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# ISOMORPHISMS OF CERTAIN TRIDIAGONAL ALGEBRAS

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ABSTRACT. We will characterize isomorphisms from the adjoint of a certain tridiagonal algebra $Alg\mathcal{L}_{2n}$ onto $Alg\mathcal{L}_{2n}$. In this paper the followings are proved:

A map $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ is an isomorphism if and only if there exists an operator $S$ in $Alg\mathcal{L}_{2n}$ with all diagonal entries are 1 and an invertible backward diagonal operator $B$ such that $\Phi(A) = SBAB^{-1}S^{-1}$.

## 1. Introduction

The study of self-adjoint operator algebras on Hilbert space is well established, with a long history including some of the strongest mathematicians of the twentieth century. By contrast, non-self-adjoint algebras, particularly reflexive algebras, are only beginning to be studied; the seminar paper of Arveson [1] in 1974 represents the beginning of widespread interest in reflexive algebras. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces. The tridiagonal algebra is one of the most important classes of non-self-adjoint reflexive algebras. These algebras possess many surprising properties. Jo [6] investigates the isometries of tridiagonal algebras. Choi [2,7] characterizes the isomorphisms of these tridiagonal algebras and another tridiagonal algebras $Alg\mathcal{L}_{2n}$.

In this paper, We will investigate the isomorphisms from the adjoint algebra of $Alg\mathcal{L}_{2n}$ onto the algebra $Alg\mathcal{L}_{2n}$.

First we will introduce the terminologies which are used in this paper. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$. $\mathcal{A}$ is called a self-adjoint algebra provided $A^*$ is in $\mathcal{A}$ for every $A$ in $\mathcal{A}$, otherwise, $\mathcal{A}$ is called a non-self-adjoint algebra.

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If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, then $Alg\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and 1. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is *reflexive* if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is *reflexive* if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is *commutative* if each pair of projections in $\mathcal{L}$ commutes. If $\mathcal{L}$ is a commutative subspace lattice, then $Alg\mathcal{L}$ is called a CSL-algebra.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism $\Phi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. An isomorphism $\Phi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is said to be spatially implemented if there is a bounded invertible operator $T$ such that $\Phi(A) = TAT^{-1}$ for all $A$ in $Alg\mathcal{L}_1$. If $x_1, x_2, \cdots, x_n$ are vectors in some Hilbert space, we denote by $[x_1, x_2, \cdots, x_n]$ the closed subspace spanned by the vectors $x_1, x_2, \cdots, x_n$. Let $i$ and $j$ be positive integers. Then $E_{ij}$ is the matrix whose $(i,j)$-component is 1 and all other entries are zero. An $n \times n$ matrix $J_n$ is said to be the backward identity matrix if the $(i, n-i+1)$-component is 1 for all $i = 1, 2, \cdots, n$ and all other entries are zero.

Let $\mathcal{H}$ be $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ and let $\mathcal{A}_{2n}$ be the tridiagonal algebras discovered by Gilfeather and Larson: that is, $\mathcal{A}_{2n} = Alg\mathcal{L}$, where $\mathcal{L}$ is the subspace lattice of orthogonal projections generated by $\{[e_1], [e_3], \cdots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \cdots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], [e_1, e_{2n-1}, e_{2n}]\}$. The isomorphisms of $\mathcal{A}_{2n}$ need not be spatially implemented. In [2], it was investigated that the necessary and sufficient condition that isomorphisms of $\mathcal{A}_{2n}$ are spatially implemented. Let $\mathcal{L}_{2n}$ be the subspace lattice of orthogonal projections generated by $\{[e_1], [e_3], \cdots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \cdots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], [e_{2n-1}, e_{2n}]\}$. Then $Alg\mathcal{L}_{2n}$ is the tridiagonal algebra consisting of all bounded operators, acting on $\mathcal{H}$, that are of the form

$$\begin{pmatrix} * & * & & & & & \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & & & \\ & & & & \ddots & & \\ & & & & & * & * \\ & & & & & & * \end{pmatrix},$$

where all non-starred entries are zero and with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Of course, $Alg\mathcal{L}_{2n}$ is a non-self-adjoint reflexive CSL-algebra. Jo and Choi [7] have proved that the isomorphisms of $Alg\mathcal{L}_{2n}$ are spatially implemented.

In this paper, we will prove that the isomorphisms from the adjoint algebra of $Alg\mathcal{L}_{2n}$ onto $Alg\mathcal{L}_{2n}$ are spatially implemented and we will find the implemented matrix $T$ of these isomorphisms.

## 2. Isomorphisms from $(Alg\mathcal{L}_{2n})^*$ onto $Alg\mathcal{L}_{2n}$

Before we investigate the general isomorphisms $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$, we will consider special isomorphisms $\rho : (Alg\mathcal{L}_{2n})^* \to (Alg\mathcal{L}_{2n})^*$ satisfying $\rho(E_{ii}) = E_{ii}$ for $i = 1, 2, \cdots, 2n$.

**Theorem 2.1.** *Let $\rho : (Alg\mathcal{L}_{2n})^* \to (Alg\mathcal{L}_{2n})^*$ be an isomorphism defined by $\rho(E_{ii}) = E_{ii}$ for $i = 1, 2, \cdots, 2n$. Then there exist $2n-1$ nonzero complex numbers $\gamma_{2i,2i-1}$ for $i = 1, 2, \cdots, n$ and $\gamma_{2j,2j+1}$ for $j = 1, 2, \cdots, n-1$ such that*

$$\rho(E_{2i,2i-1}) = \gamma_{2i,2i-1}E_{2i,2i-1} \text{ for all } i = 1, 2, \cdots, n; \text{ and}$$
$$\rho(E_{2j,2j+1}) = \gamma_{2j,2j+1}E_{2j,2j+1} \text{ for all } j = 1, 2, \cdots, n-1.$$

*Proof.* Since $E_{2i,2i-1} = E_{2i,2i}E_{2i,2i-1}E_{2i-1,2i-1}$, for all $i = 1, 2, \cdots, n$, we have

$$\rho(E_{2i,2i-1}) = \rho(E_{2i,2i}E_{2i,2i-1}E_{2i-1,2i-1}) = E_{2i,2i}\rho(E_{2i,2i-1})E_{2i-1,2i-1}.$$

Hence $\rho(E_{2i,2i-1}) = \gamma_{2i,2i-1}E_{2i,2i-1}$ for some nonzero complex number $\gamma_{2i,2i-1}$ for all $i = 1, 2, \cdots, n$.

In exactly the same way, we show that $\rho(E_{2j,2j+1}) = \gamma_{2j,2j+1}E_{2j,2j+1}$ for some nonzero complex number $\gamma_{2j,2j+1}$ for all $j = 1, 2, \cdots, n-1$. □

**Theorem 2.2.** *A map $\rho : (Alg\mathcal{L}_{2n})^* \to (Alg\mathcal{L}_{2n})^*$ is an isomorphism such that $\rho(E_{ii}) = E_{ii}$ for $i = 1, 2, \cdots, 2n$ if and only if there exists an invertible diagonal operator $D$ such that $\rho(A) = DAD^{-1}$ for all $A$ in $(Alg\mathcal{L}_{2n})^*$*

*Proof.* Suppose that $\rho : (Alg\mathcal{L}_{2n})^* \to (Alg\mathcal{L}_{2n})^*$ is an isomorphism such that $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots, 2n$. By Theorem 2.1, there exist $2n-1$ nonzero complex numbers $\gamma_{ij}$ for all $i, j (i \neq j)$ with $E_{ij}$ in $(Alg\mathcal{L}_{2n})^*$ such that

$$\rho(E_{2i,2i-1}) = \gamma_{2i,2i-1}E_{2i,2i-1} \text{ for all } i = 1, 2, \cdots, n; \text{ and}$$
$$\rho(E_{2j,2j+1}) = \gamma_{2j,2j+1}E_{2j,2j+1} \text{ forall } j = 1, 2, \cdots, n-1.$$

Let $D = [d_{ij}]$ be the invertible diagonal operator, where

$$\begin{aligned} d_{11} &= 1, \\ d_{22} &= \gamma_{21}, \\ d_{2i-1,2i-1} &= \prod_{k=1}^{i-1} \gamma_{2k,2k-1} \Big(\prod_{k=1}^{i-1} \gamma_{2k,2k+1}\Big)^{-1}, \text{ and} \\ d_{2i,2i} &= \prod_{k=1}^{i} \gamma_{2k,2k-1} \Big(\prod_{k=1}^{i-1} \gamma_{2k,2k+1}\Big)^{-1} \end{aligned}$$

for all $i = 2, \cdots, n$. Then $\rho(A) = DAD^{-1}$ for all $A$ in $(Alg\mathcal{L}_{2n})^*$. □

From now on we will prove that the isomorphism $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ is spatially implemented and we will find implemented matrix $T$ of these isomorphisms.

**Theorem 2.3** [5]. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and suppose that $\Phi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is an algebraic isomorphism. Let $\mathcal{M}$ be a maximal abelian self-adjoint subalgebra (masa) contained in $Alg\mathcal{L}_1$. Then there exists a bounded invertible operator $Y : \mathcal{H}_1 \to \mathcal{H}_2$ and an automorphism $\rho : Alg\mathcal{L}_1 \to Alg\mathcal{L}_1$ such that*

(i) *$\rho(M) = M$ for all $M$ in $\mathcal{M}$, and*
(ii) *$\Phi(A) = Y\rho(A)Y^{-1}$ for all $A$ in $Alg\mathcal{L}_1$.*

**Theorem 2.4.** *Let $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ be an isomorphism. Then there exists an invertible operator $T$ such that $\Phi(A) = TAT^{-1}$ for all $A$ in $(Alg\mathcal{L}_{2n})^*$.*

*Proof.* Since $(Alg\mathcal{L}_{2n})^* \cap Alg\mathcal{L}_{2n}$ is a masa of $(Alg\mathcal{L}_{2n})^*$ and since $E_{ii}$ is in $(Alg\mathcal{L}_{2n})^* \cap Alg\mathcal{L}_{2n}$ for all $i = 1, 2, \cdots, 2n$, by Theorem 2.3 there exists an invertible operator $Y$ in $\mathcal{B}(\mathcal{H})$ and an isomorphism $\rho : (Alg\mathcal{L}_{2n})^* \to (Alg\mathcal{L}_{2n})^*$ satisfying $\rho(E_{ii}) = E_{ii}$ for all $i = 1, 2, \cdots, 2n$ such that $\Phi(A) = Y\rho(A)Y^{-1}$. By Theorem 2.2, there exists an invertible diagonal operator $D$ such that $\rho(A) = DAD^{-1}$. Hence

$$\Phi(A) = Y\rho(A)Y^{-1} = YDAD^{-1}Y^{-1}.$$

Let $T = YD$. Then $\Phi(A) = TAT^{-1}$ for all $A$ in $(Alg\mathcal{L}_{2n})^*$. □

**Theorem 2.5.** *Let* $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ *be an isomorphism. Then, for each* $i$ $(1 \leq i \leq 2n)$,

$\Phi(E_{ii}) = E_{11} - \alpha_{12}E_{12}$ *for some complex number* $\alpha_{12}$,

$\Phi(E_{ii}) = E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k}$

*for some complex numbers* $\alpha_{2k-1,2k-2}$ *and* $\alpha_{2k-1,2k}$ $(2 \leq k \leq n)$,

$\Phi(E_{ii}) = E_{2k,2k} + \alpha_{2k-1,2k}E_{2k-1,2k} + \alpha_{2k+1,2k}E_{2k+1,2k}$

*for some complex numbers* $\alpha_{2k-1,2k}$ *and* $\alpha_{2k+1,2k}$ $(1 \leq k \leq n-1)$, *or*

$\Phi(E_{ii}) = E_{2n,2n} + \alpha_{2n-1,2n}E_{2n-1,2n}$ *for some complex number* $\alpha_{2n-1,2n}$.

*Proof.* Let $\Phi(E_{ii}) = [\alpha_{pq}]$ be in $Alg\mathcal{L}_{2n}$. Then

$$[\alpha_{pq}]^2 = \Phi(E_{ii})^2 = \Phi(E_{ii}^2) = \Phi(E_{ii}) = [\alpha_{pq}].$$

Hence $\alpha_{pp} = 1$ or $0$ for all $p = 1, 2, \cdots, 2n$. Since $\alpha_{pp} = 0$ for all $p = 1, 2, \cdots, 2n$ implies $[\alpha_{pq}]^2 = [\alpha_{pq}] = 0$, we have $\alpha_{pp} = 1$ for some $p = 1, 2, \cdots, 2n$. If $\alpha_{qq} \neq 0$ for some $q$ such that $q \neq p$ and $1 \leq q \leq 2n$, then the $(p,p)$-component and the $(q,q)$-component of $\Phi(E_{ii})$ are 1. So there exists $j (j \neq i, 1 \leq j \leq 2n)$ such that one of the $(p,p)$-component or the $(q,q)$-component of $\Phi(E_{jj})$ is 1. Hence $0 = \Phi(E_{ii}E_{jj}) = \Phi(E_{ii})\Phi(E_{jj}) \neq 0$ which contradicts. Thus $\alpha_{pp} = 1$ for one and only one $p$.

If $\alpha_{2k-1,2k-1} = 1$ for some $k = 1, 2, \cdots, n$, then

$$\Phi(E_{ii}) = E_{2k-1,2k-1} + \sum_{j=1}^{n} \alpha_{2j-1,2j}E_{2j-1,2j} + \sum_{j=1}^{n-1} \alpha_{2j+1,2j}E_{2j+1,2j}.$$

Since $\Phi(E_{ii}) = \Phi(E_{ii})^2$, we have

$$\Phi(E_{ii}) = E_{11} - \alpha_{12}E_{12}, \text{ or}$$

$$\Phi(E_{ii}) = E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k}$$

for some $k = 2, 3, \cdots, n$.

If $\alpha_{2k,2k} = 1$ for some $k = 1, 2, \cdots, n$, then

$$\Phi(E_{ii}) = E_{2k,2k} + \sum_{j=1}^{n} \alpha_{2j-1,2j}E_{2j-1,2j} + \sum_{j=1}^{n-1} \alpha_{2j+1,2j}E_{2j+1,2j}.$$

Since $\Phi(E_{ii}) = \Phi(E_{ii})^2$, we have

$$\Phi(E_{ii}) = E_{2k,2k} + \alpha_{2k-1,2k}E_{2k-1,2k} + \alpha_{2k+1,2k}E_{2k+1,2k}$$

for some $k = 1, 2, \cdots, n-1$, or

$$\Phi(E_{ii}) = E_{2n,2n} + \alpha_{2n-1,2n}E_{2n-1,2n}. \quad \square$$

**Theorem 2.6.** *Let $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ be an isomorphism.*

(1) *If the $(2k-1, 2k-1)$-component of $\Phi(E_{ii})$ is 1, then $i$ is an even number.*

(2) *If the $(2k, 2k)$-component of $\Phi(E_{ii})$ is 1, then $i$ is an odd number.*

*Proof.* (1) Suppose the $(2k-1, 2k-1)$-component of $\Phi(E_{ii})$ is 1. Then

$$\Phi(E_{ii}) = E_{11} + \alpha_{12}E_{12}, \text{ or}$$
$$\Phi(E_{ii}) = E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k},$$

for some $k = 2, 3, \cdots, n$.

Suppose $i$ is an odd number, say $i = 2l-1$. Let $\Phi(E_{2l,2l-1}) = [\lambda_{pq}]$ be in $Alg\mathcal{L}_{2n}$. If $\Phi(E_{ii}) = E_{11} + \alpha_{12}E_{12}$, then

$$\begin{aligned}\Phi(E_{2l,2l-1}) &= \Phi(E_{2l,2l})\Phi(E_{2l,2l-1})\Phi(E_{2l-1,2l-1}) \\ &= \Phi(E_{2l,2l})(\lambda_{11}E_{11} + \lambda_{11}\alpha_{12}E_{12}).\end{aligned}$$

Since $\Phi(E_{2l,2l-1}) \neq 0$, the $(1,1)$-component of $\Phi(E_{2l,2l})$ is 1. It is a contradiction to the $(1,1)$-component of $\Phi(E_{2l-1,2l-1})$ is 1. Hence $i$ is an even number. If

$$\Phi(E_{ii}) = E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k}$$

for some $k = 2, 3, \cdots, n$, then

$$\begin{aligned}\Phi(E_{2l,2l-1}) &= \Phi(E_{2l,2l})\Phi(E_{2l,2l-1})\Phi(E_{2l-1,2l-1}) \\ &= \Phi(E_{2l,2l})(\lambda_{2k-1,2k-1}E_{2k-1,2k-1} + \lambda_{2k-1,2k-1}\alpha_{2k-1,2k-2}E_{2k-1,2k-2} \\ &\qquad + \lambda_{2k-1,2k-1}\alpha_{2k-1,2k}E_{2k-1,2k}).\end{aligned}$$

Since $\Phi(E_{2l,2l-1}) \neq 0$, the $(2k-1, 2k-1)$-component of $\Phi(E_{2l,2l})$ is 1. It is a contradiction because the $(2k-1, 2k-1)$-component of $\Phi(E_{2l-1,2l-1})$ is 1. Hence $i$ is an even number.

(2) In exactly the same way, we can show that $i$ is an odd number. □

**Theorem 2.7.** *Let $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ be an isomorphism. Suppose that the $(p,p)$-component of $\Phi(E_{ii})$ is 1 and the $(q,q)$-component of $\Phi(E_{jj})$ is 1. If $|i-j| = 1$, then $|p-q| = 1$.*

*Proof.* Suppose that $p = 1$. Then $\Phi(E_{ii}) = E_{11} + \alpha_{12}E_{12}$ for some complex number $\alpha_{12}$. By Theorem 2.6, $i$ is an even number, say $i = 2l$. Let $j = i+1 = 2l+1$ and

$\Phi(E_{ij}) = \Phi(E_{2l,2l+1}) = [\lambda_{pq}]$. Then

$$\begin{aligned}\Phi(E_{ij}) &= \Phi(E_{2l,2l+1}) \\ &= \Phi(E_{2l,2l})\Phi(E_{2l,2l+1})\Phi(E_{2l+1,2l+1}) \\ &= (\lambda_{11}E_{11} + (\lambda_{12} + \alpha_{12}\lambda_{22})E_{12})\Phi(E_{2l+1,2l+1}).\end{aligned}$$

Since $\Phi(E_{ij}) = \Phi(E_{2l,2l+1}) \neq 0$,

$$\Phi(E_{2l+1,2l+1}) = \Phi(E_{jj}) = E_{22} + \beta_{12}E_{12} + \beta_{32}E_{32}$$

for some complex numbers $\beta_{12}$ and $\beta_{32}$. Hence $q = 2$ and hence $|p - q| = 1$.

Let $j = i - 1 = 2l - 1$ and $\Phi(E_{ij}) = \Phi(E_{2l,2l-1}) = [\lambda_{pq}]$. Then

$$\begin{aligned}\Phi(E_{ij}) &= \Phi(E_{2l,2l-1}) \\ &= \Phi(E_{2l,2l})\Phi(E_{2l,2l-1})\Phi(E_{2l-1,2l-1}) \\ &= (\lambda_{11}E_{11} + (\lambda_{12} + \alpha_{12}\lambda_{22})E_{12})\Phi(E_{2l-1,2l-1}).\end{aligned}$$

Since $\Phi(E_{ij}) = \Phi(E_{2l,2l+1}) \neq 0$,

$$\Phi(E_{2l-1,2l-1}) = \Phi(E_{jj}) = E_{22} + \beta_{12}E_{12} + \beta_{32}E_{32}$$

for some complex numbers $\beta_{12}$ and $\beta_{32}$. Hence $q = 2$ and hence $|p - q| = 1$.

Suppose that $p = 2k - 1$ for some $k$ $(2 \leq k \leq n)$. Then

$$\Phi(E_{ii}) = E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k}$$

for some complex numbers $\alpha_{2k-1,2k-2}$ and $\alpha_{2k-1,2k}$. By Theorem 2.6, $i$ is an even number, say $i = 2l$.

Let $j = i + 1 = 2l + 1$ and $\Phi(E_{ij}) = \Phi(E_{2l,2l+1}) = [\lambda_{pq}]$. Then

$$\begin{aligned}\Phi(E_{ij}) &= \Phi(E_{2l,2l+1}) \\ &= \Phi(E_{2l,2l})\Phi(E_{2l,2l+1})\Phi(E_{2l+1,2l+1}) \\ &= (\delta_{2k-1,2k-1}E_{2k-1,2k-1} + \delta_{2k-1,2k-2}E_{2k-1,2k-2} + \delta_{2k-1,2k}E_{2k-1,2k}) \\ &\qquad \times \Phi(E_{2l+1,2l+1}),\end{aligned}$$

where $\delta_{2k-1,2k-1} = \lambda_{2k-1,2k-1}$, $\delta_{2k-1,2k-2} = \alpha_{2k-1,2k-2}\lambda_{2k-2,2k-2} + \lambda_{2k-1,2k-2}$ and $\delta_{2k-1,2k} = \lambda_{2k-1,2k} + \alpha_{2k-1,2k}\lambda_{2k,2k}$.

Since $\Phi(E_{2l,2l+1}) \neq 0$,

$$\Phi(E_{2l+1,2l+1}) = E_{2k-2,2k-2} + \beta_{2k-3,2k-2}E_{2k-3,2k-2} + \beta_{2k-1,2k-2}E_{2k-1,2k-2}, \text{ or}$$
$$\Phi(E_{2l+1,2l+1}) = E_{2k,2k} + \beta_{2k-1,2k}E_{2k-1,2k} + \beta_{2k+1,2k}E_{2k+1,2k}$$

for some complex numbers $\beta_{ij}$. Hence $q = 2k-2$ or $2k$ and hence $|p-q| = 1$.

Let $j = i-1 = 2l-1$ and $\Phi(E_{ij}) = \Phi(E_{2l,2l-1}) = [\lambda_{pq}]$. Then

$$\begin{aligned}\Phi(E_{ij}) &= \Phi(E_{2l,2l-1})\\ &= \Phi(E_{2l,2l})\Phi(E_{2l,2l-1})\Phi(E_{2l-1,2l-1})\\ &= (\delta_{2k-1,2k-1}E_{2k-1,2k-1} + \delta_{2k-1,2k-2}E_{2k-1,2k-2} + \delta_{2k-1,2k}E_{2k-1,2k})\\ &\qquad \times\Phi(E_{2l-1,2l-1}),\end{aligned}$$

where $\delta_{2k-1,2k-1} = \lambda_{2k-1,2k-1}, \delta_{2k-1,2k-2} = \alpha_{2k-1,2k-2}\lambda_{2k-2,2k-2} + \lambda_{2k-1,2k-2}$ and $\delta_{2k-1,2k} = \lambda_{2k-1,2k} + \alpha_{2k-1,2k}\lambda_{2k,2k}$.

Since $\Phi(E_{2l,2l-1}) \neq 0$,

$$\Phi(E_{2l-1,2l-1}) = E_{2k-2,2k-2} + \beta_{2k-3,2k-2}E_{2k-3,2k-2} + \beta_{2k-1,2k-2}E_{2k-1,2k-2}, \text{ or}$$
$$\Phi(E_{2l-1,2l-1}) = E_{2k,2k} + \beta_{2k-1,2k}E_{2k-1,2k} + \beta_{2k+1,2k}E_{2k+1,2k}$$

for some complex numbers $\beta_{ij}$. Hence $q = 2k-2$ or $2k$ and hence $|p-q| = 1$. In exactly the same way, we can prove this theorem in case $p = 2k$ for $k = 1, 2, \cdots, n$. □

**Theorem 2.8.** *Let $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ be an isomorphism.*

(1) *If*

$$\Phi(E_{2i-1,2i-1}) = E_{2k,2k} + \beta_{2k-1,2k}E_{2k-1,2k} + \beta_{2k+1,2k}E_{2k+1,2k}, \text{ and}$$
$$\Phi(E_{2i,2i} \quad = E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k},$$

*then there exists a nonzero complex number $\gamma_{2k-1,2k}$ such that*

$$\Phi(E_{2i,2i-1}) = \gamma_{2k-1,2k}E_{2k-1,2k} \text{ and } \beta_{2k-1,2k} = -\alpha_{2k-1,2k}.$$

(2) *If*

$$\Phi(E_{2i,2i}) \quad = E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k}, \text{ and}$$
$$\Phi(E_{2i+1,2i+1}) = E_{2k-2,2k-2} + \beta_{2k-3,2k-2}E_{2k-3,2k-2} + \beta_{2k-1,2k-2}E_{2k-1,2k-2},$$

*then there exists a nonzero complex number $\gamma_{2k-1,2k-2}$ such that*

$$\Phi(E_{2i,2i+1}) = \gamma_{2k-1,2k-2}E_{2k-1,2k-2} \text{ and } \beta_{2k-1,2k-2} = -\alpha_{2k-1,2k-2}.$$

*Proof.* (1) Let $\Phi(E_{2i,2i-1}) = [\lambda_{pq}]$ in $Alg\mathcal{L}_{2n}$. Then

$$\begin{aligned}\Phi(E_{2i,2i-1}) &= \Phi(E_{2i,2i})\Phi(E_{2i,2i-1})\Phi(E_{2i-1,2i-1})\\ &= (E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k})[\lambda_{pq}]\\ &\qquad \times (E_{2k,2k} + \beta_{2k-1,2k}E_{2k-1,2k} + \beta_{2k+1,2k}E_{2k+1,2k})\\ &= (\lambda_{2k-1,2k-1}\beta_{2k-1,2k} + \lambda_{2k-1,2k} + \alpha_{2k-1,2k}\lambda_{2k,2k})E_{2k-1,2k}.\end{aligned}$$

So $\Phi(E_{2i,2i-1}) = \gamma_{2k-1,2k}E_{2k-1,2k}$ for some nonzero complex number $\gamma_{2k-1,2k}$.

Let $A = E_{2i-1,2i-1} + E_{2i,2i-1} + E_{2i,2i}$. Then $A^2 = E_{2i-1,2i-1} + 2E_{2i,2i-1} + E_{2i,2i}$. Hence

$$\begin{aligned}\Phi(A) &= \Phi(E_{2i-1,2i-1} + E_{2i,2i-1} + E_{2i,2i})\\ &= \Phi(E_{2i-1,2i-1}) + \Phi(E_{2i,2i-1}) + \Phi(E_{2i,2i})\\ &= (E_{2k,2k} + \beta_{2k-1,2k}E_{2k-1,2k} + \beta_{2k+1,2k}E_{2k+1,2k}) + (\gamma_{2k-1,2k}E_{2k-1,2k})\\ &\quad + (E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k})\\ &= \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + E_{2k-1,2k-1} + (\beta_{2k-1,2k} + \gamma_{2k-1,2k}\\ &\quad + \alpha_{2k-1,2k})E_{2k-1,2k} + E_{2k,2k} + \beta_{2k+1,2k}E_{2k+1,2k}\end{aligned}$$

and hence

$$\begin{aligned}\Phi(A)^2 &= \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + E_{2k-1,2k-1}\\ &\quad + 2(\beta_{2k-1,2k} + \gamma_{2k-1,2k} + \alpha_{2k-1,2k})E_{2k-1,2k} + E_{2k,2k} + \beta_{2k+1,2k}E_{2k+1,2k}.\end{aligned}$$

While

$$\begin{aligned}\Phi(A^2) &= \Phi(E_{2i-1,2i-1} + 2E_{2i,2i-1} + E_{2i,2i})\\ &= \Phi(E_{2i-1,2i-1}) + 2\Phi(E_{2i,2i-1}) + \Phi(E_{2i,2i})\\ &= (E_{2k,2k} + \beta_{2k-1,2k}E_{2k-1,2k} + \beta_{2k+1,2k}E_{2k+1,2k}) + 2\gamma_{2k-1,2k}E_{2k-1,2k}\\ &\quad + (E_{2k-1,2k-1} + \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + \alpha_{2k-1,2k}E_{2k-1,2k})\\ &= \alpha_{2k-1,2k-2}E_{2k-1,2k-2} + E_{2k-1,2k-1}\\ &\quad + (\beta_{2k-1,2k} + 2\gamma_{2k-1,2k} + \alpha_{2k-1,2k})E_{2k-1,2k} + E_{2k,2k} + \beta_{2k+1,2k}E_{2k+1,2k}.\end{aligned}$$

Since $\Phi(A)^2 = \Phi(A^2)$, we have

$$2(\beta_{2k-1,2k} + \gamma_{2k-1,2k} + \alpha_{2k-1,2k}) = \beta_{2k-1,2k} + 2\gamma_{2k-1,2k} + \alpha_{2k-1,2k}.$$

Hence $\beta_{2k-1,2k} + \alpha_{2k-1,2k} = 0$ and hence $\beta_{2k-1,2k} = -\alpha_{2k-1,2k}$.

In the same way, we can show that (2) holds. □

**Theorem 2.9.** *Let $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ be an isomorphism. Then there exist $2n-1$ nonzero complex numbers $\gamma_{2k-1,2k}, \gamma_{2k+1,2k}$ for $k = 1, 2, \cdots, n-1$ and $\gamma_{2n-1,2n}$ and $2n-1$ complex numbers $\alpha_{2k-1,2k}, \alpha_{2k+1,2k}$ for $k = 1, 2, \cdots, n-1$ and $\alpha_{2n-1,2n}$ such that*

$$\begin{aligned}
\Phi(E_{11}) &= E_{2n,2n} - \alpha_{2n-1,2n}E_{2n-1,2n};\\
\Phi(E_{2k-1,2k-1}) &= E_{2n+2-2k,2n+2-2k} - \alpha_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k}\\
&\quad - \alpha_{2n+3-2k,2n+2-2k}E_{2n+3-2k,2n+2-2k} \text{ for } k = 2, 3, \cdots, n;\\
\Phi(E_{2k,2k}) &= E_{2n+1-2k,2n+1-2k} + \alpha_{2n+1-2k,2n-2k}E_{2n+1-2k,2n-2k}\\
&\quad + \alpha_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k} \text{ for } k = 1, 2, \cdots, n-1;\\
\Phi(E_{2n,2n}) &= E_{11} + \alpha_{12}E_{12};\\
\Phi(E_{2k,2k-1}) &= \gamma_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k} \text{ for } k = 1, 2, \cdots, n; \text{ and}\\
\Phi(E_{2k,2k+1}) &= \gamma_{2n+1-2k,2n-2k}E_{2n+1-2k,2n-2k} \text{ for } k = 1, 2, \cdots, n-1.
\end{aligned}$$

*Proof.* Suppose $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ is an isomorphism. By Theorems 2.5 and 2.6, the $(2l, 2l)$-component of $\Phi(E_{11})$ is 1 for some $l (1 \le l \le n)$. If $l \ne n$, then there exist complex numbers $\alpha_{2l-1,2l}$ and $\alpha_{2l+1,2l}$ such that

$$\Phi(E_{11}) = E_{2l,2l} + \alpha_{2l-1,2l}E_{2l-1,2l} + \alpha_{2l+1,2l}E_{2l+1,2l}.$$

By Theorem 2.7,

$$\begin{aligned}
\Phi(E_{22}) &= E_{2l-1,2l-1} + \alpha_{2l-1,2l-2}E_{2l-1,2l-2} + \alpha_{2l-1,2l}E_{2l-1,2l}, \text{ or}\\
\Phi(E_{22}) &= E_{2l+1,2l+1} + \alpha_{2l+1,2l}E_{2l+1,2l} + \alpha_{2l+1,2l+2}E_{2l+1,2l+2}.
\end{aligned}$$

Suppose the $(2l-1, 2l-1)$-component of $\Phi(E_{22})$ is 1. Then by Theorem 2.7, the $(2l-j+1, 2l-j+1)$-component of $\Phi(E_{jj})$ is 1 for all $j = 1, 2, \cdots, 2l$. Hence the $(1,1)$-component of $\Phi(E_{2l,2l})$ is 1.

By Theorem 2.7, the $(2,2)$-component of $\Phi(E_{2l+1,2l+1})$ is 1. It is a contradiction. Hence the $(2l-1, 2l-1)$-component of $\Phi(E_{22})$ is not 1.

Similarly, the $(2l+1, 2l+1)$-component of $\Phi(E_{22})$ is not 1. Hence $l = n$ and hence $\Phi(E_{11}) = E_{2n,2n} + \beta_{2n-1,2n}E_{2n-1,2n}$ for some complex number $\beta_{2n-1,2n}$.

From Theorems 2.7 and 2.8, we have

$$\begin{aligned}
\Phi(E_{11}) &= E_{2n,2n} - \alpha_{2n-1,2n}E_{2n-1,2n};\\
\Phi(E_{2k,2k}) &= E_{2n+1-2k,2n+1-2k} + \alpha_{2n+1-2k,2n-2k}E_{2n+1-2k,2n-2k}\\
&\quad + \alpha_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k} \text{ for } k = 1,2,\cdots,n-1;\\
\Phi(E_{2k-1,2k-1}) &= E_{2n+2-2k,2n+2-2k} - \alpha_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k}\\
&\quad - \alpha_{2n+3-2k,2n+2-2k}E_{2n+3-2k,2n+2-2k} \text{ for } k = 2,3,\cdots,n;\\
\Phi(E_{2n,2n}) &= E_{11} + \alpha_{12}E_{12}
\end{aligned}$$

for some complex numbers $\alpha_{2k-1,2k}, \alpha_{2k+1,2k}$ for $k = 1,2,\cdots,n-1$ and $\alpha_{2n-1,2n}$.

By Theorem 2.8,

$$\begin{aligned}
\Phi(E_{2k,2k-1}) &= \gamma_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k} \text{ for } k = 1,2,\cdots,n; \text{ and}\\
\Phi(E_{2k,2k+1}) &= \gamma_{2n+1-2k,2n-2k}E_{2n+1-2k,2n-2k} \text{ for } k = 1,2,\cdots,n-1
\end{aligned}$$

for some nonzero complex numbers $\gamma_{2k,2k-1}$, $\gamma_{2k+1,2k}$ for $k = 1,2,\cdots,n-1$ and $\gamma_{2n-1,2n}$. □

Let $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ be an isomorphism. By Theorem 2.9, there exist $2n-1$ nonzero complex numbers $\gamma_{2k-1,2k}$, $\gamma_{2k+1,2k}$ for $k = 1,2,\cdots,n-1$ and $\gamma_{2n-1,2n}$ and $2n-1$ complex numbers $\alpha_{2k-1,2k}$, $\alpha_{2k+1,2k}$ for $k = 1,2,\cdots,n-1$ and $\alpha_{2n-1,2n}$ such that

$$\begin{aligned}
\Phi(E_{11}) &= E_{2n,2n} - \alpha_{2n-1,2n}E_{2n-1,2n};\\
\Phi(E_{2k-1,2k-1}) &= E_{2n+2-2k,2n+2-2k} - \alpha_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k}\\
&\quad - \alpha_{2n+3-2k,2n+2-2k}E_{2n+3-2k,2n+2-2k} \text{ for } (k = 2,3,\cdots,n);\\
\Phi(E_{2k,2k}) &= E_{2n+1-2k,2n+1-2k} + \alpha_{2n+1-2k,2n-2k}E_{2n+1-2k,2n-2k}\\
&\quad + \alpha_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k} \text{ for } (k = 1,2,\cdots,n-1);\\
\Phi(E_{2n,2n}) &= E_{11} + \alpha_{12}E_{12};\\
\Phi(E_{2k,2k-1}) &= \gamma_{2n+1-2k,2n+2-2k}E_{2n+1-2k,2n+2-2k} \text{ for } k = 1,2,\cdots,n; \text{ and}\\
\Phi(E_{2k,2k+1}) &= \gamma_{2n+1-2k,2n-2k}E_{2n+1-2k,2n-2k} \text{ for } k = 1,2,\cdots,n-1.
\end{aligned}$$

Let S be a $2n \times 2n$ matrix in $Alg\mathcal{L}_{2n}$ whose $(i,i)$-component is 1 for all $i$ and $(i,j)$-component is $-\alpha_{ij}$ for all $i,j (i \neq j)$ with $E_{ij}$ in $Alg\mathcal{L}_{2n}$ and let $B$ be a backward diagonal operator whose

(1) $(1,2n)$-component is 1,

(2) $(2, 2n-1)$-component is $\gamma_{12}^{-1}$,

(3) $(2k, 2n+1-2k)$-component is

$$\prod_{j=1}^{k-1} \gamma_{2j+1,2j} (\prod_{j=1}^{k} \gamma_{2j-1,2j})^{-1}, \text{ and}$$

(4) $(2k-1, 2n+2-2k)$-component is

$$\prod_{j=1}^{k-1} \gamma_{2j+1,2j} (\prod_{j=1}^{k-1} \gamma_{2j-1,2j})^{-1} \text{ for all } k = 2, \cdots, n,$$

and all other entries are zero.

Then $\Phi(A) = SBAB^{-1}S^{-1}$. If we put $SB = T$, then $\Phi(A) = TAT^{-1}$. From this, we have the following theorem.

**Theorem 2.10.** *A map $\Phi : (Alg\mathcal{L}_{2n})^* \to Alg\mathcal{L}_{2n}$ is an isomorphism if and only if there exist an operator $S$ in $Alg\mathcal{L}_{2n}$ with 1 as all diagonal entries and an invertible backward diagonal operator $B$ such that $\Phi(A) = SBAB^{-1}S^{-1}$.*

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