maure [15]), let H_h denote the envelope of the family of lines given by

$$x\cos\theta + y\sin\theta = p(\theta), \tag{2}$$

where $p(\theta) = h(\cos\theta, \sin\theta)$. Then H_h is said to be projective when p is a Möbius function, that is a function such that for any θ, $p(\theta+\pi) = -p(\theta)$.

Differentiating (2) partially yields

$$-x\sin\theta + y\cos\theta = p'(\theta). \tag{3}$$

From (2) and (3), the parametric equations for H_h are

$$x = p(\theta)\cos\theta - p'(\theta)\sin\theta \text{ and } y = p(\theta)\sin\theta + p'(\theta)\cos\theta. \tag{4}$$

Following Martinez-Maure [7]–[15], we present an example of a hedgehog. Suppose H_h has a well-defined tangent line at the point (x, y), say T. Then T can be expressed by (2). The unit vector $u(\theta) = (\cos\theta, \sin\theta)$ is normal to T and $p(\theta)$ may be interpreted as the signed distance from the origin to T. Thus a singularity-free plane hedgehog is simply a convex curve (see Figure 1).

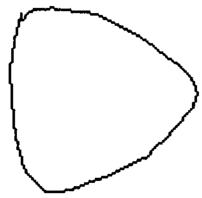

Figure 1. Singularity-free plane hedgehog

For an excellent analysis on the basic concepts of hedgehogs, one may refer to Langevin, Levitt & Roserberg [6] and Martinez-Maure [7]–[16] and references thereof.

3. Fractal hedgehogs

The following result is derived from Theorem 1.

Theorem 2 (Martinez-Maure [15]). *Let p be a real function of the form*

$$p(\theta) = \sum_{n=1}^{+\infty} \left(\frac{1}{\alpha^n}\right) \sin(\beta^n \theta),$$

where β is an odd natural number and α is a real number such that $\alpha > \beta$ and $\beta^2 > \alpha(1 + 3\pi/2)$. The function p is of class C^1 on R but its derivative is nowhere differentiable.

Using the function, $p(\theta)$ in Theorem 2, Martinez-Maure [15] gave the following significant result.

Theorem 3. *There exists a fractal projective hedgehog $H_h \subset \mathbb{R}^2$. More precisely, if*

$$p(\theta) = h(\cos\theta, \sin\theta)$$

is a Möbius function of the form

$$p(\theta) = \sum_{n=1}^{+\infty} \left(\frac{1}{\alpha^n}\right) \sin(\beta^n \theta),$$

where β is an odd natural number and α is a real number such that $\alpha > \beta$ and $\beta^2 > \alpha(1 + 3\pi/2)$ then the hedgehog H_h satisfies the following properties.

(i) *The curve H_h is continuous but nowhere differentiable.*
(ii) *The curve H_h has infinite length.*

First we present an extension of Theorem 2.

Theorem 4. *Let q be a real function of the form*

$$q(\theta) = \sum_{n=1}^{+\infty} \left(\frac{1}{\alpha^n}\right) \sin(\beta^n \theta), \tag{5}$$

where β is an odd integer and α a positive real number such that $\alpha > |\beta|$ and $\beta^2 > \alpha(1 + 3\pi/2)$. Then the function q is of class C^1 on R but its derivative is nowhere differentiable.

Proof. Let $u_n(\theta) = (1/\alpha^n)\sin(\beta^n\theta)$. Then $u_n \in C^1$ on R and

$$u_n'(\theta) = \left(\frac{\beta}{\alpha}\right)^n \cos(\beta^n\theta).$$

By the Weierstrass M-test, $\sum u_n(\theta)$ and $\sum u_n'(\theta)$ are uniformly convergent since $0 < 1/\alpha < 1$ and $0 < |\beta/\alpha| < 1$. Therefore, $q'(\theta)$ is continuous. In view of Theorem 1, $q'(\theta)$ is nowhere differentiable. □

Remark 1. If β is an odd positive integer then evidently Theorem 2 is a special case of Theorem 4.

Remark 2. Thus the derivative of $q(\theta)$ given by

$$q'(\theta) = \sum_{n=1}^{+\infty} \left(\frac{\beta}{\alpha}\right)^n \cos(\beta^n\theta) \tag{6}$$

is nowhere differentiable.

Now we are in position to present the main fractal hedgehog theorem.

Theorem 5. *There exists a fractal hedgehog $H_h \subset \mathbb{R}^2$. More precisely, if $q(\theta) = h(\cos\theta, \sin\theta)$ is a function of the form*

$$q(\theta) = \sum_{n=1}^{+\infty} \left(\frac{1}{\alpha^n}\right) \sin(\beta^n \theta), \tag{7}$$

where β is an odd integer and α a positive real number such that $\alpha > |\beta|$ and $\beta^2 > \alpha(1+3\pi/2)$, then the hedgehog satisfies the properties (i) *and* (ii) *of Theorem* 3.

Proof. The function $q(\theta)$ defined in equation (5) is of class C^1 and, by Theorem 4, $q'(\theta)$ is nowhere differentiable. Therefore the natural parameterization of H_h, viz.,

$$x_h : I = [0, 2\pi] \to H_h \subset E^2, \quad \theta \mapsto \left(x_h^1(\theta), x_h^2(\theta)\right) = q(\theta)u(\theta) + q'(\theta)u'(\theta),$$

where $u(\theta) = (\cos\theta, \sin\theta)$, is continuous and nowhere differentiable. This proves (i) of Theorem 3. □

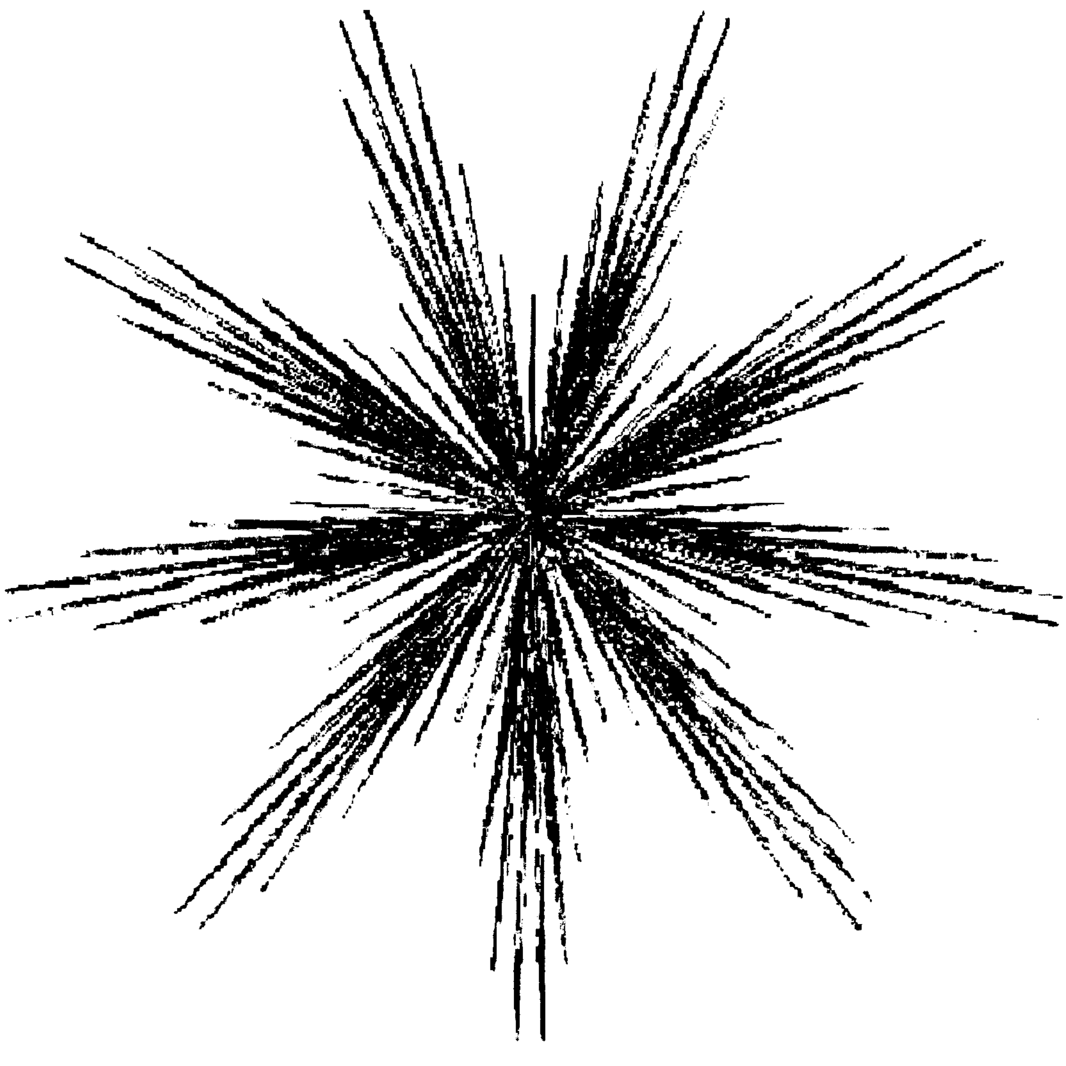

Figure 2. A fractal hedgehog for $(n, \alpha, \beta) = (4, 10, -9)$

Figure 3. Star: A fractal hedgehog for $(n, \alpha, \beta) = (1, 10, -9)$

If (ii) is not true, then let $L(h)$ be the length of H_h. Then

$$L(h) = \sup_P \sum_{j=1}^{n} \|x_h(\theta_j) - x_h(\theta_{j-1})\|,$$

where $P = \{\theta_0, \ldots, \theta_n\}$ is a partition of the closed bounded interval I. This says that $L(h)$ is the total bounded variation of the components of $x_h = (x_h^1, x_h^2)$. (For details of relationship between "bounded variation" and "differentiability", one may refer to Bachman, Narici & Beckenstein [1]). Hence functions x_h^1 and x_h^2 are almost everywhere differentiable on I. This contradicts the fact that $x_h = (x_h^1, x_h^2)$ is nowhere differentiable.

Remark 3. The function q need not be Möbious. Of course, if β is restricted to odd naturals, then q is Möbious, and in this case the hedgehog obtained in Theorem 5 becomes projective. Indeed, in such a situation, Theorem 5 reduces to Theorem 3.

4. Graphical analysis of hedgehogs

The hedgehog obtained in Theorem 5 presents fascinating approximate figures by assigning various numerical values to the parameters n, α and β. Our constructions are based on an algorithm (see Section 5), which is implemented in C++. The fractal hedgehog H_h obtained in Theorem 5 is drawn using (4) with p replaced by

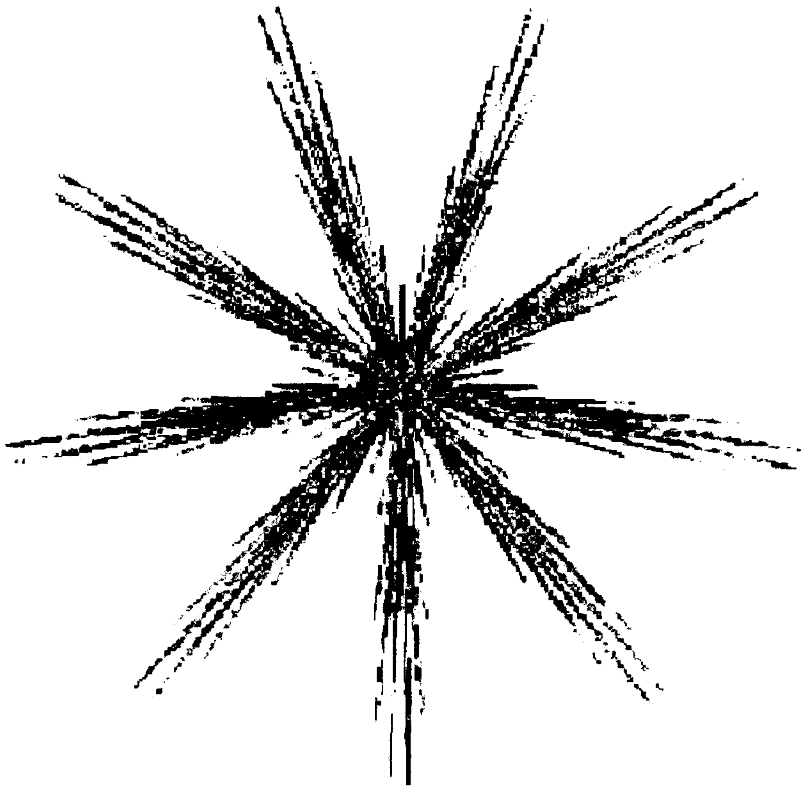

Figure 4. A fractal hedgehog for$(n, \alpha, \beta) = (4, 12, -9)$

q and relations (5) and (6). Needless to say that larger the value of n, better the approximation of H_h. For a somewhat average fair approximation of H_h, a peculiar picture is Figure 2, when $(n, \alpha, \beta) = (4, 10, -9)$.

Remark (on parameter). Here we study each of the three parameters n, α, β of the fractal hedgehog defined in Theorem 5 by graphical analysis.

4.1. Merging of spokes and size factor: n.

Following the algorithm (*cf.* Section 5), we have generated several projective and non-projective hedgehogs. We have studied their pattern and compared them. Only a few of them are included in this paper. Our observations and remarks presented below are solely based on this kind of graphical analysis.

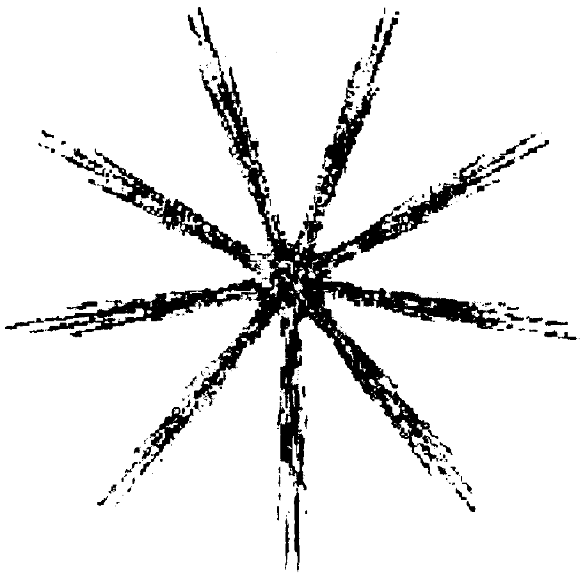

Figure 5. A fractal hedgehog for $(n, \alpha, \beta) = (4, 14, -9)$

As a first significant observation, we find that the parameter n is responsible for the size of a figure and merging of spokes in a sector of a figure. There are nine sectors in each of Figure 2–Figure 5. Each sector consists of several spokes. As the value of n increases, the spokes in each sector come closer to each other. At $n = 1$, the merging in each sector is maximum and a sector now looks like a single spoke. Comparing Figure 2 and Figure 3 for $(n, \alpha, \beta) = (4, 10, -9)$ and $(1, 10, -9)$ respectively, we see that a fractal hedgehog looking like a wild cactus converts into a star by changing the parameter n. Further, it is remarkable to see that greater the value of n, larger the size of the whole figure and its vice versa.

4.2. Scaling factor: α.

The parameter α works purely as a scaling factor for hedgehogs generated for the same value of β. Greater the value of α, smaller the size of figure and its vice-versa. It seems that figures scale down at different α's for the same β. This conclusion is based on the pattern of Figure 2, Figure 4 and Figure 5, which are generated respectively at $\alpha = 10, 12, 14$, when $(n, \beta) = (4, -9)$. Further this fact may be verified for any value of α, of course β satisfying the conditions of Theorem 5.

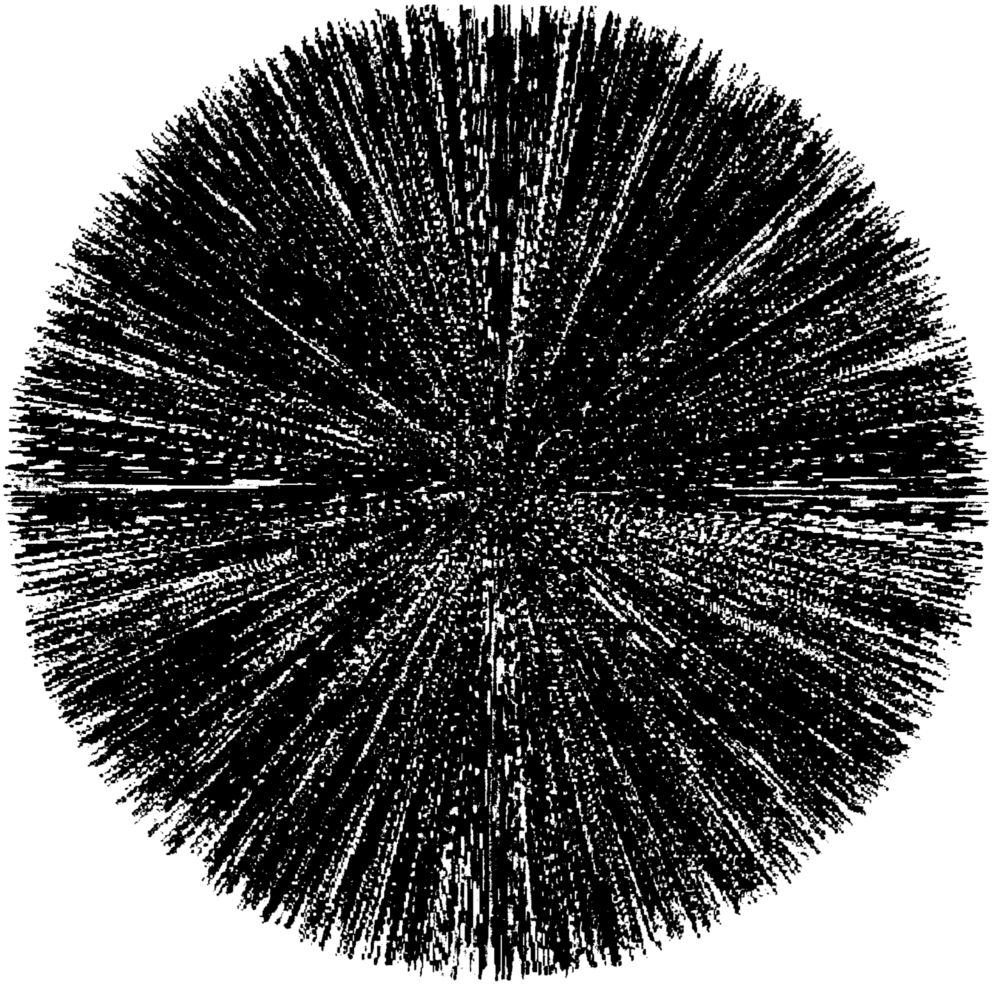

Figure 6. *Sudarśana Cakra* a fractal projective hedgehog for $(n, \alpha, \beta) = (2, 2005, -2001)$

4.3. Number of sectors and size factor: β.

Recall from Theorem 5 that:

(i) β is an odd integer,

(ii) $|\beta| < \alpha$,

(iii) $\beta^2 > \alpha(1 + 3\pi/2)$.

It has been found that the number of sectors in a fractal hedgehog is equal to $|\beta|$. Further, the size of a figure increases as $|\beta|$ increases. Notice that if β is large then the number of sectors will also be large. So, in case β is large, each sector appears to be condensed accordingly.

The parameter β will be called valid if it satisfies the conditions (i)–(iii) and invalid otherwise. We have found that invalid values of β generate nice figures as well. So we now classify the constructions into two categories, viz., those obtained from valid and invalid values of β.

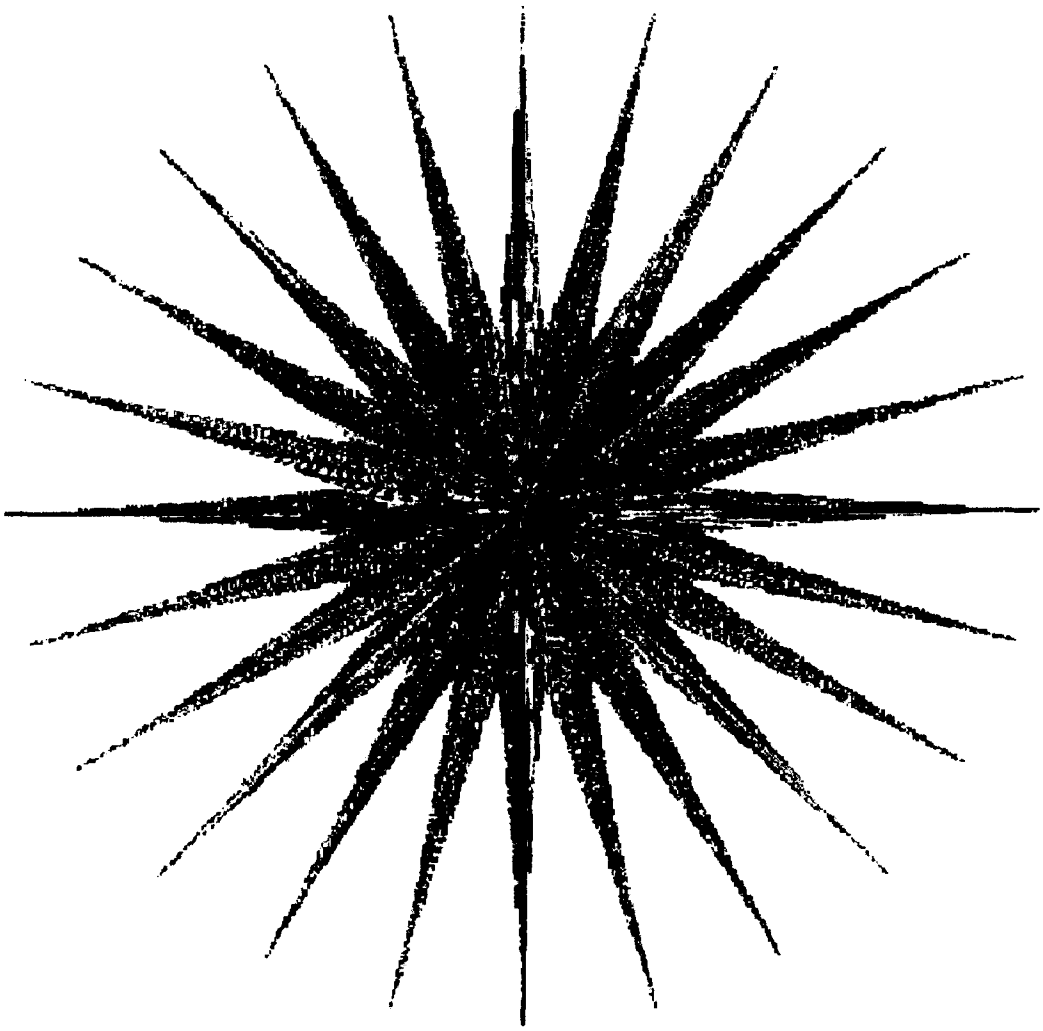

Figure 7. *Asóka Cakra* for $(n, \alpha, \beta) = (1, 14, 12)$

Remark (Valid values of β). If $\beta \geq 7$ or $\beta \leq -7$ then the algorithm works well and figures are generated for any n, $\alpha \geq 1$. For all $\beta \geq 7$, we get fractal projective hedgehogs for any n, $\alpha \geq 1$. Besides Figures 2, 3, 4 and 5, we generate a fractal hedgehog in Figure 6 when $(n, \alpha, \beta) = (2, 2005, -2001)$, which looks like Indian mythological divine weapon *Sudarśana Cakra*.

Remark (Invalid values of β). When the computer program for valid values of β is run for invalid values of β, we get some fascinating figures. In fact, figures generated for invalid values of β are so beautiful that one is tempted to have a look at them.

If β fails to satisfy any of the three conditions (i)–(iii), the function $q(\theta)$ need not belong to C^1. For $7 > \beta > -7$, there is no α to satisfy all the three conditions and this is a failure of β. By graphical analysis, we find that for an even $|\beta|$, the total number of sectors in the figure is $2|\beta|$, for any n, $\alpha \geq 1$. See Figure 7 when $(n, \alpha, \beta) = (1, 14, 12)$. Notice that this choice of β is invalid. In this figure, there are 24 sectors, and their tips are on a circle, which gives it a look of *Aśoka Cakra*, the symbol of pride in the national flag of India. Further, Figure 8 and Figure 9 show some interesting objects for invalid values of β when $(n, \alpha, \beta) = (6, 8, -2)$ and $(5, 8, 3)$ respectively.

5. Algorithm

This algorithm generates fractal hedgehogs for the function $q(\theta)$ define by (7) of Theorem 5. Comment lines are preceded by //.

```
// l and b variables replace α and β. j is the angle and xc and yc are center
// coordinates of hedgehogs. q and q1 correspond to q(θ) and q'(θ).
     n=4
     l=10
     b=-9
     xc=320
     yc=220
// Computation starts from here
   for a=1 to 360
          j=3.14*a/180
          q=0
          1=0
```

```
        for i=1 to n
              t=(b)i*j
              // calculation of q(θ)
              q=q+(0.125)^i*sin(t)
              // calculation of q'(θ)
              q1=q1+(0.125*b)^i*cos(t)
        End for
        x1=16*(q*cos(j)-q1*sin(j))
        y1=16*(q*sin(j)+q1*cos(j))
        x=xc+x1
        y=yc-y1
        line(x, y, xc, yc)
End for
```

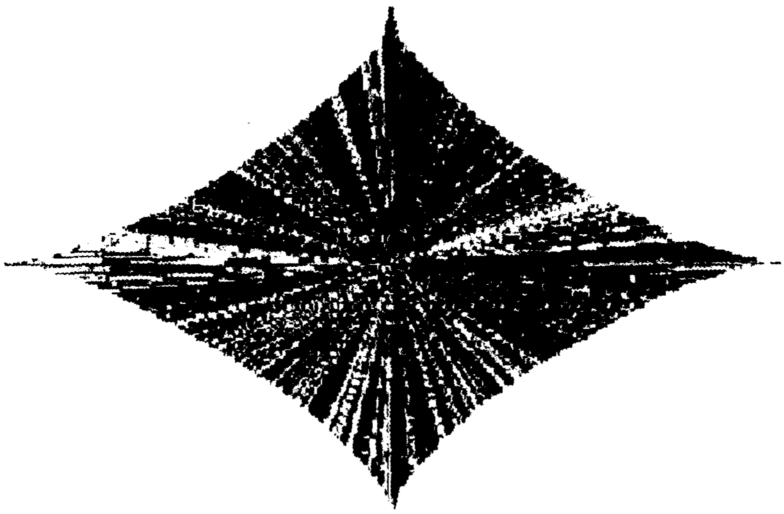

Figure 8. $(n, \alpha, \beta) = (6, 8, -2)$

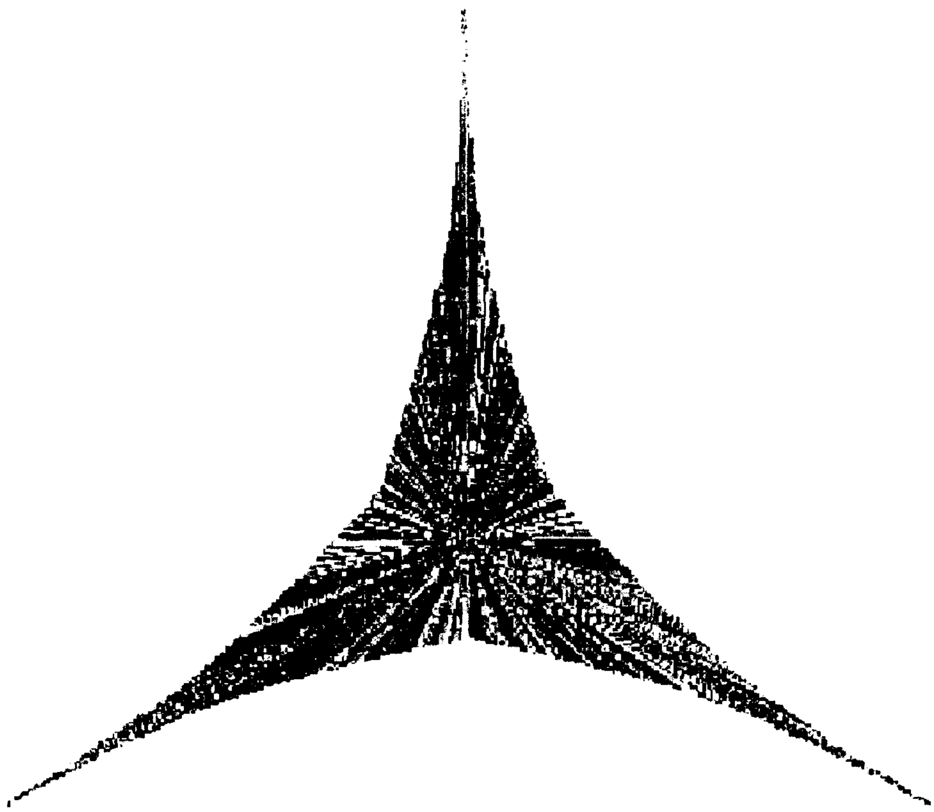

Figure 9. $(n, \alpha, \beta) = (5, 8, 3)$

6. Concluding remarks

The work in this paper is inspired by Martinez-Maure's paper [15] on fractal projective hedgehogs. Our generalization of Maure's theorem embodies non-projective fractal hedgehogs as well.

We have come up with a new algorithm to generate fractal hedgehogs. All fractal projective hedgehogs given in Martinez-Maure [15] may also be generated by our algorithm. Length of each hedgehog is infinity provided n is very large.

Weierstrass' everywhere continuous and nowhere differentiable function is the backbone of the study of hedgehogs presented in this chapter. Now the natural question is: *Can we obtain hedgehogs using "nowhere differentiable functions" other than that of Weierstarss?*

Now we pose a few questions for further work. *What is fractal or topological dimension of a fractal hedgehog? Are dimensions of projective and non-projective hedgehogs different?*

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(M. Rani) Department of Computer Science & Engineering, Gurukula Kangri University, Hardwar 249404, India
Email address: `mamtarsingh@rediffmail.com`

(V. Kumar) Department of Computer Science & Engineering, Gurukula Kangri University, Hardwar 249404, India
Email address: `sh42@rediffmail.com`