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BASICALLY DISCONNECTED SPACES AND PROJECTIVE OBJECTS

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Abstract. In this paper, we will show that every basically disconnected space is a projective object in the category $\mathbf{Tych}_\sigma$ of Tychonoff spaces and $\sigma Z^{\#}$-irreducible maps and that if X is a space such that $\beta\Lambda X = \Lambda\beta X$, then X has a projective cover in $\mathbf{Tych}_\sigma$. Moreover, observing that for any weakly Lindelöf space, $\Lambda_X : \Lambda X \longrightarrow X$ is $\sigma Z^{\#}$-irreducible, we will show that the projective objects in $\mathbf{wLind}_\sigma$ of weakly Lindelöf spaces and $\sigma Z^{\#}$-irreducible maps are precisely the basically disconnected spaces.

1. Introduction

All spaces in this paper are Tychonoff spaces and for any Tychonoff space X, $\beta_X : X \longrightarrow \beta X$ denotes the Stone-Čech compactification of X.

Gleason [4] showed that the projective objects in the category of compact spaces and continuous maps are precisely the extremally disconnected spaces and that each compact space has an essentially unique projective cover, namely its absolute (EX, k_X) (*cf.* Porter & Woods [8]).

Iliadis [6] (*resp.* Banaschewski [2]) proved similar results for the category of Hausdorff spaces (*resp.* regular spaces) and perfect continuous maps.

Henriksen, Vermeer & Woods [5] showed that the quasi-F spaces are the projective objects in the category $\mathbf{Tych}_\sigma$ of Tychonoff spaces and $Z^{\#}$-irreducible maps and that a space X has a projective cover in $\mathbf{Tych}_\sigma$ if and only if $QF(\beta X) = \beta(QFX)$.

In this paper, we will show that every basically disconnected space is a projective object in the category $\mathbf{Tych}_\sigma$ of Tychonoff spaces and $\sigma Z^{\#}$-irreducible maps and that if X is a space such that $\beta\Lambda X = \Lambda\beta X$, then X has a projective cover in $\mathbf{Tych}_\sigma$.

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Moreover, we will show that the projective objects in $\mathbf{wLind}_\sigma$ of weakly Lindelöf spaces and $\sigma Z^\#$-irreducible maps are precisely the basically disconnected spaces. For the terminology, we refer to Adámek, Herrlich & Strecker [1] and Porter & Woods [8].

2. Basically disconnected spaces

For any space X, let $Z(X)$ denote the set of all zero-sets in X, $R(X)$ the set of all regular closed sets in X and $Z(X)^\# = \{\mathrm{cl}_X(\mathrm{int}_X(A)) : A \in Z(X)\}$. Then $R(X)$ is a complete Boolean algebra in which join, meet and complemented are defined as follows:

If $A \in R(X)$ and $\{A_i : i \in I\} \subseteq R(X)$, then

$\bigvee\{A_i : i \in I\} = \mathrm{cl}_X(\bigcup\{\mathrm{int}_X(A_i) : i \in I\})$,
$\bigwedge\{A_i : i \in I\} = \mathrm{cl}_X(\mathrm{int}_X(\bigcap\{A_i : i \in I\}))$, and
$A' = \mathrm{cl}_X(X - A)$.

For any space X, $Z(X)^\#$ is a sublattice of $R(X)$.

A Boolean algebra L is called *σ-complete* if L has countable joins and hence countable meets. We note that any intersection of σ-complete Boolean subalgebras of a complete Boolean algebra L is again σ-complete and so for any sublattice M of a complete Boolean algebra L, there is the smallest σ-complete Boolean subalgebra of L containing M, which will be denoted by σM.

Definition 2.1. A space X is called *basically disconnected* if for any zero-set Z in X, $\mathrm{int}_X(Z)$ is closed in X.

It is well-known that X is basically disconnected if and only if βX is basically disconnected (*cf.* Vermeer [9]). Recall that a subspace S of a space X is called *Z-embedded in X* if for any zero-set Z in S, there is a zero-set A in X such that $Z = A \cap S$.

Proposition 2.2. *Let X be a space. Then the following are equivalent:*

(1) *X is basically disconnected.*
(2) *$Z(X)^\# = \sigma(Z(X)^\#) = B(X)$, where $B(X)$ is the set of all clopen sets in X.*
(3) *For any zero-set Z in X, $\mathrm{int}_X(Z) \cup (X - Z)$ is dense C^*-embedded in X.*

Proof. (1) $\Rightarrow$ (2). Since X is a basically disconnected space, $B(X) = Z(X)^{\#}$. Since $Z(X)^{\#} \subseteq \sigma(Z(X)^{\#})$, it is enough to show that $Z(X)^{\#}$ is a σ-complete Boolean algebra. Let $\{A_n : n \in N\}$ be a sequence in $Z(X)^{\#}$. Then

$$\bigwedge\{A_n : n \in N\} = \mathrm{cl}_X(\mathrm{int}_X(\bigcap\{A_n : n \in N\})).$$

Since a countable intersection of zero sets is a zero set, $\bigcap\{A_n : n \in N\}$ is a zero set in X and so $\mathrm{int}_X(\bigcap\{A_n : n \in N\})$ is closed because X is basically disconnected. Hence $\bigwedge\{A_n : n \in N\} \in Z(X)^{\#}$. Since $Z(X)^{\#} = B(X)$ is Boolean, $Z(X)^{\#}$ is a σ-complete Boolean algebra.

(2) $\Rightarrow$ (3). Take any zero-set Z in X. Since $\mathrm{int}_X(\mathrm{cl}_X(\mathrm{int}_X(Z))) = \mathrm{int}_X(Z)$, by (2), $\mathrm{int}_X(Z) \cup (X - Z)$ is a dense cozero-set in X and so $\mathrm{int}_X(Z) \cup (X - Z)$ is Z-embedded in X (*cf.* Blair [3]). Let $T = \mathrm{int}_X(Z) \cup (X - Z)$. Let A and B be zero-sets in T such that $\mathrm{int}_T(A) \cap \mathrm{int}_T(B) = \varnothing$. There are zero-sets C and D in X such that $A = C \cap T$ and $B = D \cap T$. Since T is dense in X, $\mathrm{int}_X(C) \cap \mathrm{int}_X(D) = \varnothing$. By (2), $\mathrm{cl}_X(\mathrm{int}_X(C)) \cap \mathrm{cl}_X(\mathrm{int}_X(D)) = \varnothing$. By Urysohn's extension theorem, T is C^*-embedded in X.

(3) $\Rightarrow$ (1). Take any zero-set$Z$ in X. By (3), $\mathrm{int}_X(Z) \cup (X - Z)$ is C^*-embedded in X. Since $\mathrm{int}_X(Z)$ and $X - Z$ are disjoint clopen stes in X,

$$\mathrm{cl}_X(\mathrm{int}_X(Z)) \cap \mathrm{cl}_X(X - Z) = \varnothing$$

and so $\mathrm{cl}_X(\mathrm{int}_X(Z)) \subseteq \mathrm{int}_X(Z)$. Hence X is basically disconnected. $\square$

3. Minimal basically disconnected covers of spaces

A map $f : Y \longrightarrow X$ is called *covering* if f is onto, continuous and perfect.

Definition 3.1. A covering map $f : Y \longrightarrow X$ is called

(a) *$Z^{\#}$-irreducible* if $\{f(A) : A \in Z(Y)^{\#}\} = Z(X)^{\#}$, and

(b) *$\sigma Z^{\#}$-irreducible* if $\{f(A) : A \in \sigma Z(Y)^{\#}\} = \sigma Z(X)^{\#}$.

For any map $f : Y \longrightarrow X$, and $\mathcal{A} \subseteq \mathcal{P}(Y)$ and $\mathcal{B} \subseteq \mathcal{P}(X)$, let

$$f(\mathcal{A}) = \{f(A) : A \in \mathcal{A}\} \text{ and } f^{-1}(\mathcal{B} = \{\mathrm{cl}_Y(f^{-1}(B)) : B \in \mathcal{B}\}.$$

For any covering map $f : Y \longrightarrow X, f^{-1}(Z(X)^{\#}) \subseteq Z(Y)^{\#}$ and hence $\sigma Z(X)^{\#} \subseteq f(\sigma Z(Y)^{\#})$. Thus a covering map $f : Y \longrightarrow X$ is $\sigma Z^{\#}$-irreducible if and only if $f(\sigma Z(Y)^{\#}) \subseteq \sigma Z(X)^{\#}$ and if f is $Z^{\#}$-irreducible, then it is $\sigma Z^{\#}$-irreducible.

Proposition 3.2. *For any covering maps $g : Y \longrightarrow W$, $h : W \longrightarrow X$, $h \circ g$ is $\sigma Z^{\#}$-irreducible if and only if h and g are $\sigma Z^{\#}$-irreducible.*

Definition 3.3. Let X be a subspace of a space Y. Then X is called *$\sigma Z^{\#}$-embedded in Y* if for any $A \in \sigma Z(X)^{\#}$, there is B $\in \sigma Z(Y)^{\#}$ such that $A = B \cap X$. A subspace X of a space Y is called *$Z^{\#}$-embedded in Y* if for any A in $Z(X)^{\#}$, there is B in $Z(Y)^{\#}$ such that $A = B \cap X$.

Proposition 3.4. *If X is a dense Z-embedded subspace of a space Y, then X is $Z^{\#}$-embedded in Y.*

Theorem 3.5. *Consider the commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ {\scriptstyle j_1}\big\downarrow & & \big\downarrow{\scriptstyle j_2} \\ Y & \xrightarrow{g} & W \end{array}$$

where X, Y, P and W are spaces with $P \subseteq Y$ and $X \subseteq W$, j_1 and j_2 are dense embeddings and f, g are covering maps. Then g is $\sigma Z^{\#}$-irreducible and P is $\sigma Z^{\#}$-embedded in Y if and only if f is $\sigma Z^{\#}$-irreducible and X is $\sigma Z^{\#}$-embedded in W.

Proof. ($\Rightarrow$) Take any $A \in \sigma Z(P)^{\#}$. Since P is $\sigma Z^{\#}$-embedded in Y, there is B in $\sigma Z(Y)^{\#}$ such that $A = B \cap P$. Note that $f(A) = f(B \cap P) = g(B) \cap X$ (*cf.* Porter & Woods [8]). Since g is $\sigma Z^{\#}$-irreducible, $f(A) \in \sigma Z(X)^{\#}$. Thus f is $\sigma Z^{\#}$-irreducible. Let C $\in \sigma Z(X)^{\#}$. Then $\mathrm{cl}_P(f^{-1}(\mathrm{int}_X(C))) \in \sigma Z(P)^{\#}$. Since P is $\sigma Z^{\#}$-embedded in Y, there is $D \in \sigma Z(Y)^{\#}$ such that $D \cap P = \mathrm{cl}_P(f^{-1}(\mathrm{int}_X(C)))$. Then $C = f(D \cap P) = g(D) \cap X$. Since g is $\sigma Z^{\#}$-irreducible, $g(D) \in \sigma Z(W)^{\#}$ and so X is $\sigma Z^{\#}$-embedded in W.

($\Leftarrow$) Take any $A \in \sigma Z(Y)^{\#}$. Since j_1 is a dense embedding, $A \cap P \in \sigma Z(P)^{\#}$ and $f(A \cap P) = g(A \cap P) = g(A) \cap X$. Since f is $\sigma Z^{\#}$-irreducible, $g(A) \cap X \in \sigma Z(X)^{\#}$. Since X is $\sigma Z^{\#}$-embedded in W, there is $B \in \sigma Z(W)^{\#}$ such that $g(A) \cap X = B \cap X$. Since j_2 is a dense embedding and $g(A)$, B are regular closed sets in W, $g(A) = B$. Thus g is $\sigma Z^{\#}$-irreducible.

Take any $C \in \sigma Z(P)^{\#}$. Since f is $\sigma Z^{\#}$-irreducible, $f(C) \in \sigma Z(X)^{\#}$. Since X is $\sigma Z^{\#}$-embedded in W, there is $D \in \sigma Z(W)^{\#}$ such that $f(C) = D \cap X$. Since g is a covering map, $\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \in \sigma Z(Y)^{\#}$. Then

$$f(\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \cap P) = g(\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D)))) \cap X = D \cap X = f(C).$$

Hence $\mathrm{cl}_Y(g^{-1}(\mathrm{int}_W(D))) \cap P = C$. Thus P is $\sigma Z^{\#}$-embedded in Y. □

Definition 3.6. Let X be a space. Then

(a) a pair (Y, f) is called *a cover of* X if $f : Y \longrightarrow X$ is a covering map,
(b) a cover (Y, f) is called *a basically disconnected cover of* X if Y is basically disconnected, and
(c) a basically disconnected cover (Y, f) of X is called *a minimal basically disconnected cover of* X if for any basically disconnected cover (Z, g) of X, there is a covering map $h : Z \longrightarrow Y$ with $f \circ h = g$.

Recall that a space X is called *weakly Lindelöf* if for any open cover $\mathcal{U}$ of X, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup \mathcal{V}$ is dense in X and a space X is called *locally weakly Lindelöf* if every element of X has a weakly Lindelöf neighborhood.

For any compact space X, there is a minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ of X such that ΛX is the Stone-space of $\sigma Z(X)^{\#}$ and $\Lambda_X(\alpha) = \bigcap \alpha$ for $\alpha \in \sigma Z(X)^{\#}$ (*cf.* Vermeer [9]). Vermeer showed that every Tychonoff space has a minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and in Kim [7], it was shown that if X is locally weakly Lindelöf space, then ΛX is given by the fixed $\sigma Z(X)^{\#}$-ultrafilter space.

Lemma 3.7. *Let X be a weakly Lindelöf space. Then for any $A \in \sigma Z(X)^{\#}$, $\mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(A))) = A^*$, where $A^* = \{\alpha \in \Lambda X : A \in \alpha\}$.*

Proof. Let $A \in \sigma Z(X)^{\#}$. Then for any $\alpha \in A^*, \Lambda_X(\alpha) \in A$. Take any $x \in \mathrm{int}_X(A)$. Since Λ_X is onto, there is $\alpha \in \Lambda X$ such that $\Lambda_X(\alpha) = x$. For any $C \in \alpha$, $x \in \mathrm{cl}_X(\mathrm{int}_X(C))$ and hence $\mathrm{int}_X(C) \cap \mathrm{int}_X(A) \neq \varnothing$. Thus for any $C \in \alpha, C \wedge A \neq \varnothing$. Since α is a $\sigma Z(X)^{\#}$-ultrafilter, $A \in \alpha$ (*cf.* Porter & Woods [8]). Thus $x = \Lambda_X(\alpha) \in \Lambda_X(A^*)$ and so $\mathrm{int}_X(A) \subseteq \Lambda_X(A^*)$. Since Λ_X is closed, $\mathrm{cl}_X(\mathrm{int}_X(A)) \subseteq \Lambda_X(A^*)$. Hence $A = \mathrm{cl}_X(\mathrm{int}_X(A)) = \Lambda_X(A^*)$. Since Λ_X is a covering map and

$$\Lambda_X(\mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(A)))) = A, \quad \mathrm{cl}_{\Lambda X}(\Lambda_X^{-1}(\mathrm{int}_X(A))) = A$$

(*cf.* Kim [7]). □

Corollary 3.8. *For any weakly Lindelöf space X, $\Lambda_X : \Lambda X \longrightarrow X$ is $\sigma Z^{\#}$-irreducible.*

For any space X, let $(\Lambda\beta X, \Lambda)$ be the minimal basically disconnected cover of βX. Kim [7] has shown that for any space X, $\Lambda^{-1}(X)$ is C^*-embedded in $\Lambda\beta X$ if and only if $\Lambda^{-1}(X)$ is $Z^{\#}$-embedded (or Z-embedded) in $\Lambda\beta X$.

Theorem 3.9. *Let X be a space. Then $\Lambda_X : \Lambda X \longrightarrow X$ is $\sigma Z^{\#}$-irreducible if and only if $\Lambda^{-1}(X)$ is C^*-embedded in $\Lambda\beta X$.*

Proof. Suppose that Λ_X is $\sigma Z^{\#}$-irreducible. Let Λ_0 be the restriction and corestriction of Λ to $\Lambda^{-1}(X)$ and X, respectively. Then there is a covering map $g : \beta\Lambda X \longrightarrow \beta X$ such that $\beta_X \circ \Lambda_X = g \circ \beta_{\Lambda X}$. Since $\beta\Lambda X$ is a basically disconnected space, there is a covering map $h : \beta\Lambda X \longrightarrow \Lambda\beta X$ with $\Lambda \circ h = g$. Hence there is a covering map $k : \Lambda X \longrightarrow \Lambda^{-1}(X)$ with $\Lambda_0 \circ k = \Lambda_X$ and $j \circ k = h \circ \beta_{\Lambda X}$, where $j : \Lambda^{-1}(X) \longrightarrow \Lambda\beta X$ is the inclusion map. Since Λ_X is $\sigma Z^{\#}$-irreducible, by Proposition 3.2, k and Λ_0 are $\sigma Z^{\#}$-irreducible maps. By Theorem 3.5, $\Lambda^{-1}(X)$ is $Z^{\#}$-embedded in $\Lambda\beta X$ and hence $\Lambda^{-1}(X)$ is C^*-embedded in $\Lambda\beta X$.

If $\Lambda^{-1}(X)$ is C^*-embedded in $\Lambda\beta X$, then $\Lambda^{-1}(X)$ is $\sigma Z^{\#}$-embedded in $\Lambda\beta X$ and so, by Theorem 3.5 and Corollary 3.8, Λ_X is $\sigma Z^{\#}$-irreducible. □

4. Projective objects and basically disconnected spaces

Definition 4.1. Let $\mathbf{C}$ be a topological subcategory of the category **Top** of topological spaces and continuous maps:

(a) An object X in $\mathbf{C}$ is called *a projective object* in $\mathbf{C}$ if for any objects $Y, Z \in \mathbf{C}$, morphism $f : X \longrightarrow Y$ in $\mathbf{C}$ and onto morphism $g : Z \longrightarrow Y$ in $\mathbf{C}$, there is a morphism $h : X \longrightarrow Z$ in $\mathbf{C}$ with $g \circ f = h$.

(b) A pair (X, f) is called a projective cover of an object Y in $\mathbf{C}$ if X is a projective object in $\mathbf{C}$ and $f : X \longrightarrow Y$ is a morphism of $\mathbf{C}$ that is an onto, closed and irreducible map.

Gleason [4] showed that the projective objects in the category of compact spaces and continuous maps are precisely the extremely disconnected spaces and that each compact space has an essentially unique projective cover, namely its absolute (EX, k_X). Iliadis [6] (*resp.* Banaschewski [2]) proved similar results for the category of Hausdorff spaces (*resp.* regular spaces) and perfect continuous maps.

A Tychonoff topological space is called a quasi F-space if each dense cozero-set of X is C^*-embedded in X. Henriksen, Vermeer & Woods [5] showed that the quasi-F spaces are the projective objects in the category of Tychonoff spaces and $Z^{\#}$-irreducible maps. In this section, we will investigate projective objects for the category of basically disconnected spaces and $\sigma Z^{\#}$-irreducible maps.

Let $\mathbf{wLind}_c$ (*resp.* $\mathbf{Tych}_c$) be the category of weakly Lindelöf spaces (*resp.* Tychonoff spaces) and $\sigma Z^{\#}$-irreducible maps.

Lemma 4.2. *Let $X \in \mathbf{wLind}_c$ and $f : Y \longrightarrow X$ be a $\sigma Z^{\#}$-irreducible map. Then there is a $\sigma Z^{\#}$-irreducible map $k : \Lambda X \longrightarrow Y$ with $f \circ k = \Lambda_X$.*

Proof. Since X is weakly Lindelöf and $f : Y \longrightarrow$ X is a covering map, Y is weakly Lindelöf. Since $f \circ \Lambda_Y$ is a covering map, there is a covering map $g : \Lambda Y \longrightarrow \Lambda X$ with $f \circ \Lambda_Y = \Lambda_X \circ g$. By Proposition 3.2, g is $\sigma Z^{\#}$-irreducible. Since ΛX and ΛY are basically disconnected spaces, g is a homeomorphism. Let $k = \Lambda_Y \circ g^{-1}$. Then k is a $\sigma Z^{\#}$-irreducible map and $f \circ k = \Lambda_X$. □

Theorem 4.3. *Let X be a space. Then*

(a) *if X is a basically disconnected space, then X is a projective object in $\mathbf{Tych}_\sigma$,*

(b) *if X is a projective object in $\mathbf{Tych}_\sigma$ and for any zero-set Z in X, $\mathrm{int}_X(Z) \cup (X - Z)$ is Z-embedded in X, then X is a basically disconnected space, and*

(c) *if $\beta\Lambda X = \Lambda\beta X$, then X has a projective cover in $\mathbf{Tych}_\sigma$.*

Proof. (a) Suppose that X is a basically disconnected space. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be $\sigma Z^{\#}$-irreducible maps and $Y, Z \in \mathbf{Tych}_\sigma$. Since $\Lambda\beta X$ is a basically disconnected space, there is a covering map $h : \beta Z \longrightarrow \beta Y$ such that $h \circ \beta_Z = \beta_Y \circ g$. By Theorem 3.5, h is $\sigma Z^{\#}$-irreducible. Since βX is basically disconnected, there is a covering map $k : \beta X \longrightarrow \Lambda\beta Y$ such that $\Lambda_{\beta Y} \circ k = l$, where $l : \beta X \longrightarrow \beta Y$ is the Stone extension of $\beta_Y \circ f$. Since l is $\sigma Z^{\#}$-irreducible, k is $\sigma Z^{\#}$-irreducible. Hence k is a homeomorphism, because βX and $\Lambda\beta Y$ are basically disconnected. Note that there is a covering map $t : \Lambda\beta Z \longrightarrow \Lambda\beta Y$ such that $h \circ \Lambda_{\beta Z} = \Lambda_{\beta Y} \circ t$. Since h is $\sigma Z^{\#}$-irreducible, t is $\sigma Z^{\#}$-irreducible and hence t is a homeomorphism. Let $m = \Lambda_{\beta Z} \circ t^{-1} \circ k \circ \beta_X$. Since $\Lambda_{\beta Z} \circ t^{-1} \circ k$ is a covering map, $m(X) = Z$ (*cf.* Porter & Woods [8]). Hence $g \circ m = f$ and m is $\sigma Z^{\#}$-irreducible. Thus X is a projective object in $\mathbf{Tych}_\sigma$.

(b) Suppose that X is not basically disconnected. By Proposition 2.2, there is a zero-set$Z$ in X such that $(X - Z) \cup \mathrm{int}_X(Z)$ is not C^*-embedded in X. Let $T = (X - Z) \cup \mathrm{int}_X(Z)$. By the assumption, T is Z-embedded in X. There is a continuous map $f : \beta T \longrightarrow \beta X$ such that $f \circ \beta_T = \beta_X \circ j_T$, where $j_T : T \longrightarrow X$ is the inclusion map. Clearly, f is $\sigma Z^{\#}$-irreducible. Let $Y = f^{-1}(X)$. Let $k : Y \longrightarrow X$ be the restriction (*resp.* corestriction) of f to Y (*resp.* X). By Theorem 3.5, k is $\sigma Z^{\#}$-irreducible. Since the identity map $1_X : X \longrightarrow X$ is $\sigma Z^{\#}$-irreducible and X

is a projective object in $\mathbf{Tych}_\sigma$, there is a $\sigma Z^{\#}$-irreducible map $h : X \longrightarrow Y$ such that $k \circ h = 1_X$. Hence h is a homeomorphism and so k is a homeomorphism. Moreover, there is a continuous map $l : T \longrightarrow Y$ with $k \circ l = j_T$. Hence l is an embedding and so T is C^*-embedded in Y. Since T is not C^*-embedded in X, there are disjoint zero-sets A_1 and A_2 in T such that $\mathrm{cl}_X(A_1) \cap \mathrm{cl}_X(A_2) \neq \varnothing$. Pick $p \in \mathrm{cl}_X(A_1) \cap \mathrm{cl}_X(A_2)$. Since h is continuous, $h(p) \in \mathrm{cl}_Y(h(A_1)) \cap \mathrm{cl}_Y(h(A_2))$ and hence $\mathrm{cl}_Y(h(A_1)) \cap \mathrm{cl}_Y(h(A_2)) \neq \varnothing$. This is a contradiction that T is C^*-embedded in Y. Hence X is a basically disconnected space.

(c) Suppose that $\beta\Lambda X = \Lambda\beta X$. Then there is a homeomorphism $h : \beta\Lambda X \longrightarrow \Lambda\beta X$ such that $\beta_X \circ \Lambda_X = \Lambda \circ h \circ \beta_{\Lambda X}$. By Theorem 3.5, Λ_X is $\sigma Z^{\#}$-irreducible. By (a), $(\Lambda X, \Lambda_X)$ is the projective cover of X in $\mathbf{Tych}_\sigma$. □

Corollary 4.4.

(a) *The projective objects in* $\mathbf{wLind}_\sigma$ *are precisely the basically disconnected spaces.*

(b) *If* (Y, g) *is a projective cover of* X *in* $\mathbf{wLind}_\sigma$*, then* Y *and* ΛX *are homeomorphic.*

Proof. (a) If X is a basically disconnected and weakly Lindelöf space, then by (a) in Theorem 4.3, X is a projective object in $\mathbf{wLind}_\sigma$.

Suppose that X is a projective objective in $\mathbf{wLind}_\sigma$. Then $\Lambda_X : \Lambda X \longrightarrow X$ is $\sigma Z^{\#}$-irreducible and hence there is a $\sigma Z^{\#}$-irreducible map $h : X \longrightarrow \Lambda \mathrm{X}$ such that $\Lambda_X \circ h = 1_X$. Hence h is a homeomorphism and so X is a basically disconnected space.

(b) By Lemma 4.2, there is a $\sigma Z^{\#}$-irreducible map $k : \Lambda X \longrightarrow Y$ such that $\Lambda_X = g \circ k$. By (a), Y is a basically disconnected space and hence k is a homeomorphism. □

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