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PEBBLING NUMBERS OF THE COMPOSITIONS OF TWO GRAPHS

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ABSTRACT. Let G be a connected graph. A pebbling move on a graph G is the movement of taking two pebbles off from a vertex and placing one of them onto an adjacent vertex. The pebbling number $f(G)$ of a connected graph G is the least n such that any distribution of n pebbles on the vertices of G allows one pebble to be moved to any specified, but arbitrary vertex by a sequence of pebbling moves. In this paper, the pebbling numbers of the compositions of two graphs are computed.

1. INTRODUCTION

Pebbling in graphs was first considered by Chung [1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A *pebbling move* is defined as the process of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. We say that we can pebble a vertex v, the target vertex, if we can apply pebbling moves repeatedly so that it is possible to reach a configuration with at least one pebble at v. The *pebbling number of a vertex* v in a graph G, denoted by $f(G,v)$, is defined to be the smallest integer m which guarantees that any starting pebble configuration with m pebbles allows pebbling v. We define the *pebbling number of* G, denoted by $f(G)$, as the maximum of $f(G,v)$ over all vertices v.

A graph G is called *demonic* if $f(G)$ is equal to the number of its vertices. So far, very little is known regarding $f(G)$ (see [1]–[6]). If one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Also, if w is at distance d from v, and $2^d - 1$ pebbles are placed on w, then no pebble can be moved

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to v. So it is clear (see Chung [1]) that $f(G) \geq \max\{|V(G)|,\ 2^D\}$, where $|V(G)|$ is the number of vertices of the graph G and D is the diameter of G.

Definition 1. Let G and H be connected graphs. If $V(H) = V(G), E(H) \subset E(G)$, then H is called a *spanning subgraph* of G.

If H is a spanning subgraph of a graph G, then $f(H) \geq f(G)$. Furthermore, we know that K_n and $K_{s,t}$ are demonic when $s > 1$ and $t > 1$ (see Chang [1] and Feng et al. [3]), where K_n is the complete graph on n vertices, and $K_{s,t}$ is the complete bipartite graph such that two partition sets have s and t vertices respectively. But $f(P_n) = 2^{n-1}$ (see Chang [1]), i.e., the graph P_n is not demonic when $n > 2$, where P_n is the path on n vertices.

In this paper, we study the pebbling number of the composition of two graphs. Throughout this paper, G denotes a simple connected graph with vertex set $V(G)$ and edge set $E(G)$.

2. THE PEBBLING NUMBER OF THE COMPOSITION OF TWO GRAPHS

We now define the composition of two graphs, and discuss some results on the pebbling numbers of such graphs.

Definition 2. If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are two graphs, the *composition* of G and H is the graph $G[H]$, whose vertex set is the Cartesian product

$$V(G[H]) = V(G) \times V(H) = \{(x, y) : x \in V(G),\ y \in V(H)\}$$

and whose edge set is given as

$$\begin{aligned} &E(G[H]) \\ &\quad = \{((x, y), (x', y')) : \text{either } (x, x') \in E(G), \text{or } x = x' \text{and } (y, y') \in E(H)\} \end{aligned}$$

The *complement* $\overline{G}$ of a graph G has $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{(u, v) : u \neq v \text{ and } (u, v) \notin E(G)\}$. Consider $\overline{K}_n$. Then $|V(\overline{K}_n)| = n$ and $E(\overline{K}_n) = \varnothing$. If G is a simple connected graph, then $G[\overline{K}_n]$ is also a simple connected graph. In general $G[H]$ is not equal to $H[G]$.

Example. $P_2[P_3] \neq P_3[P_2]$.

Lemma 1. *Let G be a simple connected graph with $|V(G)| \geq 2$. Then*

$$f(G[\overline{K}_n]) \leq nf(G).$$

Proof. Suppose that $nf(G)$ pebbles are distributed on the vertices of $G[\overline{K}_n]$. Let $V(\overline{K}_n)$ be $\{y_1, \cdots, y_n\}$ and the target vertex be (x, y_i) for some $i \in \{1, \ldots, n\}$ and some $x \in V(G)$. $p(x, y_i)$ refers to the number of pebbles on the vertex (x, y_i). We may assume that $p(x, y_i) = 0$.

According to the distribution of the pebbles on $\{x\} \times \overline{K}_n$, there are three possible cases.

Case 1. If two or more pairs of pebbles are on $\{x\} \times \overline{K}_n$, then two pebbles can be moved to (x', y_i) with $(x, x') \in E(G)$ from $\{x\} \times \overline{K}_n$. Using these two pebbles on (x', y_i), we can pebble (x, y_i).

Case 2. If only one pair of pebbles is placed on $\{x\} \times \overline{K}_n$, then the total number of pebbles on $\{x\} \times \overline{K}_n$ is at most $n + 1$. We write p_x for the total number of pebbles on $\{x\} \times \overline{K}_n$. If $p_x \leq n$, the following holds. One pebble can be moved to (x', y_i) with $(x, x') \in E(G)$ from $\{x\} \times \overline{K}_n$. In the sequel,

$$\sum_{\substack{x' \neq x \\ x' \in V(G)}} p_{x'} \geq nf(G) - n + 1.$$

In each fiber $\{x'\} \times \overline{K}_n$ with $x' \neq x$, choose a vertex (x', y') such that

$$p(x', y') = \max\{p(x', y') \mid y' \in V(\overline{K}_n)\}.$$

Then $p(x', y') \geq \lceil \frac{p_{x'}}{n} \rceil$, where $\lceil x \rceil$ denotes the ceiling integer for x, i. e., the result of rounding x up to the nearest integer. Let S be the subset of $V(G[\overline{K}_n])$ consisting of the selected vertices and the target vertex (x, y_i). Let H be the induced subgraph $\langle S \rangle$ of $G[\overline{K}_n]$. Then this subgraph H is isomorphic to G. Let $p(H)$ be the total number of pebbles on this subgraph H. Then

$$p(H) \geq \sum_{x' \neq x,\, x' \in V(G)} \left\lceil \frac{p_{x'}}{n} \right\rceil \geq \left\lceil \frac{1}{n} \sum_{x' \neq x,\, x' \in V(G)} p_{x'} \right\rceil = \left\lceil \frac{\{f(G) - 1\}n + 1}{n} \right\rceil = f(G).$$

So we can pebble the target vertex. If $p_x = n + 1$, then we consider two possible subcases.

Subcase (a). If $p_{x'} \geq 1$ for some x' with $(x, x') \in E(G)$, then we can pebble the target vertex (x, y_i).

Let $p(x', y_j) = 1$. Then we can put one more pebble on (x', y_j) from $\{x\} \times \overline{K}_n$. Using these two pebbles on (x', y_j) we can pebble the target vertex (x, y_i).

Subcase (b). If $p_{x'} = 0$ for all x' with $(x, x') \in E(G)$, then we can pebble the target vertex (x, y_i).

Let $(x, x') \in E(G)$. Using one pair of pebbles on $\{x\} \times \overline{K}_n$ we can put one pebble on $\{x'\} \times \overline{K}_n$. In the sequel $p_{x'} = 1$. Set $W = V(G) - \{x, x'\}$. Then

$$\sum_{x'' \in W} p_{x''} = nf(G) - n - 1.$$

$$p(H) \geq p_{x'} + \sum_{x'' \in W} \left\lceil \frac{p_{x''}}{n} \right\rceil \geq 1 + \left\lceil \frac{1}{n} \sum_{x'' \in W} p_{x''} \right\rceil = 1 + \left\lceil \frac{\{f(G) - 2\}n + (n-1)}{n} \right\rceil = f(G).$$

So we can pebble the target vertex.

Case 3. If no pair of pebbles is placed on $\{x\} \times N_n$, the total number of pebble on $\{x\} \times N_n$ is at most $n - 1$. By the same process as Case 2, we can pebble the target vertex (x, y_i). □

Theorem 2. *If G is a simple connected graph with $|V(G)| \geq 2$ and H is any graph, then*

$$f(G[H]) \leq |V(H)| f(G).$$

Proof. Suppose that G is a simple connected graph with $|V(G)| \geq 2$ and H is any graph with $|V(H)| = n$.

Then $G[\overline{K}_n]$ is a spanning subgraph of $G[H]$. So $f(G[\overline{K}_n]) \geq f(G[H])$. By Lemma 1, the result follows. □

Corollary 3. $f(K_{n,n}) = 2n$.

Proof. Since $K_{n,n} = P_2[\overline{K}_n]$, so the result follows. □

It would be interesting to know which graph is demonic. We mention the following results.

Corollary 4. *If both G and H are demonic, then G[H] is also demonic.*

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