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SOME RESULTS FROM THE SPACES OF ALMOST CONTINUOUS FUNCTIONS

JOUNG NAM LEE

ABSTRACT. In this paper, we study the space of almost continuous functions with the topology of uniform convergence. And we investigate some properties of this space.

1. INTRODUCTION

The concept of almost continuous functions was introduced by Stallings [13] in 1959. An almost continuous function is one whose graph can be approximated by graphs of continuous functions(*cf.* Hocking & Young [5], Kelley [7]). Thereafter many investigations have been carried out, in general theoretical fields and also in different application sides, based on this concept. The idea of "the graph topology" was introduced by Naimpally [10] in 1964. The idea enabled him to tackle almost continuous functions (*cf.* Naimpally [11]). Since then, there have appeared a large number of observations of the fact that these functions, in the case of the mapping of closed intervals into themselves, possess a fixed point caused the investigation of topological properties of those mappings (*cf.* Brown [3], Kelley [8], Naimpally [10], Singal & Singal [12]). On the other hand, several investigators have studied algebraic operations performed on almost continuous functions (*cf.* Kellum [9], Stronska [14]). and then have been considered the structure of the space of almost continuous functions with the topology (*cf.* Beer [2], Hueber [6]). It is also well known that every almost continuous function is a Darboux function, but need not be a function of the first class of Baire (*cf.* Brown [3]). In this paper, we study the space of almost continuous functions with the topology of uniform convergence. And we investigate

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some properties of this space. That is, we investigate that Darboux function (in the sense of Brown [3]) of the first class of Baire is an almost continuous function.

2. ALMOST CONTINUOUS FUNCTIONS

Let X and Y be topological spaces and let F denote the set of all functions on X to Y. Let C denote the subset of F consisting of all continuous functions. For $f \in F$, the graph of f, denoted by $G(f)$, is the set

$$\{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

Let $X \times Y$ be assigned the usual product topology. A function $f \in F$ is called almost continuous if for any open U in $X \times Y$ containing $G(f)$, there exists a $g \in C$ such that $G(g) \subset U$ (*cf.* Stalling [13]). Whereas every continuous function is almost continuous, there exist almost continuous functions which are not continuous. If $X = Y = \mathbb{R}$ where $\mathbb{R}$ denotes the set of all real numbers with the usual topology, then the function $f \in F$ defined by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

is almost continuous but not continuous.

A subset C of $X \times Y$ is called a blocking set of f in $X \times Y$ if C is closed in $X \times Y$, $C \cap G(f) = \varnothing$ and C intersects $G(g)$ whenever g is a continuous function. If C has no proper subset which is a blocking set of f then C is called a minimal blocking set of f in $X \times Y$.

Throughout the work, we denote by I the closed unit interval of the real line. And ρ denotes the metric in I^2. The set of all almost continuous functions $f : I \to I$ and the metric space consisting of these functions with the metric ρ^* of uniform convergence is denoted by W. And let V denote the set of all $f \in W$ which are functions of the first class of Baire.

For a set $A \subset I^2$, $\pi_X(A)$ denotes the projection of A onto X. In particular, if A is closed, we write

$$m(A) = \{(x^*, y^*) \in A \mid y^* = \inf\{y \mid (x^*, y) \in A\}\}.$$

In case of $A \subset I$ and $f \in W$, we say that $f|_A$ is the restriction of f to A. If $f(I) \subset B$, we denote by $\tilde{f}|_B$ a function from I to B as a subspace of I.

Let X be a metric space, with a metric d. We define an open d-ball in X by $B_d(a,r) = \{x \mid d(x,a) < r\}$, where a is referred to as the center of the ball and r the radius. Clearly, $B_d(a,r) \subset B_d(a,r^*)$ if $r \leq r^*$ and $B_d(a,0) = \varnothing$. In the future, we will omit the distinguishing d whenever the metric is clear from the context.

For $D \subset X$, $x \in X$ and $R > 0$, we shall denote by $\lambda(x,R,D)$ the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,R) \setminus D$. Remember that a number

$$d(D,x) = 2 \limsup_{R \to 0+} \frac{\lambda(x,R,D)}{R}$$

is called the *porosity* of D at x. D is porous at x if $d(D,x) > 0$. Also $E \subset X$ is called *superporous* at $x \in X$ if $E \cup F$ is porous at x whenever E is porous at x.

Let X, Y be topological spaces. If $f : X \to Y$ is a continuous function such that $f(X) \subset B \subset Y$, then $\tilde{f}|_B$ is a continuous function. Of course, it is possible to put a metric on I, which gives use to the usual topology τ and an almost continuous function $g : I \to (I, \tau)$ such that $g(I) = [\frac{1}{2}, 1]$ and $\tilde{g}|_{[\frac{1}{2},1]}$ is not almost continuous. Hence we have the following immediately:

Proposition 2.1. *Let $f : I \to I$ be a function such that $f(I) \subset [m,n]$ where $m < n$. Then $f \in V$ if and only if $\tilde{f}|_{[m,n]}$ is almost continuous.*

Finally, we close this section with the following proposition showing some properties of almost continuous functions.

Proposition 2.2 (Kellum [8]). *A function f from I to I is almost continuous if and only if $f|_{[a,b]}$ is almost continuous for any $a, b \in [0,1]$.*

It is well known that there exists an almost continuous real function defined on some compact subset E of the plane, such that the restriction of this function to the interior of E, written as E^i is not almost continuous (*cf.* Kellum [8]).

3. Main results

Theorem 3.1. *Let $f : I \to [m,n]$ be a function for which there exists $(a,b) \subset I$ such that $f|_{I \setminus (a,1]}$ and $f|_{I \setminus [0,b)}$ are almost continuous and f is not almost continuous. Then $\pi_X(E_f) \cap (a,b)$ is a non-degenerate interval, where $E_f \subset I \times [m,n]$ is the minimal blocking set of f.*

Proof. It suffices to prove that $\pi_X(E_f) \cap (a,b) \neq \varnothing$. For this purpose, assume that $\pi_X(E_f) \cap (a,b) = \varnothing$. By Kellum [8] and Stronska [14], we have either $E_f \subset \{(x,y)|\ x \leq a\}$ or $E_f \subset \{(x,y)|\ x \geq b\}$. Without loss of generality, we may assume that $E_f \subset \{(x,y)|\ x \leq a\}$. Of course, $E_f \cap G(f) = \varnothing$, and so, $E_f \cap G(f^*) = \varnothing$, where $f^* = f|_{I\setminus(a,1)}$. Let $A = ((I \setminus (a,1]) \times [m,n]) \setminus E_f$. Then A is a neighborhood of $G(f^*)$. This implies that A contains the graph of some continuous function $g^* : I \setminus (a,1] \to [m,n]$. Let g be as follows: for any x of I,

$$g(x) = \begin{cases} g^*(x) & \text{if } x \in I \setminus (a,1] \\ g^*(a) & \text{if } x \geq a. \end{cases}$$

Then g is continuous and $E_f \cap G(g) = \varnothing$, which is impossible because E_f is the blocking set of f. This contradictions prove the theorem. □

Now let us establish the following theorem.

Theorem 3.2. *V is perfect and superporous at each point of the space W.*

Proof. Assume that $f \in V, \mu > 0$ and let f be continuous at $x^* \in (0,1)$. Then we must have $f(x^*) < 1$ or $f(x^*) > 0$. Without loss of generality, assume that $f(x^*) < 1$. Now suppose that $0 < \mu^* < \frac{\mu}{3}$ satisfying $f(x^*) + 2\mu^* < 1$. Then there exists $\delta > 0$ such that $[x^* - \delta, x^* + \delta] \subset (0,1)$ and

$$f([x^* - \delta, x^* + \delta]) \subset (f(x^*) - \mu^*, f(x^*) + \mu^*).$$

Let $g : I \to I$ be a function such that $g(x) = 0$ for every $x \notin [x^* - \delta, x^* + \delta]$, $g(x^*) = \mu^*$ and g is linear in $[x^* - \delta, x^*]$ and $[x^*, x^* + \delta]$. Then g is continuous, and so by Bruckner [4, Theorem 2.3.2], we have $h = f + g \in V$. It is trivial to check that $f \neq h$ and $\rho^*(f,h) < \mu$. By the method of the consideration similar as above, we conclude that V is perfect.

Now let us show that V is superporous at each point t of W. Let D be any porous set at t. And also suppose the condition of $d(D,t) > 0$. Let $d(D,t) = 2m > 0$. Then this implies that there exists a sequence $\{R_n\}$ such that $R_n \to 0$ and

$$\lim_{n\to 0} \frac{\lambda(t, R_n, D)}{R_n} = \alpha.$$

Let n be a fixed number. By the definition of $\lambda(t, R_n, D)$, there exist s in W and

$$\sigma_n \geq \lambda(t, R_n, D) - \frac{1}{2^n} \cdot R_n > 0$$

with $B(s,\sigma_n) \subset B(t,R_n) \setminus D$. In addition, if

$$\overline{B\left(s,\frac{\sigma_n}{2}\right)} \cap V = \varnothing, \tag{1}$$

then $\lambda(t,R_n,D\cup V) \geq \frac{\sigma_n}{2}$. We may assume that $\eta \in \overline{B(s,\frac{\sigma_n}{2})} \cap V$. From this assumption we deduce that

$$B\left(\eta,\frac{\sigma_n}{2}\right) \subset B\left(s,\frac{\sigma_n}{2}\right) \subset B(t,R_n). \tag{2}$$

Put $\theta = \frac{\sigma_n}{2}$. Let us assume that η is continuous at x^*. This implies the existence of a non-degenerate interval $[a,b]$ such that

$$\eta([a,b]) \subset \left(\eta(x^*) - \frac{\theta}{4},\ \eta(x^*) + \frac{\theta}{4}\right).$$

For points x, y in (a,b), we define $x \sim y$ if and only if $x - y \in \mathbb{Q}$ where $\mathbb{Q}$ denotes the set of all rational numbers. One can easily verify that this relation $\sim$ is an equivalence relation in (a,b).

Let Γ denote the set of all equivalence classes of the above relation. For every x in (a,b), let $M \in \Gamma$ be a set with $x \in M$. Let $\phi : \Gamma \to M$ be a map, which is onto, represent N as the set of all functions of the first class of Baire from I to $[\eta(x^*) - \frac{\theta}{4}, \eta(x^*) + \frac{\theta}{4}]$. Consider the function g defined by

$$g(x) = \begin{cases} \eta(x) & \text{if } x \leq a \ \text{ or } \ x \geq b \\ (\phi(M))(x) & \text{if } x \in (a,b). \end{cases}$$

Now let us show that $g \in W$. For this purpose, we may assume that $g \notin W$. Then, according to Proposition 2.2, we can easily show that the function $j : I \to I$ defined by

$$j(x) = \begin{cases} \eta(a) & \text{if } x \leq a \\ g(x) & \text{if } x \in (a,b) \\ \eta(b) & \text{if } x \geq b \end{cases}$$

does not belong to W.

By Proposition 2.1, it follows that $g^* = \tilde{j}|_{[\eta(x^*)-\frac{\theta}{4},\eta(x^*)+\frac{\theta}{4}]}$ is not almost continuous, and so, there exists a minimal blocking set $E_{g^*} \subset I \times [\eta(x^*) - \frac{\theta}{4}, \eta(x^*) + \frac{\theta}{4}]$ of g^*. From Proposition 2.2 we deduce that the assumption of Theorem 3.1 are fulfilled, and so $(a, b) \cap E_{g^*}$ contains some non-degenerate interval $[r, s]$. Hence we have that

$$\begin{aligned}(3)\quad &\{(x, \min\{y \in I : (r, s) \in E_{g^*}\}) : x \leq r\} \\ &\qquad \cup \{(x, \min\{y \in I : (s, y) \in E_{g^*}\}) : x \geq s\} \\ &\qquad\qquad \cup \min\{(E_{g^*} \cap \{(x, y) : r \leq x \leq s\})\}\end{aligned}$$

is the graph of some function $\zeta \in N$. Let P be any set in Γ which satisfies $\phi(P) = \zeta$ and $c \in P \cap [r, s]$. Then $g^*(c) = g(c) = \zeta(c)$ and hence $(c, g^*(c)) \in E_{g^*}$. This contradicts the fact that E_{g^*} is the blocking set of g^*. Consequently, we obtain $g \in W$.

We may assume that $\rho^*(\eta, g) \leq \frac{\theta}{2}$. From this assumption we have

$$B\left(g, \frac{\theta}{2}\right) \subset B(\eta, \theta). \tag{4}$$

Next let us prove that, for every function $\varphi \in V$, $\varphi \notin B(g, \frac{\theta}{5})$. For this purpose, assume that there exists φ with $\varphi \in B(g, \frac{\theta}{5}) \cap V$. Since φ is continuous, then there exists a point y^* of φ in (a, b). Now we deduce that $\varphi(y^*) \in [\eta(x^*) - \frac{\theta}{4}, \eta(x^*) + \frac{\theta}{4}]$. Without loss of generality assume that $\varphi(y^*)$ is a member of $[\eta(x^*), \eta(x^*) + \frac{\theta}{4}]$. In case $\varphi(y^*) \in [\eta(x^*) - \frac{\theta}{4}, \eta(x^*)]$, the proof is analogous. Hence there exists a non-degenerate interval $(p, q) \subset (a, b)$ with $\varphi((p, q)) \subset (\eta(x^*) - \frac{\theta}{20}, 1]$.

Let $S \in \Gamma$ be a set such that $\phi(S)$ is constant and equal to $\eta(x^*) - \frac{\theta}{4}$. Moreover, let $z \in S \cap (p, q)$. Then we have $g(z) = \eta(x^*) - \frac{\theta}{4}$ and $\phi(z) > \eta(x^*) - \frac{\theta}{20}$. Consequently, we obtain $\varphi^*(g, \varphi) \geq \frac{\theta}{5}$. This contradicts the assumption that $\varphi \in B(g, \frac{\theta}{5})$.

According to (2), (3) and the above, we have $\lambda(t, R_n, D \cup V) \geq \frac{\sigma_n}{10}$. Applying (1), we have also

$$\frac{\lambda(t, R_n, D \cup V)}{R_n} \geq \frac{\lambda(t, R_n, D) - \frac{1}{2^n} R_n}{10 R_n}.$$

Thus, to complete the proof, it suffices to observe that

$$\limsup_{R \to 0+} \frac{\lambda(t, R, D \cup V)}{R} > 0.$$

According to the definition of porosity, this can be see easily. □

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School of the Liberal Arts, Seoul National University of Technology, Gongneung-2-dong, Nowon-gu, Seoul 139-743, Korea
Email address: `ljnam@duck.snut.ac.kr`