J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 8, Number 2 (November 2001), Pages 87–99

NOTES ON THE MCSHANE-STIELTJES INTEGRABILITY

BYONG IN SEUNG

ABSTRACT. In this paper, we define the McShane-Stieltjes integral for Banach-valued functions, and will investigate some of its properties and comparison with the Pettis integral.

1. INTRODUCTION

In 1990, Gordon [7] introduced the concepts of the McShane integral of Banach-valued functions.

Fremlin and Mendoza [5] improved some results of Gordon as follows: A function $\phi : [0,1] \to X$ is McShane integrable whenever a tagged partition P is sub δ on $[0,1]$ if and only if it is Pettis integrable.

We are concerned with the McShane-Stieltjes integral for Banach-valued functions which is a generalization of the McShane-Stieltjes integral for real-valued functions.

In this paper, for Banach-valued functions we shall introduce the McShane-Stieltjes integral that is encouraged naturally by the idea of Riemann-Stieltjes or Lebesgue-Stieltjes integral. Also, we will investigate some properties of McShane-Stieltjes integrability and comparison with the Pettis integral.

2. PRELIMINARIES

Throughout this paper, (Ω, Σ, μ) is a finite measure space and X, Y will denote Banach spaces with dual X^*, Y^* and unless otherwise stated, α is an increasing function on $[0,1]$ into $\mathbb{R}$.

Received by the editors March 20, 2001.
2000 *Mathematics Subject Classification.* 28B05, 46G10.
Key words and phrases. McShane integral, McShane-Stieltjes integral, McShane-Stieltjes integrability.
This paper is supported by Kyonggi University Research Grant in 2000.

Definition 2.1 (Diestel and Uhl [1]). A function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \cdots, x_n \in X$ and $E_1, E_2, \cdots, E_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \notin E_i$.

A function $f : \Omega \to X$ is called *μ-measurable* if there exists a sequence of simple functions $\{f_n\}$ with $\lim_n \|f_n - f\| = 0$ μ-almost everywhere.

A function $f : \Omega \to X$ is called *weakly μ-measurable* if for each $x^* \in X^*$ the numerical function $x^* f$ is μ-measurable. More generally, if $\Gamma \subseteq X^*$ and $x^* f$ is measurable for each $x^* \in \Gamma$, then f is called *Γ-measurable.* If $f : \Omega \to X^*$ is X-measurable (when X is identified with its image under the natural imbedding of X into X^{**}), then f is called *weak*-measurable.*

Theorem 2.2 (Pettis's Measurability Theorem). *A function $f : \Omega \to X$ is μ-measurable if and only if*

(i) *f is μ-essentially separably valued, i.e., there exists $E \in \Sigma$ with $\mu(E) = 0$ and such that $f(\Omega \setminus E)$ is a (norm) separable subset of X, and*

(ii) *f is weakly μ-measurable.*

Definition 2.3. A μ-measurable function $f : \Omega \to X$ is called *Bochner integrable* if there exists a sequence of simple functions $\{f_n\}$ such that

$$\lim_n \int_\Omega \|f_n - f\| = 0.$$

In this case, $\int_E f d\mu$ is defined for each $E \in \Sigma$ by

$$\int_E f d\mu = \lim_n \int_E f_n d\mu,$$

where $\int_E f_n d\mu$ is defined in the obvious way.

Definition 2.4. If f is a weakly μ-measurable X-valued function on Ω such that $x^* f \in L_1(\mu)$ for all $x^* \in X^*$, then f is called *Dunford integrable.* The Dunford integral over $E \in \Sigma$ is defined by the element x_E^{**} of X^{**} such that $x_E^{**}(x^*) = \int_E x^* f d\mu$ for all $x^* \in X^*$, and we write $x_E^{**} = (D)\text{-}\int_E f d\mu$.

In the case that $(D)\text{-}\int_E f d\mu \in X$ for each $E \in \Sigma$, then f is called *Pettis integrable* and we write $(P)\text{-}\int_E f d\mu$ instead of $(D)\text{-}\int_E f d\mu$ to denote the Pettis integral of f over $E \in \Sigma$.

The above definitions and theorem are given in Diestel and Uhl [1].

Definition 2.5 (Gordon [7]). Let $\delta(\cdot)$ be a positive function defined on the interval $[0, 1]$. A *tagged interval* $(x, [a, b])$ consists of an interval $[a, b] \subset [0, 1]$ and a point x in

$[0,1]$. This x may not be a point in $[a,b]$. The tagged interval $(x,[a,b])$ is subordinate to δ if $[a,b] \subset (x-\delta(x), x+\delta(x))$. The capital letter P will be used to denote a finite collection of non-overlapping tagged intervals. Let $P = \{(x_i,[a_i,b_i]) : 1 \le i \le n\}$ be such a collection in $[0,1]$. We adopt the following terminology:

(a) The points $\{x_i\}$ are called *tags* of P and the intervals $\{[a_i,b_i]\}$ are called *intervals* of P.
(b) If $(x_i,[a_i,b_i])$ is subordinate to δ for each i, then we write *P is sub δ*.
(c) If P is subordinate to δ and $[0,1] = \bigcup_{i=1}^n [a_i,b_i]$, then P is called a *tagged partition* (or *McShane partition*) of $[0,1]$.
(d) If P is a tagged partition of $[0,1]$ and if P is sub δ, then we write *P is sub δ on* $[0,1]$.
(e) If $f : [0,1] \to X$, then $f(P) = \sum_{i=1}^n f(x_i)(b_i - a_i)$.
(f) If F is defined on the intervals of $[0,1]$, then $F(P) = \sum_{i=1}^n F([a_i,b_i])$.
(g) We will write $\mu(P)$ for $\sum_{i=1}^n (b_i - a_i)$ and $\int_P f$ for $\sum_{i=1}^n \int_{a_i}^{b_i} f$.

Definition 2.6 (Gordon [7]). The function $f : [0,1] \to X$ is *McShane integrable* on $[0,1]$ if there exists a vector z in X with the following property: for each $\epsilon > 0$ there exists a positive function δ on $[0,1]$ such that $\|f(P) - z\| < \epsilon$ whenever P is sub δ on $[0,1]$, and z is denoted by

$$(M)\text{-}\int_0^1 f \quad \text{or} \quad (M)\text{-}\int_0^1 f(x)dx.$$

The function f is McShane integrable on the set $E \subset [0,1]$ if the function $f\chi_E$ is McShane integrable on $[0,1]$.

We now present the definition of the McShane-Stieltjes integral for Banach-valued functions.

Let $f : [0,1] \to X$ and let α be an increasing function on $[0,1]$. Then we will use the following notation:

$$f_\alpha(P) = \sum_{i=1}^n f(x_i)[\alpha(b_i) - \alpha(a_i)]$$

where a tagged partition $P = \{(x_i,[a_i,b_i] : 1 \le i \le n\}$ of $[0,1]$ is sub δ on $[0,1]$.

Definition 2.7. A function $f : [0,1] \to X$ is *McShane-Stieltjes integrable with respect to α on* $[0,1]$ if there exists a vector z in X with the following property: for each $\epsilon > 0$ there exists a positive function δ on $[0,1]$ such that $\|f_\alpha(P) - z\| < \epsilon$ whenever a tagged partition P is sub δ on $[0,1]$.

A function f is McShane-Stieltjes integrable on a measurable set $E \subset [0,1]$ with respect to α if $f\chi_E$ is a McShane-Stieltjes integrable function with respect to α on $[0,1]$. We note that when such a number z in X exists, it is uniquely determined and is denoted by

$$(MS)\text{-}\int_0^1 f(x)d\alpha(x) \quad \text{or} \quad (MS)\text{-}\int_0^1 f d\alpha$$

and we also say that McShane-Stieltjes integral $(MS)\text{-}\int_0^1 f d\alpha$ with respect to α on $[0,1]$ exists. The function f and α are referred to as the integrand function and integrator function, respectively.

And otherwise, all notions and notations used in this paper, unless mentioned, can be found in [1], [2], and [7].

3. Properties of the McShane-Stieltjes integral

The next propositions and theorems record some of the basic computational properties of McShane-Stieltjes integral for Banach-valued functions and the proofs of these facts are virtually identical to the proofs for real-valued functions, and sometimes the concept of norm in Banach space will be required.

Proposition 3.1. *Let α be an increasing function on $[0,1]$. A function $f : [0,1] \to X$ is a McShane-Stieltjes integrable with respect to α on $[0,1]$ if and only if for each $\epsilon > 0$ there exists a positive function δ on $[0,1]$ such that $\|f_\alpha(P_1) - f_\alpha(P_2)\| < \epsilon$ whenever P_1 and P_2 are sub δ on $[0,1]$.*

Proof. Suppose that f is a McShane-Stieltjes integrable function with respect to α on $[0,1]$. Then there exists a vector z in X with the following property: for each $\epsilon > 0$, there exists a positive function δ on $[0,1]$ such that

$$\|f_\alpha(P) - z\| < \frac{\epsilon}{2}$$

whenever a tagged partition P is sub δ on $[0,1]$. If P_1 and P_2 are sub δ on $[0,1]$, then

$$\|f_\alpha(P_1) - z\| < \frac{\epsilon}{2} \quad \text{and} \quad \|f_\alpha(P_2) - z\| < \frac{\epsilon}{2}.$$

Hence, we get

$$\begin{aligned} \|f_\alpha(P_1) - f_\alpha(P_2)\| &= \|(f_\alpha(P_1) - z) + (z - f_\alpha(P_2))\| \\ &\leq \|f_\alpha(P_1) - z\| + \|z - f_\alpha(P_2)\| \\ &< \epsilon. \end{aligned}$$

Conversely, suppose that for each $\epsilon > 0$ there exists a positive function δ on $[0,1]$ such that $\|f_\alpha(P_1) - f_\alpha(P_2)\| < \epsilon$ whenever P_1 and P_2 are sub δ on $[0,1]$. Then for each positive integer n, there exists a positive function δ'_n on $[0,1]$ such that

$$\|f_\alpha(P') - f_\alpha(P'')\| < \frac{1}{n}$$

whenever each tagged partition P' and P'' are sub δ'_n on $[0,1]$.

If we take $\delta_n = \min\{\delta'_1, \delta'_2, \cdots, \delta'_n\}$ for $n = 1,2,3,\cdots$, then δ_n is a positive function on $[0,1]$ for $n = 1,2,3,\cdots$ and

$$\|f_\alpha(P') - f_\alpha(P'')\| < \frac{1}{n}$$

whenever each P' and P'' are sub δ_n on $[0,1]$.

For each positive integer n, choose a tagged partition P_n which is sub δ_n on $[0,1]$. For each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ with $\frac{1}{N} < \epsilon$. If $m, n > N$, then

$$\|f_\alpha(P_m) - f_\alpha(P_n)\| < \frac{1}{N}$$

since P_m and P_n are sub δ_N on $[0,1]$. Thus, a sequence $(f_\alpha(P_n))$ is Cauchy sequence in X. Since X is a Banach space, the sequence $(f_\alpha(P_n))$ converges to any one vector in X. Let z be the limit of this sequence and let $\epsilon > 0$. Then there exists a positive integer N_1 with $\frac{1}{N_1} < \frac{\epsilon}{2}$ such that if $N > N_1$, then

$$\|f_\alpha(P_N) - z\| < \frac{1}{N_1} < \frac{\epsilon}{2}.$$

Choose a positive integer $N \in \mathbb{N}$ such that $N > N_1$ and $\frac{1}{N} < \frac{\epsilon}{2}$. Then

$$\|f_\alpha(P) - z\| \leq \|f_\alpha(P) - f_\alpha(P_N)\| + \|f_\alpha(P_N) - z\| < \frac{1}{N} + \frac{\epsilon}{2} < \epsilon$$

where P and P_N are sub δ_N on $[0,1]$. Therefore f is a McShane-Stieltjes integrable with respect to α on $[0,1]$. □

Proposition 3.2. *Let α be an increasing function on $[0,1]$. If $f : [0,1] \to X$ is a McShane-Stieltjes integrable with respect to α on $[0,1]$, then f is a McShane-Stieltjes integrable with respect to α on every subinterval of $[0,1]$.*

Proof. Let $[a,b]$ be any subinterval of $[0,1]$ and let $\epsilon > 0$. Then, there exists a positive function δ_1 such that $\|f_\alpha(P_1) - f_\alpha(P_2)\| < \epsilon$ whenever P_1 and P_2 are sub δ_1 on $[0,1]$ by Proposition 3.1.

Let δ be the restriction of δ_1 to subinterval $[a,b]$ of $[0,1]$. Then δ be a positive function on $[a,b]$. Also, let P_1 and P_2 be any tagged partitions of $[a,b]$ which are sub δ on $[a, b]$. Choose a tagged partition P_a of $[0, a]$ which is sub δ_1, and choose a tagged partition P_b of $[b, 1]$ which is sub δ_1. Then two tagged partitions

$$P' = P_a \cup P_1 \cup P_b \quad \text{and} \quad P'' = P_a \cup P_2 \cup P_b$$

are sub δ_1 on $[0, 1]$. Hence

$$\|f_\alpha(P_1) - f_\alpha(P_2)\| = \|f_\alpha(P') - f_\alpha(P'')\| < \epsilon.$$

By Proposition 3.1, f is McShane-Stieltjes integrable with respect to α on $[0, 1]$. That is, f is McShane-Stieltjes integrable with respect to α on every subinterval of $[0, 1]$. □

It is easy to prove the following fact from the above two properties.

Proposition 3.3. *Let $f : [0, 1] \to X$ and let $t \in (0, 1)$. If f is a McShane-Stieltjes integrable with respect to α on each of the interval $[0, t]$ and $[t, 1]$, then f is a McShane-Stieltjes integrable with respect to α on $[0, 1]$ and*

$$(MS)\text{-}\int_0^1 f d\alpha = (MS)\text{-}\int_0^t f d\alpha + (MS)\text{-}\int_t^1 f d\alpha.$$

Moreover, the Propositions above give, mutatis mutandis, the following remarkable property:

Proposition 3.4. *Let f and let g be McShane-Stieltjes integrable with respect to α on $[0, 1]$ into X. Then for any real numbers s and t, the function $sf + tg$ is McShane-Stieltjes integrable with respect to α on $[0, 1]$ and*

$$(MS)\text{-}\int_0^1 (sf + tg) d\alpha = s\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] + t\Big[(MS)\text{-}\int_0^1 g d\alpha\Big].$$

The following theorem shows the linearity of the McShane-Stieltjes integrator functions.

Theorem 3.5. *Let α and β be increasing functions on $[0, 1]$ and let c_1 and c_2 be nonnegative real numbers. If $f : [0, 1] \to X$ is a McShane-Stieltjes integrable with respect to α and β respectively, then*

$$(MS)\text{-}\int_0^1 f d(c_1\alpha + c_2\beta) = c_1\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] + c_2\Big[(MS)\text{-}\int_0^1 f d\beta\Big].$$

Proof. We first show that $(MS)\text{-}\int_0^1 f d(\alpha + \beta) = (MS)\text{-}\int_0^1 f d\alpha + (MS)\text{-}\int_0^1 f d\beta$ and next show that $(MS)\text{-}\int_0^1 f d(k\alpha) = k[(MS)\text{-}\int_0^1 f d\alpha]$ for a negative real number k.

First, suppose that $f : [0,1] \to X$ is a McShane-Stieltjes integrable with respect to both α and β. Then for each $\epsilon > 0$, there exists a positive function δ_1 on $[0,1]$ such that $\|f_\alpha(P_1) - (MS)\text{-}\int_0^1 f d\alpha\| < \frac{\epsilon}{2}$ whenever a tagged partition P_1 on $[0,1]$ is sub δ_1, and a positive function δ_2 on $[0,1]$ such that $\|f_\beta(P_2) - (MS)\text{-}\int_0^1 f d\beta\| < \frac{\epsilon}{2}$ whenever a tagged partition P_2 of $[0,1]$ is sub δ_2.

Choose a tagged partition $P = \{(x_i, [a_i, b_i]) : 1 \le i \le n\}$ of $[0,1]$ consisting of elements that is the intersection of every element of P_1 and P_2.

Let $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ for x in $[0,1]$. Then δ is a positive function on $[0,1]$ and also P is sub δ on $[0,1]$. Thus the following properties hold: for given $\epsilon > 0$,

$$\left\|f_\alpha(P) - (MS)\text{-}\int_0^1 f d\alpha\right\| < \frac{\epsilon}{2}$$

whenever a tagged partition P is sub δ on $[0,1]$ and

$$\left\|f_\beta(P) - (MS)\text{-}\int_0^1 f d\beta\right\| < \frac{\epsilon}{2}$$

whenever a tagged partition P is sub δ on $[0,1]$. Hence,

$$\begin{aligned}
&\left\|f_{\alpha+\beta}(P) - \left[(MS)\text{-}\int_0^1 f d\alpha + (MS)\text{-}\int_0^1 f d\beta\right]\right\| \\
= &\left\|\sum_{i=1}^{n} f(x_i)[(\alpha+\beta)(b_i) - (\alpha+\beta)(a_i)] - \left[(MS)\text{-}\int_0^1 f d\alpha + (MS)\text{-}\int_0^1 f d\beta\right]\right\| \\
= &\Bigg\|\left[\sum_{i=1}^{n} f(x_i)[\alpha(b_i) - \alpha(a_i)] + \sum_{i=1}^{n} f(x_i)[\beta(b_i) - \beta(a_i)]\right] \\
&\qquad - \left[(MS)\text{-}\int_0^1 f d\alpha + (MS)\text{-}\int_0^1 f d\beta\right]\Bigg\| \\
\le &\left\|f_\alpha(P) - (MS)\text{-}\int_0^1 f d\alpha\right\| + \left\|f_\beta(P) - (MS)\text{-}\int_0^1 f d\beta\right\| \\
< &\ \epsilon
\end{aligned}$$

whenever a tagged partition $P = \{(x_i, [a_i, b_i]) : 1 \le i \le n\}$ is sub δ on $[0,1]$. Thus, we get

$$(MS)\text{-}\int_0^1 f d(\alpha+\beta) = (MS)\text{-}\int_0^1 f d\alpha + (MS)\text{-}\int_0^1 f d\beta.$$

Second, suppose that f is a McShane-Stieltjes integrable with respect to α on $[0,1]$ and k is a nonnegative real number.

Case 1: $k = 0$. It is trivial.

Case 2: $k > 0$. Given $\epsilon > 0$, there exists a positive function δ on $[0,1]$ such that

$$\left\| f_\alpha(P) - (MS)\text{-}\int_0^1 f d\alpha \right\| < \frac{\epsilon}{k}$$

whenever a tagged partition P is sub δ on $[0,1]$. Thus we obtain

$$\begin{aligned}
&\left\| f_{(k\alpha)}(P) - k\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] \right\| \\
= &\left\| \sum_{i=1}^n f(x_i)[k\alpha(b_i) - k\alpha(a_i)] - k\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] \right\| \\
= &\left\| k\sum_{i=1}^n f(x_i)[\alpha(b_i) - \alpha(a_i)] - k\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] \right\| \\
= &\left\| k f_\alpha(P) - k\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] \right\| \\
< &\ \epsilon
\end{aligned}$$

whenever a tagged partition P is sub δ on $[0,1]$. Hence,

$$(MS)\text{-}\int_0^1 f d(k\alpha) = k\Big[(MS)\text{-}\int_0^1 f d\alpha\Big].$$

Consequently, we have the following result as required:

$$(MS)\text{-}\int_0^1 f d(c_1\alpha + c_2\beta) = c_1\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] + c_2\Big[(MS)\text{-}\int_0^1 f d\beta\Big]. \qquad \square$$

Theorem 3.6. *If $f : [0,1] \to X$ is a McShane-Stieltjes integrable with respect to α and if $T : X \to Y$ is a bounded linear operator, then the composition $T \circ f : [0,1] \to Y$ is a McShane-Stieltjes integrable with respect to α and*

$$T\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] = (MS)\text{-}\int_0^1 T \circ f \, d\alpha$$

Proof. There exists $M \in \mathbb{R}^+$ such that $\|T\| \le M$ since $T : X \to Y$ is a bounded linear operator. Now let $(MS)\text{-}\int_0^1 f d\alpha = z$. Then, for given $\epsilon > 0$ there exists a positive function δ on $[0,1]$ such that

$$\left\| z - \sum_{i=1}^n f(x_i)[\alpha(b_i) - \alpha(a_i)] \right\| < \frac{\epsilon}{M}$$

whenever a tagged partition $\{(x_i, [a_i, b_i]) : 1 \leq i \leq n\}$ is sub δ on $[0,1]$. Thus,

$$\begin{aligned}
&\left\| T[(MS)\text{-}\int_0^1 f d\alpha] - (MS)\text{-}\int_0^1 (T \circ f) d\alpha \right\| \\
&= \left\| Tz - \sum_{i=1}^{n} (T \circ f)(x_i)[\alpha(b_i) - \alpha(a_i)] \right\| \\
&= \left\| T \cdot \left[z - \sum_{i=1}^{n} f(x_i)[\alpha(b_i) - \alpha(a_i)\right] \right\| \\
&= \|T\| \cdot \left\| z - \sum_{i=1}^{n} f(x_i)[\alpha(b_i) - \alpha(a_i)] \right\| \\
&< M \cdot \frac{\epsilon}{M} = \epsilon
\end{aligned}$$

whenever also $\{(x_i, [a_i, b_i]) : 1 \leq i \leq n\}$ is sub δ on $[0,1]$. Therefore, $T \circ f : [0,1] \to Y$ is a McShane-Stieltjes integrable operator with respect to α and

$$(MS)\text{-}\int_0^1 T \circ f d\alpha = T(z) = T\Big[(MS)\text{-}\int_0^1 f d\alpha\Big]. \qquad \square$$

Corollary 3.7. *If f is a McShane-Stieltjes integrable with respect to α on $[0,1]$ into X and for each t in $[0,1]$, then x^*f is a McShane-Stieltjes integrable with respect to α on $[0,1]$ and for each x^* in X^**

$$(MS)\text{-}\int_0^t x^* f d\alpha = x^*\Big[(MS)\text{-}\int_0^t f d\alpha\Big].$$

Proof. If $x^* = 0$, then the result follows immediately. Now we consider the case that x^* is not zero. Since f is a McShane-Stieltjes integrable with respect to α on $[0,1]$, there exists a positive function δ on $[0,1]$ such that for each $\epsilon > 0$,

$$\left\| f_\alpha(P) - (MS)\text{-}\int_0^1 f d\alpha \right\| < \frac{\epsilon}{\|x^*\|},$$

whenever a tagged partition P is sub δ on $[0,1]$. And then,

$$\begin{aligned}
&\left\| (x^*f)_\alpha(P) - x^*\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] \right\| \\
&= \left\| \sum_{i=1}^{n} (x^*f)(x_i)[\alpha(b_i) - \alpha(a_i)] - x^*\Big[(MS)\text{-}\int_0^1 f d\alpha\Big] \right\|
\end{aligned}$$

$$= \left\| x^*\Big[\sum_{i=1}^{n} f(x_i)[\alpha(b_i) - \alpha(a_i)] - (MS)\text{-}\int_0^1 f d\alpha\Big]\right\|$$
$$= \|x^*\| \cdot \left\| f_\alpha(P) - (MS)\text{-}\int_0^1 f d\alpha \right\|$$
$$< \|x^*\| \cdot \frac{\epsilon}{\|x^*\|} = \epsilon$$

whenever also $P = \{(x_i, [a_i, b_i]) : 1 \le i \le n\}$ is sub δ on $[0,1]$. Hence, x^*f is a McShane-Stieltjes integrable with respect to α on $[0,1]$ and for each $x^* \in X^*$,

$$(MS)\text{-}\int_0^1 (x^*f)d\alpha = x^*\Big[(MS)\text{-}\int_0^1 f d\alpha\Big].$$

Moreover, for each $t \in [0,1]$, f is a McShane-Stieltjes integrable with respect to α on $[0,t]$ by Proposition 3.2.

Considering the above argument carefully, x^*f is a McShane-Stieltjes integrable with respect to α on $[0,t]$ and

$$(MS)\text{-}\int_0^t x^* f d\alpha = x^*\Big[(MS)\text{-}\int_0^t f d\alpha\Big]$$

for $x^* \in X^*$. □

4. Comparison with the Pettis integral

We now proceed to prove that every measurable and Pettis integrable function is McShane-Stieltjes integrable.

Theorem 4.1. *Let $f : [0,1] \to X$ be McShane-Stieltjes integrable with respect to α on $[0,1]$. If $f = g$ almost everywhere on $[0,1]$, then g is McShane-Stieltjes integrable with respect to α on $[0,1]$ and $(MS)\text{-}\int_0^1 f d\alpha = (MS)\text{-}\int_0^1 g\, d\alpha$.*

Proof. It is sufficient to prove that if $f = \theta$ (the zero of X) almost everywhere on $[0,1]$ then f is McShane-Stieltjes integrable with respect to α on $[0,1]$ and $(MS)\text{-}\int_0^1 f d\alpha = \theta$. Since $\|f\| = 0$ a.e. on $[0,1]$, the function $\|f\|$ is McShane-Stieltjes integrable with respect to α on $[0,1]$ and it is Lebesgue integrable since $\int_0^1 \|f\| = 0$. Let $\epsilon > 0$ and choose a positive function δ on $[0,1]$ such that $\|f\|_\alpha(P) < \epsilon$ whenever P is sub δ on $[0,1]$. Let P be sub δ on $[0,1]$ and compute $\|f_\alpha(P) - \theta\| = \|f_\alpha(P)\| \le \|f\|_\alpha(P) < \epsilon$. This shows that f is McShane-Stieltjes integrable with respect to α on $[0,1]$ and $(MS)\text{-}\int_0^1 f d\alpha = \theta$. □

The next definition and the proof of the theorems can be found in Gordon [7]. We shall have necessarily any modifications about them.

Definition 4.2. Let $\{f_n\}$ be a collection of McShane-Stieltjes integrable functions with respect to α on $[0,1]$. The collection $\{f_n\}$ is *uniformly McShane-Stieltjes integrable* with respect to α on $[0,1]$ if there exists a set E in $[0,1]$ such that $\mu(E) = 1 - 0 = 1$ and for each $\epsilon > 0$ there exists a positive function δ on $[0,1]$ such that

$$\left\|(f_n)_\alpha\chi_E(P) - (MS)\text{-}\int_0^1 f_n d\alpha\right\| < \epsilon$$

for all n and whenever P is sub δ on $[0,1]$.

Theorem 4.3. *Let $f_n : [0,1] \to X$ be a McShane-Stieltjes integrable function with respect to α on $[0,1]$ for each positive integer n. If $f_n \to f$ uniformly on $[0,1]$, then f is McShane-Stieltjes integrable with respect to α on $[0,1]$ and*

$$(MS)\text{-}\int_0^1 f d\alpha = \lim_{n\to\infty} (MS)\text{-}\int_0^1 f_n d\alpha.$$

Theorem 4.4. *Let $\{E_n\}$ be a sequence of disjoint measurable sets in $[0,1]$, let $\{x_n\}$ be a sequence in X, and let $f : [0,1] \to X$ be defined by $f(t) = \sum_n x_n\chi_{E_n}(t)$.*

If the series $\sum_n \mu(E_n)x_n$ is unconditionally convergent, then the function f is McShane-Stieltjes integrable with respect to α on $[0,1]$ and

$$(MS)\text{-}\int_0^1 f d\alpha = \sum_n \mu(E_n)x_n[\alpha(b_i) - \alpha(a_i)].$$

Now we are ready to verify the following two theorems that will be used to prove Theorem 4.9.

Theorem 4.5. *If $f : [0,1] \to X$ is Bochner integrable on $[0,1]$, then f is McShane-Stieltjes integrable with respect to α on $[0,1]$.*

Proof. Since f is measurable, there exist $E \subset [0,1]$ with $\mu(E) = 1 - 0 = 1$ and a sequence $\{f_n\}$ of countably-valued functions such that for each n the inequality $\|f_n(t) - f\chi_E(t)\| \le \frac{1}{n}$ holds for all t in $[0,1]$. It is clear that each f_n is Bochner integrable on $[0,1]$. For each n, let $f_n = \sum_n x_k^n\chi_{E_k^n}$ where the sets $\{E_k^n : k \ge 1\}$ are disjoint and measurable. The series $\sum_k \mu(E_k^n)x_k^n$ is absolutely convergent and hence unconditionally convergent for each n. By Theorem 4.4, each of the functions f_n is McShane-Stieltjes integrable with respect to α on $[0,1]$.

Since $f\chi_E$ is the uniform limit of $\{f_n\}$ on $[0,1]$, the function $f\chi_E$ is McShane-Stieltjes integrable with respect to α on $[0,1]$ by Theorem 4.3. And by Theorem 4.1 the function f is McShane-Stieltjes integrable with respect to α on $[0,1]$. □

Theorem 4.6. *Let $f : [0,1] \to X$ be measurable. If f is Pettis integrable on $[0,1]$, then f is McShane-Stieltjes integrable with respect to α on $[0,1]$.*

Proof. Since f is measurable, there exist $E \subset [0,1]$ with $\mu(E) = 1 - 0 = 0$ and a countably-valued function $g : [0,1] \to X$ such that $\|g(t) - f\chi_E(t)\| \leq 1$ for all t in $[0,1]$. It is easy to see that $g - f\chi_E$ is Bochner integrable on $[0,1]$ and that g is Pettis integrable on $[0,1]$. By Theorem 4.5 the function $g - f\chi_E$ is McShane-Stieltjes integrable with respect to α on $[0,1]$. Let $g = \sum_n x_n \chi_{E_n}$ where the E_n's are disjoint, measurable sets in $[0,1]$. Since g is Pettis integrable on $[0,1]$, every subseries of $\sum_n \mu(E_n)x_n$ is weakly convergent. By a theorem of Orlicz and Pettis (cf. Diestel and Uhl [1, p. 22]), the series $\sum_n \mu(E_n)x_n$ is unconditionally convergent. By Theorem 4.4, the function g is McShane-Stieltjes integrable with respect to α on $[0,1]$, and it follows that $f\chi_E = g - (g - f\chi_E)$ is McShane-Stieltjes integrable with respect to α on $[0,1]$. By Theorem 4.1, the function f is McShane-Stieltjes integrable with respect to α on $[0,1]$. □

Theorem 4.7. *Suppose that X contains no copy of c_0 and let $f : [0,1] \to X$ be Dunford integrable on $[0,1]$. If $\int_I f \in X$ for every interval $I \subset [0,1]$, then f is Pettis integrable on $[0,1]$.*

Proof. The proof is a consequence of the Bessaga-Pelczński characterization of Banach spaces that do not contain a copy of c_0 (cf. Diestel and Uhl [1, p. 22]). □

From the fact that every McShane integrable function is Dunford integrable and X-valued on intervals, we obtain the corollary below:

Corollary 4.8. *Suppose that X contains no copy of c_0. If $f : [0,1] \to X$ is McShane-Stieltjes integrable with respect to α on $[0,1]$, then f is Pettis integrable on $[0,1]$.*

Combining Theorem 4.6 and Corollary 4.8, we have the important following result:

Theorem 4.9. *Suppose that X is separable and contains no copy of c_0. A function $f : [0,1] \to X$ is McShane-Stieltjes integrable with respect to α on $[0,1]$ if and only if f is Pettis integrable on $[0,1]$.*

Proof. Suppose that X is separable and contains no copy of c_0. If $f : [0,1] \to X$ is McShane-Stieltjes integrable with respect to α on $[0,1]$, then f is Pettis integrable on $[0,1]$ by Corollary 4.8.

Conversely, if $f : [0,1] \to X$ is Pettis integrable on $[0,1]$, then f is measurable by Theorem 2.2. Therefore f is McShane-Stieltjes integrable with respect to α on $[0,1]$ by Theorem 4.6. $\square$

REFERENCES

1. J. Diestel and J. J. Uhl. Jr.: *Vector measure.* With a foreword by B. J. Pettis. Math. Surveys, No. 15. Amer. Math. Soc., Providence, R. I., 1977. MR **56#**12216
2. N. Dunford and J. J. Schwartz: *Linear Operators*, I. General Theory. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, 7. Interscience, New York, 1958. MR **22#**8302
3. D. H. Fremlin: The Henstock and McShane integrals of vector-valued functions. *Illinois J. Math.* **38** (1994), no. 3, 471–479. MR **95d:**28015
4. ______: The generalized McShane integral. *Illinois J. Math.* **39** (1995), no. 1, 39–67. MR **95j:**28008
5. D. H. Fremlin and J. Mendoza: On the integration of vector-valued functions. *Illinois J. Math.* **38** (1994), no. 1, 127–147. MR **94k:**46083
6. R. A. Gordon: The Denjoy extension of the Bochner, Pettis, and Dunford integral. *Studia Math.* **92** (1989), no. 1, 73–91. MR **90b:**28011
7. ______: The McShane integral of Banach-valued functions. *Illinois J. Math.* **34** (1990), no. 3, 557–567. MR **91m:**26013
8. ______: *The integrals of Lebesgue, Denjoy, Perron, and Henstock.* Graduate Studies in Mathematics, 4. Amer. Math. Soc., Providence, R. I., 1994. MR **95m:**26010
9. R. M. McLeod: *The generalized Riemann integral.* Carus Mathematical Monographs, 20. Mathematical Association of America, Washington, D. C., 1980. MR 82h;26015
10. E. J. McShane: *Unified integration.* Pure and Applied Mathematics, 107. Academic Press, San Diego, 1983. MR **86c:**28002
11. J. M. Park and D. H. Lee: The Denjoy extension of the McShane integral. *Bull. Korean Math. Soc.* **33** (1996), no. 3, 411–417. MR **97i:**28008

DEPARTMENT OF MATHEMATICS, KYONGGI UNIVERSITY, SAN 94-6 IUI-DONG, PALDAL-GU, SUWON, GYEONGGI-DO 442-760, KOREA

E-mail address: biseung@kuic.kyonggi.ac.kr