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ON FUZZY QUOTIENT RINGS AND CHAIN CONDITIONS

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Abstract. We prove some characterizations of rings with chain conditions in terms of fuzzy quotient rings and fuzzy ideals. We also show that a ring R is left Artinian if and only if the set of values of every fuzzy ideal on R is upper well-ordered.

1. Introduction

The pioneering work of Zadeh on fuzzy subsets of a set in [20] and Rosenfeld on fuzzy subgroups of a group in [18] led to the fuzzification of algebraic structures. For example, Liu [9] introduced the notion of fuzzy ideal of a ring. Since then, the notions of prime fuzzy ideal, maximal fuzzy ideal, fractionary fuzzy ideal and fuzzy invertible fractionary fuzzy ideal were introduced in [6, 12, 19] and applied in [5, 7, 13, 14]. Also, Malik [11], Mukererjee and Sen [15] studied rings with chain conditions with the help of fuzzy ideals and the notions of fuzzy quotient rings were introduced by Kumar [3], Kuroaka and Kuroki [4].

In this paper, we examine some properties of fuzzy quotient rings. We use these results to charaterize rings with chain conditions in terms of fuzzy quotient rings and finite (or well-ordered) valued fuzzy ideals. We also show that a ring R is left Artinian if and only if the set of values of every fuzzy ideal on R is upper well-ordered, which is a generalization of Theorem 3.2 of Malik [11] and a new characterization of left Artinian rings.

Let R be a ring with identity. A *fuzzy subset* of R is a function from R to [0,1]. Let μ and ν be fuzzy subsets of R. We write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)\ \forall x \in R$. If $\mu \subseteq \nu$ and there exists a $x \in R$ such that $\mu(x) < \nu(x)$, then we write $\mu \subset \nu$. We denote

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the image of μ by $\mathrm{Im}\,(\mu)$, and for $t \in [0,1]$, let $\mu_t = \{x \in R \mid \mu(x) \geq t\}$, a *level subset* of μ.

We let χ_W denote the characteristic function of a subset W of R. A fuzzy subset μ of R is a *fuzzy left (right) ideal* of R if for every $x, y \in R$, $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \mu(y)\,(\mu(xy) \geq \mu(x))$. A fuzzy subset μ is a fuzzy left ideal if and only if $\mu(0) \geq \mu(x)\,\forall x \in R$ and μ_t is a left ideal of R $\forall t \in [0, \mu(0)]$. A fuzzy subset μ of R is a *fuzzy ideal* of R if it is a left and right fuzzy ideal. We say that μ is *finite-valued* if $\mathrm{Im}\,(\mu)$ is a finite set. For a function f from a set S to a set T, a fuzzy subset μ of S is called *f-invariant* if for all $x, y \in S$, $f(x) = f(y)$ implies $\mu(x) = \mu(y)$. A ring R is left Noetherian (Artinian) if it satisfies the ascending chain condition (resp. descending chain condition) on left ideals of R; that is, every strictly ascending (resp. descending) chain of left ideals of R is finite.

2. Preliminaries

In this section, we explain some basic definitions and results which will be used in the later section.

Definition 2.1 [4]. Let X and Y be two sets and f a function of X into Y. Let μ and ν be fuzzy subsets of X and Y, respectively. Then $f(\mu)$, the *image* of μ under f, is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{f(x)=y} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

for all $y \in Y$. The *preimage* $f^{-1}(\nu)$ of ν under f is a fuzzy subset of X defined by

$$f^{-1}(\nu)(x) = \nu(f(x)) \text{ for all } x \in X.$$

Lemma 2.2. *Given a homomorphism of rings $f : R \to R'$ and a finite-valued fuzzy left ideal μ of R, $f(\mu_t) = f(\mu)_t$.*

Proof. Let $x \in f(\mu_t)$. Then $x = f(z)$ for some $z \in \mu_t$ and $f(\mu)(x) = \sup_{f(z)=x} \mu(z) \geq t$. Thus $f(\mu_t) \subseteq f(\mu)_t$.

Conversely if $y \in f(\mu)_t$, then $f(\mu)(y) = \sup_{f(x)=y} \mu(x) \geq t$. Since μ is finite-valued, there exists a $x \in f^{-1}(y)$ such that $\mu(x) \geq t$. Thus $f(\mu)_t \subseteq f(\mu_t)$. □

Definition 2.3 [3]. Let μ be a fuzzy ideal of R and let $x \in R$. Then the fuzzy subset μ_x^* of R defined by

$$\mu_x^*(r) = \mu(r - x) \quad \text{for all } r \in R$$

is termed as the *fuzzy coset* determined by x and μ. The set of all fuzzy cosets of μ in R is a ring under the binary operations

$$\mu_x^* + \mu_y^* = \mu_{x+y}^* \text{ and } \mu_x^* \mu_y^* = \mu_{xy}^* \quad \forall x, y \in R$$

and it is denoted by R_μ. We call it the *fuzzy quotient ring* of R induced by the fuzzy ideal μ.

Theorem 2.4 [3]. *Let μ be a fuzzy ideal of a ring R. Then the map $f : R \to R_\mu$ defined by $f(x) = \mu_x^*$ for all $x \in R$, is a surjective homomorphism with kernel μ_t, where $t = \mu(0)$.*

The following two theorems are characterizations of rings with chain conditions in terms of finite (or well-ordered) valued fuzzy ideals.

Theorem 2.5 [11]. *Let R be a ring with unity. Then R is left Artinian if and only if every fuzzy left ideal of R is finite-valued.*

Theorem 2.6 [15]. *A ring R is left Notherian if and only if the set of values of every fuzzy left ideal on R is a well-ordered subset of $[0, 1]$.*

3. Rings with chain conditions

In this section, we characterize rings with chain conditions with the help of fuzzy ideal theory and fuzzy quotient rings.

Lemma 3.1. *Let R and R' be rings and $f : R \to R'$ be a ring homomorphism. If f is surjective and μ is a fuzzy left ideal of R, then so is $f(\mu)$. If θ is a fuzzy left ideal of R' , then so is $f^{-1}(\theta)$.*

Proof. The proof is similar to the proof of Lemma 2.11 of [4]. □

Lemma 3.2 [4]. *Given a homomorphism $f : R \to R'$ and a fuzzy left ideal μ of R, $f^{-1}(f(\mu)) = \mu + \chi_{\mathrm{Ker} f}$.*

Theorem 3.3. *Let μ be a fuzzy ideal of R. Then a ring R is left Artinian if and only if R_μ is left Artinian and the set $\theta(\mu_t) = \{\theta(x) | x \in \mu_t, \text{ where} t = \mu(0)\}$ is a finite subset of* [0,1] *for every fuzzy left ideal θ of R.*

Proof. Let R be a left Artinian ring. Then for every fuzzy left ideal θ of R, $\theta(\mu_t)$ is a finite subset of [0,1] since θ is finite-valued by Theorem 2.5. Now μ' be any fuzzy left ideal of R_μ. Define a map $\theta : R \to [0,1]$ by $\theta(x) = \mu'(\mu_x^*)$ for every $x \in R$. Then θ is a fuzzy left ideal of R, since

$$\theta(x-y) = \mu'(\mu_{x-y}^*) = \mu'(\mu_x^* - \mu_y^*) \geq \min(\mu'(\mu_x^*), \mu'(\mu_y^*)) = \min(\theta(x), \theta(y))$$

and

$$\theta(xy) = \mu'(\mu_{xy}^*) = \mu'(\mu_x^* \mu_y^*) \geq \mu'(\mu_y^*) = \theta(y)$$

for all $x, y \in R$. Since R is left Artinian, θ is finite-valued by Theorem 2.5. And μ' is also finite-valued since the set of values of θ is same to the set of values of μ', so that R_μ is left Artinian.

To prove the converse, let $f : R \to R_\mu$ be the surjection defined by $f(x) = \mu_x^*$ and θ a fuzzy left ideal of R. Then by Lemma 3.1, $f^{-1}(f(\theta))$ is a fuzzy left ideal of R and by Theorem 2.4 and Lemma 3.2,

$$\begin{aligned} f^{-1}(f(\theta))(x) &= (\theta + \chi_{\mathrm{Ker}f})(x) \\ &= \sup \min_{x=a+b} \{\theta(a), \chi_{\mu_t}(b)\} \\ &= \sup_{x=a+b,\, b\in\mu_t} \{\theta(a)\} \\ &= \sup_{b\in\mu_t} \{\theta(x-b)\} \end{aligned}$$

for every $x \in R$. Let $x \in R$ and assume that $\theta(x) \neq \theta(b)$ for every $b \in \mu_t$. Since $\theta(\mu_t)$ is finite, let $\theta(\mu_t) = \{\theta(b_i) | 1 \leq i \leq n, b_i \in \mu_t, \theta(b_1) < \theta(b_2) < \cdots < \theta(b_n)\}$. Then either $\theta(x) > \theta(b_n)$ or $\theta(x) < \theta(b_n)$. If $\theta(x) > \theta(b_n)$, then $\theta(x-b) \geq \min(\theta(x), \theta(b)) = \theta(b)$ and $\theta(b) = \theta(x - x + b) \geq \min(\theta(x), \theta(x-b)) = \theta(x-b)$ for $\forall b \in \mu_t$. Hence $\theta(x-b) = \theta(b)$ for $\forall b \in \mu_t$ and

$$\begin{aligned} \sup_{b\in\mu_t} \{\theta(x-b)\} &= \sup_{b\in\mu_t - \{0\}} \{\theta(x), \theta(x-b)\} \\ &= \sup_{b\in\mu_t - \{0\}} \{\theta(x), \theta(b)\} \\ &= \theta(x), \end{aligned}$$

that is,

$$\theta(x) = f^{-1}(f(\theta))(x) = f(\theta)(f(x))$$

for $x \in R$. In the case $\theta(x) < \theta(b_j)$ for some $j, 1 \leq j \leq n$ and $\theta(x) > \theta(b_{j-1})$, then $\theta(x-b_i) \geq \min(\theta(x), \theta(b_i)) = \theta(x)$ and $\theta(x) = \theta(x-b_i+b_i) \geq \min(\theta(x-b_i), \theta(b_i)) = \theta(x-b_i)\,\forall i \geq j$. So $\theta(x-b_i) = \theta(x)\ \forall i, i \geq j$ and hence

$$\begin{aligned}\sup_{b\in\mu_t}\{\theta(x-b)\} &= \sup_{b\in\mu_t-\{0\}}\{\theta(x), \theta(x-b)\} \\ &= \sup_{1\leq k\leq j-1}\{\theta(x), \theta(x-b_k)\} \\ &= \sup_{1\leq k\leq j-1}\{\theta(x), \theta(b_k)\} \\ &= \theta(x).\end{aligned}$$

Thus we can see that $\theta(x) = \theta(b)$ for some $b \in \mu_t$ or $\theta(x) = f^{-1}(f(\theta))(x) = f(\theta)(f(x))$ for a fuzzy left ideal $f(\theta)$ of R_μ. Since $\theta(\mu_t)$ is finite and $\operatorname{Im} f(\theta)$ is finite by Theorem 2.5, it follows that $\operatorname{Im}(\theta)$ is also a finite set. Therefore, R is left Artinian. □

As a corollary of Theorem 3.3 we can prove a well-known result about left Artinian rings [1, Proposition 10.12] using only fuzzy ideal theoretic technique.

Corollary 3.4. *Let I be a left ideal of R. Then R is left Artinian if and only if R/I and I are both left Artinian.*

Proof. Let R be a left Artinian ring and I a left ideal of R. Then there exists a fuzzy left ideal $\mu = \chi_I$ with $\mu_t = I$, where $t = \mu(0)$. Since $R_\mu \simeq R/\mu_t$ is left Artinian by Theorem 3.3, R/I is left Artinian. Now assume that $I = \mu_t$ is not left Artinian. Then there exists a descending chain of left ideals $I = I_0 \supset I_1 \supset I_2 \supset \cdots$. Define a fuzzy subset θ of R by

$$\theta(x) = \begin{cases} 0 & \text{if } x \in R - I_0 \\ \frac{n}{n+1} & \text{if } x \in I_{n-1} - I_n \\ 1 & \text{if } x \in \cap_{n=0}^{\infty} I_n. \end{cases}$$

Then we can see that θ is a fuzzy left ideal of R such that $\theta(\mu_t)$ is not finite. This is a contradiction to Theorem 3.3, so that I is left Artinian. The converse is proved by the similar method to the proof of Theorem 3.3. □

Similarly to the proof of Theorem 3.3 and by Theorem 2.6, we obtain the following characterization of left Noetherian rings.

Theorem 3.5. *A ring R is left Noetherian if and only if R_μ is left Noetherian and the set $\theta(\mu_t)$ is a well-ordered subset of* [0,1] *for every fuzzy left ideal θ of R.*

Definition 3.6 [10]. A set B is said to be *upper well-ordered* if for all nonempty subsets $C \subset B$, sup $C \in C$.

We can see that a set $B \subset [0,1]$ is upper well-ordered if and only if it is without any increasing monotonic limit.

Now we give a new characterization of left Artinian rings which is a generalization of Theorem 2.5.

Theorem 3.7. *A ring R is left Artinian if and only if the set of values of any fuzzy left ideal on R is a upper well-ordered subset of* [0,1].

Proof. The sufficiency follows from Theorem 2.5. To prove the converse, suppose that R is not left Artinian. Then there exists a strictly descending chain $I_0 \supset I_1 \supset I_2 \supset \cdots$ of left ideals of R. Define a fuzzy subset μ by

$$\mu(x) = \begin{cases} \frac{n}{n+1} & \text{if} \quad x \in I_n - I_{n+1},\ n = 0, 1, 2, \cdots \\ 1 & \text{if} \quad x \in \cap_{n=0}^{\infty} I_n, \end{cases}$$

where $I_0 = R$. Then μ is a fuzzy left ideal and $\text{Im}(\mu) = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots, 1\}$ is not upper well-ordered, since it has an increasing monotonic limit 1. This contradicts to the hypothesis. Hence R is left Artinian. □

The following result is a consequence of Theorem 2.5 and Theorem 3.7. Also, we can prove as follows:

Theorem 3.8. *Every fuzzy left ideal of R is finite-valued if and only if* Im (μ) *is upper well-ordered for every fuzzy left ideal μ of R.*

Proof. The sufficiency is obvious. Now suppose that Im (μ) is upper well-ordered for every fuzzy left ideal μ of R and there exists a fuzzy left ideal ν of R which is not finite-valued. Then $\text{Im}(\nu)$ has a strictly decreasing sequence $t_1 > t_2 > t_3 > \cdots$. Then $\nu_{t_1} \subset \nu_{t_2} \subset \nu_{t_3} \subset \cdots$ is a strictly ascending chain of left ideals of R. Let $A_n = \cup_{i=n}^{\infty} \nu_{t_i}$. Then $A_1 \supset A_2 \supset A_3 \supset \cdots$ is a strictly descending chain of left ideals of R. Define a fuzzy left ideal θ by

$$\theta(x) = \begin{cases} 0 & x \in R \\ a_n & x \in A_n - A_{n+1} \\ 1 & x \in \cap_{n=1}^{\infty} A_n, \end{cases}$$

where $\{a_n\} \subset [0,1]$ is a strictly increasing sequence converging to 1. Then $\text{Im}(\theta)$ is not upper well-ordered, a contradiction to the hypothesis. Thus every fuzzy left ideal of R is finite-valued. □

Next, we characterize rings with chain conditions using only fuzzy quotient rings. In the remaining of this paper, R denotes a commutative ring with identity.

Proposition 3.9. *A ring R is Artinian if and only if R_μ is Artinian for every fuzzy ideal μ of R.*

Proof. Let μ be a fuzzy ideal of R and μ' be any fuzzy ideal of R_μ. To show that μ' is finite-valued, define a map $\theta : R \to [0,1]$ by $\theta(x) = \mu'(\mu_x^*)$ for every $x \in R$. Then θ is a fuzzy ideal of R and it is finite-valued by Theorem 2.5. And μ' is also finite-valued since the set of values of θ is same to the set of values of μ', so that R_μ is Artinian.

Conversely, let μ be a fuzzy ideal of a ring R. Then the fuzzy ideal μ' of R_μ defined by $\mu'(\mu_x^*) = \mu(x)$ for every $x \in R$ is finite-valued, so that μ is also finite-valued. By Theorem 2.5, R is Artinian. □

Proposition 3.10. *A ring R is Noetherian if and only if R_μ is Noetherian for every fuzzy ideal μ of R.*

Proof. Suppose that R is Noetherian. Then by the similar method to the proof of Proposition 3.9, it is proved that R_μ is Noetherian for every fuzzy ideal μ of R. Conversely, let μ be any fuzzy ideal of R. Then for the fuzzy ideal μ' of R_μ defined by $\mu'(\mu_x^*) = \mu(x)$ for $\forall x \in R$, the set of values of μ' is a well-ordered subset of [0,1] by Theorem 2.6. Since the set of values of μ' is same to the set of values of $\mu, \text{Im}(\mu)$ is also well-ordered. Thus R is Noetherian by Theorem 2.6. □

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