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COLUMN-REDUCED ORTHOGONAL RATIONAL MATRIX FUNCTIONS WITH PRESCRIBED ZERO-POLE STRUCTURE

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ABSTRACT. An inverse interpolation problem for rational matrix functions with a certain type of symmetricity in zero-pole structure is studied.

1. INTRODUCTION

Let σ be a subset of the complex plane $\mathbb{C}$ and V be an invertible $m \times m$ constant matrix which is either symmetric or antisymmetric:

$$V^T = \alpha V$$

where $\alpha = \pm 1$. (It then follows that m is even in the antisymmetric case where $\alpha = -1$.)

By a *σ-admissible Sylvester data set*, it is meant a set of matrices

$$\tau = (C_\pi,\, A_\pi\, ;\, A_\zeta,\, B_\zeta\, ;\, \Gamma) \tag{1.1}$$

of sizes $m \times n_\pi$, $n_\pi \times n_\pi$; $n_\zeta \times n_\zeta$, $n_\zeta \times m$; $n_\zeta \times n_\pi$, respectively, where

$$\sigma(A_\pi) \cup \sigma(A_\zeta) \subset \sigma,$$

(C_π, A_π) is a *null-kernel pair* (i. e., $\bigcap_{j=0}^{n_\pi-1} \operatorname{Ker} C_\pi A_\pi^j = \{0\}$), (A_ζ, B_ζ) is a *full-range pair* (i. e., $\sum_{j=0}^{n_\zeta-1} \operatorname{Im} A_\zeta^j B_\zeta = \mathbb{C}^{n_\zeta}$) and γ satisfies the following matrix equation

$$\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi.$$

Here, $\sigma(A_\pi)$ is the spectrum of the matrix A_π.

For a given τ, we associate another set of matrices

$$\tau^T = (-V^{-1}B_\zeta^T,\, A_\zeta^T;\, A_\pi^T,\, C_\pi^T V;\, \Gamma^T). \tag{1.2}$$

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It is easy to check that τ^T is a σ-admissible Sylvester data set if and cnly if τ is. For two σ-admissible Sylvester data sets $\tau = (C_\pi,\, A_\pi\,;\, A_\zeta, B_\zeta\,;\, \gamma)$ and $\tau' = (C'_\pi,\, A'_\pi\,;\, A'_\zeta, B'_\zeta\,;\, \gamma')$, τ is *similar* to τ' if there exist invertible matrices Φ and Ψ such that

$$\begin{aligned} C_\pi &= C'_\pi \Phi, \\ A_\pi &= \Phi^{-1} A'_\pi \Phi, \\ A_\zeta &= \Psi^{-1} A'_\zeta \Psi, \\ B_\zeta &= \Psi^{-1} B'_\zeta, \\ \Gamma &= \Psi^{-1} \Gamma' \Phi. \end{aligned}$$

If we want to emphasize the matrices Φ and Ψ we say that τ is (Φ, Ψ)-similar to τ'. If τ is similar to τ^T, τ is said to be *symmetric* and is (Φ, $\alpha\Phi^T$)-similar to τ^T for an invertible matrix Φ (see Ball & Kim [5]).

Let $\Theta(z)$ be an $M \times M$ rational matrix function. For a Sylvester data set τ, Θ is said to have τ as its *$\mathbb{C}$-null-pole triple* if

$$\Theta P_M = \Big\{ C_\pi(zI - A_\pi)^{-1}x + h(z) \;\Big|\; x \in \mathbb{C}^{n_\pi}, h \in P_M \text{ such that } \sum_{z_0 \in \mathbb{C}} \mathrm{Res}_{z=z_0}(zI - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x \Big\},$$

where P_M is the set of polynomials with coefficients in $\mathbb{C}^M$.

Motivation for similarity of Sylvester data sets can be seen from the formula for ΘP_M, that is, changing τ to a similar τ' does not change ΘP_M, and it is the only way two $\mathbb{C}$-admissible Sylvester data sets give rise to the same ΘP_M (see Ball, Kaashoek, Groenewald & Kim [4]).

In this paper, our aim is to prove the following theorem.

Main Theorem. *If τ is a given σ-admissible Sylvester data set which is similar to τ^T, then there exists an $m \times m$ rational matrix function $\Theta(z)$ for which*

(i) *Θ has τ as its $\mathbb{C}$-null-pole triple,*

(ii) *Θ is column reduced at infinity, and*

(iii) *$\Theta^T(z)V\Theta(z) = P$, for all $z \in \mathbb{C}_\infty$, where $P = [p_{ij}]$ is the particular canonical choice of invertible symmetric (resp. antisymmetric) $m \times m$ constant matrix*

$$p_{ij} = \begin{cases} 1, & j = m+1-i \text{ and } 1 \le i \le m, \\ 0, & \text{otherwise} \end{cases}$$

for the symmetric case ($\alpha = 1$), (*resp.*

$$p_{ij} = \begin{cases} 1, & 1 \leq i \leq \frac{m}{2} \text{ and } j = m+1-i, \\ -1, & \frac{m}{2} < i \leq m \text{ and } j = m+1-i, \\ 0, & \text{otherwise} \end{cases}$$

for the antisymmetric case ($\alpha = -1$ *and* m *even*)).

In this case the column indices of Θ *are*

$$-\alpha_1, -\alpha_2, \cdots, -\alpha_t, 0, \cdots, 0, \alpha_t, \cdots, \alpha_1,$$

where $(m - 2t)$ *zeros and* $\alpha_1 \geq \cdots \geq \alpha_t$ *are the nonzero observability indices of* (C_π, A_π).

A rational matrix function satisfying (iii) is said to be *V-orthogonal.*

The problem of finding a rational matrix function having the prescribed null-pole structure (that is, satisfying (i)), known as the inverse spectral problem for rational matrix functions, is studied in literature (e. g., [1, 3, 6, 7, 8, 9]). In particular, the problem of finding Θ satisfying (i) and (ii) is studied in [2, 4]. Without the condition (ii), Ball & Kim [5] solved a problem of nonhomogeneous symmetric interpolation.

As an application of Main Theorem, we can think of a parametrization of the set of symmetric (or anti-symmetric) rational solutions of a set of bitangential interpolation conditions having also the minimal possible McMillan degree, as in Ball, Kaashoek, Groenewald & Kim [4], where the parametrization is done for the problem without the symmetricity condition. Also, in Ball & Kim [5], a bitangential interpolation problem for symmetric (or anti-symmetric) rational matrix functions is solved and a parametrization of all solutions is given (but, without considering the condition on their McMillan degree). To derive the parametrization from which the McMillan degrees of the solutions of nonhomogeneous interpolation problem can be read, we need to find a solution of the homogeneous problem which is column reduced at infinity, as it is done in the main theorem of this paper.

2. Preliminaries

Let z_0 be a complex number such that

$$z_0 \notin \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\},$$

N be a complement of $\ker\Gamma$ in $\mathbb{C}^{n_\pi}$, and K be a complement of $\operatorname{Im}\Gamma$ in $\mathbb{C}^{n_\zeta}$. We choose $\Gamma^+ : \mathbb{C}^{n_\zeta} \to \mathbb{C}^{n_\pi}$ to be a generalized inverse of Γ such that $\operatorname{Im}\Gamma^+ = N$ and $\operatorname{Ker}\Gamma^+ = K$. Let ρ_π be the projection of $\mathbb{C}^{n_\pi}$ onto $\operatorname{Ker}\Gamma$ along N and ρ_ζ be the projection of $\mathbb{C}^{n_\zeta}$ onto K along $\operatorname{Im}\Gamma$. We write ρ_π for the embedding of $\operatorname{Ker}\Gamma$ onto $\mathbb{C}^{n_\pi}$ and η_ζ for the embedding of K into $\mathbb{C}^{n_\zeta}$. We may choose bases $\{d_{jk} : k = 1, \cdots, \alpha_j \ ; \ j = 1, \cdots, t\}$ and $\{f_{jk} : k = 1, \cdots, \alpha_j \ ; \ j = 1, \cdots, s\}$, respectively, in $\operatorname{Ker}\Gamma$ and K, respectively, such that the following hold:

(i) $\{d_{jk} : k = 1, \cdots, \alpha_j \ ; \ j = 1, \cdots, t\}$ is a basis of $\operatorname{Ker}\Gamma \cap \operatorname{Ker} C_\pi$.

(ii) $A_\pi d_{j,k+1} = d_{j,k}, \ k = 1, \cdots, \alpha_j - 1$.

(iii) $\{f_{j,\omega_j} : j = 1, \cdots, s\}$ is a basis for a complement of $\operatorname{Im}\Gamma$ in $\operatorname{Im}\Gamma + \operatorname{Im} B_\zeta$.

(iv) $A_\zeta f_{j,k+1} - f_{jk} \in \operatorname{Im}\Gamma + \operatorname{Im} B_\zeta, \ k = 0, \cdots, \omega_j - 1$ with $f_{j0} := 0$.

Without loss of generality, we assume that $\alpha_1 \geq \cdots \geq \alpha_t > 0$ and $\omega_1 \geq \cdots \geq \omega_s > 0$. Such a basis $\{d_{jk}\}$ (*resp.* $\{f_{jk}\}$) is called an *outgoing* (*resp. incoming*) *basis* for τ at infinity (see Gohberg, Kaashoek & Ran [8]). With these bases we associate the following two operators.

$$S : \operatorname{Ker}\Gamma \to \operatorname{Ker}\Gamma, \qquad Sd_{jk} = d_{j,k+1} \, (d_{j,\alpha_j+1} := 0), \tag{2.1}$$

$$T : K \to K, \qquad Tf_{jk} = f_{j,k+1} \, (f_{j,w_j+1} := 0). \tag{2.2}$$

In the sequel $\Gamma_1 : K \to \operatorname{Ker}\Gamma$ is an arbitrary linear transformation which we may choose freely.

From Ball, Kaashoek, Groenewald & Kim [4], we can choose operators

$$F : K \to \mathbb{C}^m, \qquad A_{12} : K \to \mathbb{C}^{n_\pi},$$
$$H : \mathbb{C}^m \to \operatorname{Ker}\Gamma, \qquad A_{21} : \mathbb{C}^{n_\zeta} \to \operatorname{Ker}\Gamma$$

such that the following identities are fulfilled:

$$(z_0 I - A_\zeta)\Gamma A_{12} = A_\zeta \eta_\zeta T - \eta_\zeta - B_\zeta F, \tag{2.3}$$

$$A_{21}\Gamma(z_0 I - A_\pi) = S\rho_\pi A_\pi - \rho_\pi - HC_\pi, \tag{2.4}$$

$$A_{21}\eta_\zeta(I - z_0 T) - (I - z_0 S)\rho_\pi A_{12} = \Gamma_1 T - S\Gamma_1 - HF. \tag{2.5}$$

Let

$$X := -\sum_{j=0}^{\omega_1 - 1} A_\pi^j A_{12} T^j : K \to \mathbb{C}^{n_\pi}, \tag{2.6}$$

$$Y := -\sum_{j=0}^{\alpha_1 - 1} S^j A_{21} A_\zeta^j : \mathbb{C}^{n_\zeta} \to \operatorname{Ker}\Gamma, \tag{2.7}$$

and

$$\tau_\infty = (C_\pi X(I - z_0 T) - F, T; S, (I - z_0 S)YB_\zeta + H; \Gamma_\infty) \tag{2.8}$$

with

$$\Gamma_\infty = -Y(z_0 I - A_\zeta)\Gamma(z_0 I - A_\pi)X + \rho_\pi(z_0 I - A_\pi)X + Y(z_0 I - A_\zeta)\eta_\zeta - \Gamma_1. \tag{2.9}$$

Then τ_∞ is a minimal complement of τ at infinity. Also, it is similar to τ_∞^T when τ is similar to τ^T (*cf.* Kim [10]). More specifically, if τ is similar to τ^T, there exists an invertible matrix Φ for which

$$C_\pi = -V^{-1}B_\zeta^T\Phi, \qquad A_\pi = \Phi^{-1}A_\zeta^T\Phi, \tag{2.10}$$

$$\Gamma = -\alpha\Phi^{-T}\Gamma^T\Phi. \tag{2.11}$$

Let $\{d_{jk} : k = 1, \cdots, \alpha_j \ ; \ j = 1, \cdots, t\}$ be an outgoing basis for $\operatorname{Ker}\Gamma$ and U be an $n_\pi \times n_\pi$ invertible matrix having d_{jk} as its $(\alpha_1 + \alpha_2 + \cdots + \alpha_{j-1} + k)^{\text{th}}$ column. If we set $\mathbf{N}$ to be the subspace generated by the last $(m - \dim \operatorname{Ker}\Gamma)$-columns of U and

$$f_{jk} = \Phi^{-T}U^{-T}e_{\alpha_1 + \cdots + \alpha_{j-k+1}}, \quad 1 \le j \le t, \quad 1 \le k \le \alpha_j, \tag{2.12}$$

then $\{f_{jk} : j = 1, \cdots, t \ ; \ k = 1, \cdots, \alpha_j\}$ is an incoming basis for $K := \Phi^{-T}(\bar{\mathbf{N}}^\perp)$ which is a complement of $\operatorname{Im}\Gamma$.

Define operators S, T, H, A_{21}, X, and Y as in (2.1)–(2.7) and let

$$\begin{aligned} F &= -\alpha V^{-1}H^T\Phi_{|K}^T, \\ A_{12} &= -\alpha\Phi^{-1}A_{21}^T\Phi_{|K}^T, \\ \eta_\zeta &= \Phi^{-T}\rho_\pi^T\Phi_{|K}^T, \end{aligned}$$

where ρ_π is a projection onto $\operatorname{Ker}\Gamma$ along $\mathbf{N}$. It can be easily checked that F and A_{12} satisfy the relations (2.3), (2.5), and

$$T = \Phi^{-T}S^T\Phi_{|K}^T, \tag{2.13}$$

$$Y = -\alpha\rho_\pi\Phi^{-1}X^T\Phi^T.$$

Finally, we choose an arbitrary operator Γ_1 so that

$$\Gamma_1 = -\alpha\rho_\pi\Phi^{-1}\Gamma_1^T\Phi_{|K}^T.$$

Then, τ_∞ defined by (2.8) is a minimal complement of τ at infinty and is similar to τ_∞^T.

With the choices of β, ρ_ζ, Γ_1, S, H, η_π, T, F, we have the following theorem (*cf.* Kim [10]).

Theorem 2.1. *For a given σ-admissible Sylvester data set τ similar to τ^T, there exists a rational matrix function $\tilde{\Theta}(z)$ for which*

(i) *$\tilde{\Theta}$ has τ at its $\mathbb{C}$-null-pole triple, and*

(ii) *$\tilde{\Theta}^T(z)V\tilde{\Theta}(z) = V$, for $z \in \mathbb{C}_\infty$.*

If τ is the same as (1.1), such a $\tilde{\Theta}(z)$ is given by

$$\begin{aligned}\tilde{\Theta}(z)&\\ &= I-(z-z_0)C_\pi(zI-A_\pi)^{-1}\{\Gamma^+ + (z_0I-A_\pi)X\rho_\zeta - \Gamma_1\rho_\zeta\}(z_0I-A_\zeta)^{-1}B_\zeta\\ &\quad + \eta_\pi(z_0S-I)^{-1}H\\ &\quad + (z-z_0)[C_\pi X(I-z_0T)-F](I-zT)^{-1}\rho_\zeta(z_0I-A_\zeta)^{-1}B_\zeta, \end{aligned} \tag{2.14}$$

where Γ^+ is a generalized inverse of Γ satisfying

$$(\Gamma^+)^T = -\alpha\Phi^{-1}(\Gamma^+)^T\Phi^T.$$

3. Main theorem

To prove our main theorem, we need the next theorem.

Theorem 3.1. *Given is*

$$\tau = (C_\pi, A_\pi; A_\zeta, B_\zeta; \Gamma)$$

which is similar to τ^T. Let $\{d_{jk} : j = 1, \cdots, t\ ;\ k = 1, \cdots, \alpha_j\}$ be an outgoing basis for τ at infinity and $\{f_{jk} : j = 1, \cdots, t\ ;\ k = 1, \cdots, \alpha_j\}$ be given by (2.12) and let z_0 be a complex number satisfying

$$z_0 \notin \sigma(A_\pi) \cup \sigma(A_\zeta) \cup \{0\}.$$

If we choose $z_j, y_j \in \mathbb{C}^m$ so that

$$z_j = C_\pi(z_0I - A_\pi)^{-1}d_{j\alpha_j}, \quad j = 1, \cdots, t, \tag{3.1}$$

$$(A_\zeta - \alpha I)^{-1}B_\zeta y_j = (I - z_0T)^{-1}f_{j1}, \quad j = 1, \cdots, t, \tag{3.2}$$

then

$$z_i^TVz_j = 0, \quad 1 \le i, \quad j \le t, \tag{3.3}$$

$$z_i^TVy_j = \alpha\delta_{ij}, \quad 1 \le i, \quad j \le t, \tag{3.4}$$

$$y_i^TVy_j = 0, \quad 1 \le i, \quad j \le t, \tag{3.5}$$

where T is chosen as in (2.13).

Proof. Since τ is similar to τ^T we have (2.10) and (2.11) for an invertible matrix Φ. Upon substituting the transposed version of (2.10) in places of A_ζ, B_ζ in Sylvester equation $\Gamma A_\pi - A_\zeta\Gamma = B_\zeta C_\pi$ and premultiplying by Φ^T, we get

$$(z_0 I - A_\pi^T)\Phi^T\Gamma - \Phi^T\Gamma(z_0 I - A_\pi) = -\alpha C_\pi^T V C_\pi. \tag{3.6}$$

By the choice of z_j in (3.1), for all $i, j = 1, \cdots, t$,

$$z_i^T V z_j = d_{i\alpha_i}^T (z_0 I - A_\pi^T)^{-1} C_\pi^T V C_\pi (z_0 I - A_\pi)^{-1} d_{j\alpha_j}.$$

Multiplying both sides of (3.6) by $-\alpha$ and plug it in the above, we obtain

$$z_i^T V z_j = -\alpha\{d_{i\alpha_i}^T \Phi^T\Gamma(z_0 I - A_\pi)^{-1} d_{j\alpha_j} - d_{i\alpha_i}^T (z_0 I - A_\pi^T)^{-1}\Phi^T\Gamma d_{j\alpha_j}\}.$$

Recalling that $d_{k\alpha_k} \in \operatorname{Ker}\Gamma$ for all $k = 1, \cdots, t$ and $\Phi^T\Gamma = -\alpha\Gamma^T\Phi$, the right hand side of the above equality collapses to 0. Hence, (3.3) is proved.

Next, we substitute (2.10) in (3.1) and take transpose to obtain

$$z_i^T V y_j = -\alpha d_{i\alpha_i}^T \Phi^T \Gamma (z_0 I - A_\zeta)^{-1} B_\zeta \, y_j.$$

Plug (3.1) in the right hand side of above and note that

$$(I - z_0 T)^{-1} f_{j1} = f_{j1} + z_0 f_{j2} + \cdots + z_0^{\alpha_j - 1} f_{j\alpha_j} \tag{3.7}$$

to get

$$z_i^T V y_j = \alpha d_{i\alpha_i}^T \Phi^T (f_{j1} + z_0 f_{j2} + \cdots + z_0^{\alpha_j-1} f_{j\alpha_j}).$$

Because f_{jk} is given by (2.12) and $d_{i\alpha_i} = U e_{\alpha_1+\cdots+\alpha_i}$, $z_i^T V y_j = \alpha\delta_{ij}$. Upon substituting (3.7) in (3.2) and taking transpose, we get

$$y_j^T B_\zeta^T (A_\zeta^T - z_0 I)^{-1} = (g_{j1}^T + z_0 g_{j2}^T + \cdots + z_0^{\alpha_j-1} g_{j\alpha_j}^T).$$

Plug (2.10) in places of B_ζ^T, A_ζ^T and (2.12) in places of g_{jk} and then postmultiply both sides of the resulting equality, we obtain

$$\begin{aligned} &y_j^T V C_\pi (z_0 I - A_\pi)^{-1} U \\ &\quad = (e_{\alpha_1+\cdots+\alpha_j}^T + z_0 e_{\alpha_1+\cdots+\alpha_j-1}^T + \cdots + z_0^{\alpha_j-1} e_{\alpha_1+\cdots+\alpha_{j-1}+1}^T). \end{aligned} \tag{3.8}$$

Let M be an $m \times m$ matrix

$$M = [z_1 \cdots z_t \tilde{z}_{t+1} \cdots \tilde{z}_{\ell_2} \tilde{y}_{\ell_1} \cdots \tilde{y}_1],$$

where $\ell_1 = \ell_2 = \frac{m}{2}$ if m is even and $\ell_1 = \frac{m-1}{2}$, $\ell_2 = \frac{m+1}{2}$ if m is odd, $\tilde{z}_k$, $\tilde{y}_k$ are chosen so that

$$\begin{aligned}
\tilde{z}_i^T V \tilde{z}_j &= 0, \quad 1 \le i,\ j \le \ell_1; \\
\tilde{z}_i^T V \tilde{y}_j &= \delta_{ij}, \quad 1 \le i \le \ell_2,\ 1 \le j \le \ell_1; \\
\tilde{y}_i^T V \tilde{y}_j &= 0, \quad 1 \le i,\ j \le \ell_1; \\
\tilde{z}_{\ell_1+1}^T V \tilde{z}_{\ell_1+1} &= 1 \quad \text{when } m \text{ is odd.}
\end{aligned}$$

Here, we choose $\tilde{z}_i = z_i$ for $i = 1, \cdots, t$. Such vectors $\tilde{z}_i$, $\tilde{y}_j$ can be chosen. By the choice of M,

$$M^T V M = \tilde{P} = [\tilde{p}_{ij}], \tag{3.9}$$

where

$$\tilde{p}_{ij} = \begin{cases} 1, & 1 \le i \le \ell_2,\ j = m+1-i; \\ \alpha, & \ell_2 + 1 < i \le m,\ j = m+1-i; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, due to (2.14) and the construction of U, we derive

$$\text{the } (\alpha_1 + \cdots + \alpha_k)^{\text{th}}\text{-column of } M^{-1} C_\pi (z_0 I - A_\pi)^{-1} U = e_k.$$

Hence, for the equality (3.7) to be fulfilled

$$y_j^T V M = e_j^T,$$

equivalently, $y_j = \alpha V^{-1} M^{-T} e_j$. Finally, we obtain

$$y_i^T V y_j = e_i^T M^{-1} V^{-1} M^{-T} e_j = 0$$

because $i, j = 1, \cdots, t$ and $M^{-1} V^{-1} M^{-T} = \tilde{P}$. This completes the proof. □

Proof of **Main Theorem**. Suppose τ is a given σ-Sylvester data set similar to τ^T and τ_∞ is the minimal complement of τ constructed as in (2.8). If we define z_j, y_j by (3.1) and (3.2) for $j = 1, \cdots, t$, there exist $z_{t+1}, \cdots, z_{\ell_1}$, $y_{t+1}, \cdots, y_{\ell_1}$ so that (3.3)–(3.5) with $1 \le i,\ j \le \ell_1$ hold, where $\ell_1 = [\frac{m}{2}]$. If m is odd, we can find z_{ℓ_1+1} so that

$$z_{\ell_1+1}^T V z_{\ell_1+1} = 1$$

and (3.6) hold for $1 \le i \le \ell_1$, $j = \ell_1 + 1$.

Let

$$\ell_2 = \begin{cases} \ell_1, & \text{if } m \text{ is even;} \\ \ell_1 + 1, & \text{if } m \text{ is odd;} \end{cases}$$

and let

$$\Theta(z) = \tilde{\Theta}(z) E,$$

where $\tilde{\Theta}(z)$ is given by (2.14) and

$$E := [z_1 z_2 \cdots z_{\ell_2}\ y_{\ell_1} \cdots y_1].$$

Upon recalling Theorem 2.1 and the fact that $\Theta P_M = \tilde{\Theta} P_M$, we see that $\Theta(z)$ satisfies (i) and that it is enough to show $E^T V E = P$ to prove (iii), where $P = [p_{ij}]$ is given by (2.1). But the above equality is obvious from the fact that

$$\text{the } (i,j)\text{-entry of } E^T V E = \begin{cases} z_i^T V z_j, & 1 \le i,\ j \le \ell_2; \\ z_k^T V y_j, & \ell_2 + k \le i \le m,\ 1 \le j \le \ell_1; \\ y_k^T V z_j, & \ell_2 + k \le i \le m,\ 1 \le j \le \ell_2; \\ z_k^T V z_h, & \ell_2 + k \le i \le m,\ \ell_2 + h \le j \le m. \end{cases}$$

Here we also note that E is invertible.

The only thing left is to show that $\Theta(z)$ is column reduced at infinity with column indices $-\alpha_t - \alpha_{t-1}, \cdots, -\alpha_1, 0, \cdots, 0, \alpha_t, \cdots, \alpha_1$.

If we set

$$\hat{E} = [z_1, \cdots, z_t, \cdots, z_{\ell_2}, y_{\ell_1}, \cdots, y_{t+1}\, y_1, \cdots, y_s],$$

by Theorem 3.1 of Ball, Kaashoek, Groenewald & Kim [4],

$$\hat{\Theta}(z) := \tilde{\Theta}(z)\hat{E}$$

is column reduced at infinity with column indices

$$-\alpha_1, \cdots, -\alpha_t, 0, \cdots, 0, \alpha_1, \cdots, \alpha_t.$$

Noting that

$$E = \hat{E} \begin{pmatrix} I_{m-t} & 0 \\ 0 & \hat{P} \end{pmatrix},$$

where $\hat{P} = [\hat{p}_{ij}]_{t \times t}$ with

$$\hat{p}_{ij} = \begin{cases} 1, & i + j = t + 1, \\ 0, & \text{otherwise}, \end{cases}$$

it can be easily seen that $\Theta(z)$ can be obtained from $\hat{\Theta}(z)$ by interchanging some columns of $\hat{\Theta}(z)$ which is column reduced at infinity. Necessarily, $\Theta(z)$ is column reduced at infinity with column indices

$$-\alpha_1 \le \cdots \le -\alpha_t < 0 \le \cdots \le 0 < \alpha_t \le \alpha_{t-1} \le \cdots \le \alpha_1.$$

This completes the proof. □

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