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LORENTZIAN ALMOST PARACONTACT MANIFOLDS AND THEIR SUBMANIFOLDS

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ABSTRACT. This is a survey article on almost Lorentzian paracontact manifolds. The study of Lorentzian almost paracontact manifolds was initiated by Matsumoto [On Lorentzian paracontact manifolds, *Bull. Yamagata Univ. Nat. Sci.* **12** (1989), 151-156]. Later on several authors studied Lorentzian almost paracontact manifolds and their different classes, viz. *LP*-Sasakian and *LSP*-Sasakian manifolds. Different types of submanifolds, for example invariant, semi-invariant and almost semi-invariant, of Lorentzian almost paracontact manifold have been studied. Here, we present a brief survey of results on Lorentzian almost paracontact manifolds with their different classes and their different kind of submanifolds.

1. INTRODUCTION

The study of Lorentzian almost paracontact manifolds was initiated by Matsumoto [11]. Later on several authors studied Lorentzian almost paracontact manifolds and their different classes, viz. *LP*-Sasakian and *LSP*-Sasakian manifolds (cf. [5], [12], [13], [14], [15], [19], [20], [27]). Different types of submanifolds, for example invariant, semi-invariant and almost semi-invariant, of Lorentzian almost paracontact manifold have been studied in [3], [4], [6], [7], [18], [21], [22], [24], [29] and [30].

Here, we present a brief survey of results on Lorentzian almost paracontact manifolds with their different classes and their different kind of submanifolds. In Section 2, the definition of Lorentzian almost paracontact manifolds is given, while in Section 3 different classes of Lorentzian almost paracontact manifolds are defined. In Section 4, some example(s) and their construction is given. In Section 5, sectional curvature in an *LP*-Sasakian manifold is discussed. Results concerned with

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infinitesimal *CL*-transformation in an *LP*-Sasakian manifold are presented in Section 6. Some *CL*-relations are given in Section 7. In Section 8, two results on null structure conformal vector fields on an *LP*-Sasakian manifold are collected, while some results on ξ-null geodesic gradient vector fields on an *LP*-Sasakian manifold are given in Section 9. Section 10 contains two properties of *LP*-Sasakian manifolds with η-parallel Ricci tensor. In Section 11, results on η-Einstein *LP*-Sasakian manifolds are compiled. Some transformations in *LP*-Sasakian manifolds are subject matter of Section 12. Section 13 deals with 3-dimensional *LP*-Sasakian manifolds. Later part gives a brief account of results, on submanifolds of Lorentzian almost paracontact manifolds and their different classes, so far. Some basic formulas on submanifolds of Lorentzian almost paracontact manifolds are given in Section 14. Different types of submanifolds are discussed in Section 15. Some results on submanifolds of Lorentzian *s*-paracontact manifold are given in Section 16. Certain integrability conditions for natural distributions on submanifolds of Lorentzian almost paracontact manifolds and their different classes are given in Section 17. Totally umbilical and totally geodesic submanifolds are dealt in Section 18. In Section 19, non-existence of an anti-invariant distribution on certain submanifold of an *LP*-Sasakian manifold and non-existence of proper mixed foliated semi-invariant submanifolds of a Lorentzian *s*-paracontact manifold are demonstrated.

2. LORENTZIAN ALMOST PARACONTACT MANIFOLDS

Let an n-dimensional smooth connected paracompact Hausdorff manifold $\bar{M}$ be equipped with a *Lorentzian metric* g, that is, g is a smooth symmetric tensor field of type $(0,2)$ such that at every point $p \in \bar{M}$, the tensor $g_p : T_p\bar{M} \times T_p\bar{M} \to \mathbb{R}$ is a non-degenerate innerproduct of signature $(-,+,\cdots,+)$, where $T_p\bar{M}$ is the tangent space of $\bar{M}$ at p and $\mathbb{R}$ is the real line. In other words, a matrix representation of g_p has one eigenvalue negative and all other eigenvalues positive. Then $\bar{M}$ is called a *Lorentzian manifold.* A non-zero vector $X_p \in T_p\bar{M}$ is known to be *spacelike, null, non-spacelike* or *timelike* if it satisfies

$$g_p(X_p, X_p) > 0, \quad = 0, \quad \leq 0 \quad \text{or} \quad < 0$$

respectively.

Let $\bar{M}$ be an n-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on $\bar{M}$ such that (cf.

Matsumoto [11])

$$\eta(\xi) = -1, \tag{1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2}$$

where I denotes the identity map of $T_p\bar{M}$ and the symbol $\otimes$ is the tensor product. These two equations imply that

$$\eta \circ \phi = 0, \tag{3}$$

$$\phi\xi = 0, \tag{4}$$

$$\operatorname{rank}(\phi) = n - 1. \tag{5}$$

Then $\bar{M}$ admits a *Lorentzian metric* g, such that

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

and $\bar{M}$ is said to admit a *Lorentzian almost paracontact structure* (ϕ, ξ, η, g). In this case, we get

$$g(X, \xi) = \eta(X), \tag{7}$$

$$\Phi(X, Y) \equiv g(X, \phi Y) = g(\phi X, Y) = \Phi(Y, X), \tag{8}$$

$$(\bar{\nabla}_X \Phi)(Y, Z) = g(Y, (\bar{\nabla}_X \phi)Z) = (\bar{\nabla}_X \Phi)(Z, Y), \tag{9}$$

where $\bar{\nabla}$ is the covariant differentiation with respect to g. The Lorentzian metric g makes ξ a timelike unit vector field, that is, $g(\xi, \xi) = -1$. The manifold $\bar{M}$ equipped with a Lorentzian almost paracontact structure (ϕ, ξ, η, g) is said to be a *Lorentzian almost paracontact manifold* (briefly *LAP-manifold*) (cf. [11], [12]).

In equations (1) and (2) if we replace ξ by $-\xi$, then the triple (ϕ, ξ, η) is an *almost paracontact structure* on $\bar{M}$ defined by Satō [25]. The Lorentzian metric given by (6) stands analogous situation to almost paracontact Riemannian metric for any almost paracontact manifold (cf. [25], [26]).

3. DIFFERENT CLASSES OF LAP-MANIFOLDS

An LAP-manifold $\bar{M}$ equipped with the structure (ϕ, ξ, η, g) is called a *Lorentzian paracontact manifold* (briefly *LP-manifold*) [11] if

$$\Phi(X, Y) = \frac{1}{2}\left(\left(\bar{\nabla}_X \eta\right) Y + \left(\bar{\nabla}_Y \eta\right) X\right). \tag{10}$$

An LAP-manifold $\bar{M}$ equipped with the structure (ϕ, ξ, η, g) is said to be *Lorentzian para Sasakian* (in brief, *LP-Sasakian*) *manifold* [11] if

$$(\bar{\nabla}_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X, \tag{11}$$

or equivalently,

$$(\bar{\nabla}_X \phi)Y = \eta(Y)X + g(X, Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{12}$$

or equivalently,

$$(\bar{\nabla}_X \Phi)(Y, Z) = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z). \tag{13}$$

In an LP-Sasakian manifold the 1-form η is closed.

Also in [11] it is proved that if an n-dimensional Lorentzian manifold $(\bar{M}, g)$ admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\bar{\nabla}_X \bar{\nabla}_Y \eta)Z = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z), \tag{14}$$

then $\bar{M}$ admits an LP-Sasakian structure.

An n-dimensional Lorentzian manifold $(\bar{M}, g)$ is said to be a *Lorentzian special para Sasakian* (in brief, *LSP-Sasakian*) *manifold* [11] if $\bar{M}$ admits a timelike unit vector field ξ with its associated 1-form η satisfying

$$\Phi(X, Y) = (\bar{\nabla}_X \eta)Y = \varepsilon\left(g\left(X, Y\right) + \eta\left(X\right)\eta\left(Y\right)\right), \quad \varepsilon^2 = 1. \tag{15}$$

Of course, an LSP-Sasakian manifold is an LP-Sasakian manifold.

On the other hand, the eigen values of ϕ are -1, 0 and 1; and the multiplicity of 0 is one. Let k and l be the multiplicities of -1 and 1 respectively. Then $\operatorname{trace}(\phi) = l - k$. So, if $(\operatorname{trace}(\phi))^2 = (n-1)^2$, then either $l = 0$ or $k = 0$. In this case we call the structure a trivial LP-Sasakian structure [5].

An LAP-manifold is called an *LP-cosymplectic manifold* [24] if

$$\bar{\nabla}\phi = 0, \tag{16}$$

and an *LP-nearly cosymplectic manifold* [24] if

$$(\bar{\nabla}_X \phi)\, X = 0, \quad X \in T\bar{M}. \tag{17}$$

4. Examples

A beautiful example of a 5-dimensional *LP*-Sasakian manifold is given as follows.

Example 4.1 (Matsumoto, Mihai and Rosca [13]). Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). Defining

$$\eta = ds - ydx - tdz, \quad \xi = \frac{\partial}{\partial s}, \quad g = \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2,$$

and

$$\begin{aligned}
&\phi\left(\frac{\partial}{\partial x}\right) = -\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial s}, \quad \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y},\\
&\phi\left(\frac{\partial}{\partial z}\right) = -\tfrac{\partial}{\partial z} - t\tfrac{\partial}{\partial s}, \quad \phi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \quad \phi\left(\frac{\partial}{\partial s}\right) = 0
\end{aligned}$$

the structure (ϕ, ξ, η, g) becomes an *LP*-Sasakian structure in $\mathbb{R}^5$. The metric tensor g can be expressed by matrix

$$g = \begin{pmatrix} 1+y^2 & 0 & ty & 0 & -y \\ 0 & -1 & 0 & 0 & 0 \\ ty & 0 & -1+t^2 & 0 & -t \\ 0 & 0 & 0 & -1 & 0 \\ -y & 0 & -t & 0 & 1 \end{pmatrix}.$$

Recently, Tripathi and Shukla [32] have found the examples of Lorentzian almost paracontact structures on an almost paracontact Riemannian manifold. A differentiable manifold $\bar{M}$ is said to admit an *almost paracontact Riemannian structure* (ϕ, ξ, η, g), where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\bar{M}$ such that

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{18}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{19}$$

for all vector fields X and Y on $\bar{M}$ (see [25, 26]). Here, we state the following theorem which interrelates the Riemannian and Lorentzian almost paracontact structures.

Theorem 4.2 (Tripathi and Shukla [32]). *On a differentiable manifold $\bar{M}$, (ϕ, ξ, η, g) is an LAP-structure if and only if (ϕ, ξ, η', g') is an almost paracontact Riemannian structure, where η, g, η', g' are related by*

$$\eta(X) = -\eta'(X), \tag{20}$$

$$g(X,Y) = g'(X,Y) - 2\eta(X)\eta(Y). \tag{21}$$

In view of the preceding theorem, it is now easy to construct a Lorentzian almost paracontact structure by an almost paracontact Riemannian structure and vice-versa.

An LAP-structure is not unique on a differentiable manifold. In fact, we have

Theorem 4.3 (Prasad and Ojha [24]). *If $\bar{M}$ admits an LAP-structure (ϕ, ξ, η, g), then (ϕ', ξ', η', g') is also an LAP-structure, where for a non-singular $(1,1)$ tensor ψ*

$$\xi' = \psi^{-1}\xi, \quad \eta' = \eta \circ \psi, \quad \phi' = \psi^{-1}\phi\psi, \quad g'(X,Y) = g(\psi X, \psi Y).$$

5. SECTIONAL CURVATURE IN AN LP-SASAKIAN MANIFOLD

Let $(\bar{M}, g)$ be an n-dimensional Lorentzian manifold. A 2-dimensional linear subspace E of $T_p\bar{M}$ is called a *plane section*. A plane section E is said to be non-degenerate if for each non-trivial vector $V \in E$, there exists a vector $U \in E$ such that $g(U,V) \neq 0$. If V and U form a basis of a non-degenerate plane section E, then $g(V,V)\,g(U,U) - g(V,U)^2$ is a non-zero quantity. The *sectional curvature* $K(p,E)$ of a non-degenerate plane section E at p with a basis $\{V,U\}$ is given by

$$K(p,E) = \frac{-g(R(V,U)V,U)}{g(V,V)\,g(U,U) - g(V,U)^2},$$

where R denotes the curvature tensor with respect to g.

A non-degenerate plane section E is said to be timelike if it is spanned by a spacelike vector and a timelike vector [1]. The sectional curvature of an LP-Sasakian manifold is determined by the following theorem [11]:

Theorem 5.1 (Matsumoto [11]). *In an n-dimensional LP-Sasakian manifold $\bar{M}$ with structure (ϕ, ξ, η, g) the sectional curvature of each plane section which is spanned by ξ and a spacelike vector is equal to 1.*

Theorem 5.2 (Matsumoto [11]). *Let $(\bar{M}, g)$ be a Lorentzian manifold of dimension n. Suppose that*

(i) *$\bar{M}$ admits a timelike unit vector field ξ,*
(ii) *the 1-form η associated to ξ is closed,*
(iii) *$\mathcal{L}_\xi \bar{\nabla}\xi = 0$, where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ; and*
(iv) *the sectional curvature for timelike planes containing ξ are equal to 1 at every point of $\bar{M}$.*

Then $\bar{M}$ has a Lorentzian paracontact structure.

6. Infinitesimal CL-transformations in an LP-Sasakian manifold

Definition 6.1. A vector field V^i in an LP-Sasakian manifold $\bar{M}$ is said to be an *infinitesimal CL-transformation* if it satisfies

$$\mathcal{L}_V \left\{ {h \atop j\ i} \right\} = \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha \left(\eta_j \phi_i^h + \eta_i \phi_j^h \right) + \beta \phi_{ji} \xi^h$$

for certain vector field ρ_i which is called an associated vector field and certain constants α and β, where $\mathcal{L}_V$ denotes the Lie derivative with respect to V^i and $\left\{ {h \atop j\ i} \right\}$ is the Christoffel symbol of g_{ji}.

The nature of the associated vector field for an infinitesimal CL-transformation in an LP-Sasakian manifold is determined by the followings:

Proposition 6.2 (Matsumoto and Mihai [12]). *For an infinitesimal CL-transformation V^i in an LP-Sasakian manifold, the associated vector field ρ_i is closed, that is, $\bar{\nabla}_j \rho_i = \bar{\nabla}_i \rho_j$.*

Theorem 6.3 (Matsumoto and Mihai [12]). *For an infinitesimal CL-transformation V^i in an LP-Sasakian manifold, we have*

$$\mathcal{L}_V g_{ji} = -\bar{\nabla}_j \rho_i + (\alpha + \beta)\, g_{ji} + (\beta - \alpha)\, \eta_j \eta_i.$$

In particular, if $\alpha = \beta$, then the vector field $V^i + \frac{1}{2}\rho^i$ is a homothetic conformal Killing vector field.

7. CL-RELATIONS

A symmetric affine connection in an *LP*-Sasakian manifold $\bar{M}$ with the structure (ϕ, ξ, η, g) is said to be *CL-related* with the connection $\left\{ {}_{j}{}^{h}{}_{i} \right\}$, if it satisfies

$$\Gamma_{j}{}^{h}{}_{i} = \left\{ {}_{j}{}^{h}{}_{i} \right\} + \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha \left(\eta_j \phi_i^h + \eta_i \phi_j^h \right) + 2\xi^h \phi_{ji} \tag{22}$$

for a certain vector field ρ^i and a certain constant α.

The curvature tensor of $\Gamma_{j}{}^{h}{}_{i}$ is obtained by the followings.

Proposition 7.1 (Matsumoto and Mihai [12]). *If a symmetric affine connection $\Gamma_{j}{}^{h}{}_{i}$ is CL-related with the connection $\left\{ {}_{j}{}^{h}{}_{i} \right\}$ in an LP-Sasakian manifold $\bar{M}$, then the curvature tensor R'^{h}_{kji} of $\Gamma_{j}{}^{h}{}_{i}$ is given by*

$$\begin{aligned} R'^{h}_{kji} = {} & R^h_{kji} + (P_{kj} - P_{jk})\,\delta_i^h + P_{ki}\delta_j^h - P_{ji}\delta_k^h + \\ & + 2(\alpha - 1)\left(\phi_k^h \phi_{ji} - \phi_j^h \phi_{ki} \right) + (3\alpha + 2)\,(g_{ki}\eta_j - g_{ji}\eta_k)\,\xi^h, \end{aligned} \tag{23}$$

where the tensor P_{ji} is defined by

$$P_{ji} = \bar{\nabla}_j \rho_i + \alpha(\alpha - 2)\,\eta_j \eta_i - \rho_j \rho_i - 2\xi^l \rho_l \phi_{ji} - (\widetilde{\rho}_j \eta_i + \widetilde{\rho}_i \eta_j) \tag{24}$$

with $\widetilde{\rho}_j = \rho_l \phi_j^l$.

Proposition 7.2 (Matsumoto and Mihai [12]). *If the connection $\Gamma_{j}{}^{h}{}_{i}$ defined by (22) is flat, then the vector field ρ^i is closed.*

Theorem 7.3 (Matsumoto and Mihai [12]). *In an LP-Sasakian manifold $\bar{M}$, if the connection $\Gamma_{j}{}^{h}{}_{i}$ defined by (22) is flat, then the curvature tensor R^h_{kji} with respect to g_{ji} satisfies*

$$\begin{aligned} R^h_{kji} = {} & (3\alpha + 1)\left(g_{ki}\delta_j^h - g_{ji}\delta_k^h \right) + 2(\alpha - 1)\left(\phi_{ki}\phi_j^h - \phi_{ji}\phi_k^h \right) \\ & + (3\alpha + 2)\left(\eta_k \eta_i \delta_j^h - \eta_j \eta_i \delta_k^h + g_{ki}\eta_j \xi^h - g_{ji}\eta_k \xi^h \right). \end{aligned} \tag{25}$$

During the study of an invariant tensor field under *CL*-relation in an *LP*-Sasakian manifold, the authors of [12] obtained ann invariant tensor field under a *CL*-relation given by (22). Then they introduced the notion of a *CL*-relation in an *LP*-Sasakian manifold.

Theorem 7.4 (Matsumoto and Mihai [12]). *Let a symmetric affine connection $\Gamma_{j}{}^{h}{}_{i}$ in an LP-Sasakian manifold $\bar{M}$ be CL-related with the Lorentzian metric connection $\left\{ {}_{j}{}^{h}{}_{i} \right\}$ and it be given by (22). If the tensor field P_{ji} is symmetric with respect to* *j and i, then a tensor field W_{kji}^h is invariant under this relation, where the tensor field W_{kji}^h is given by*

$$
\begin{aligned}
(26) \quad W_{kji}^h = T_{kji}^h &- (3\alpha+2) A_1 \left(T_{ki}\delta_j^h - T_{ji}\delta_k^h\right) \\
&- (3\alpha+2) B_1 \left(T_{tki}{}^s\delta_j^h - T_{tji}{}^s\delta_k^h\right)\phi_s^t - 2(\alpha-1) A_2 \left(\phi_k^h T_{ji} - \phi_j^h T_{ki}\right) \\
&- 2(\alpha-1) B_2 \left(\phi_k^h T_{tji}{}^s - \phi_j^h T_{tki}{}^s\right)\phi_s^t \\
&- (3\alpha+2) A_1 \xi^h \left(\left(T_{ki}\eta_j - T_{ji}\eta_k\right) + \left(T_{tki}{}^s\eta_j - T_{tji}{}^s\eta_k\right)\phi_s^t\right)
\end{aligned}
$$

for certain constants A_1 ,A_2, B_1 and B_2, T_{kji}^h is defined as

$$
(27) \qquad T_{kji}^h = Z_{kji}^h - Z_{tki}^s\eta_s\xi^t\delta_j^h + Z_{tji}^s\eta_s\xi^t\delta_k^h,
$$

$$
(28) \qquad Z_{kji}^h = R_{kji}^h + \frac{1}{n-1}\left(R_{ki}\delta_j^h - R_{ji}\delta_k^h\right)
$$

and $T_{ji} = T_{lji}^l$.

An LP-Sasakian manifold $\bar{M}$ is said to be *CL-flat* if the tensor field W_{kji}^h defined by (26) vanishes identically. Then the following is proved.

Theorem 7.5 (Matsumoto and Mihai [12]). *Let $\bar{M}$ be an n-dimensional CL-flat LP-Sasakian manifold. Then we have the following two cases:*

(A) *the manifold $\bar{M}$ is* LSP-*Sasakian and the curvature tensor R_{kji}^h has the form*

$$
R_{kji}^h = c\left(g_{ki}\delta_j^h - g_{ji}\delta_k^h\right) + (c+1)\left(\left(\delta_j^h\eta_k - \delta_k^h\eta_j\right)\eta_i - \left(g_{ji}\eta_k - g_{ki}\eta_j\right)\xi^h\right)
$$

for certain constant c, or

(B) *the curvature tensor R_{kji}^h has the form*

$$
\begin{aligned}
R_{kji}^h = a&\left(g_{ki}\delta_j^h - g_{ji}\delta_k^h\right) + (a-1)\left(\phi_{ki}\phi_j^h - \phi_{ji}\phi_k^h\right) \\
&+ (a+1)\left(\left(\delta_j^h\eta_k - \delta_k^h\eta_j\right)\eta_i - \left(g_{ji}\eta_k - g_{ki}\eta_j\right)\xi^h\right) \\
&+ b\left(\phi_{ki}\phi_j^h - \phi_{ji}\phi_k^h + g_{ki}\phi_j^h - g_{ji}\phi_k^h + \left(\phi_{ki}\eta_j - \phi_{ji}\eta_k\right)\xi^h \right. \\
&\qquad \left. - \left(\phi_j^h\eta_k - \phi_k^h\eta_j\right)\eta_i\right)
\end{aligned}
$$

for certain constant a and b.

8. NULL STRUCTURE CONFORMAL VECTOR FIELDS ON AN LP-SASAKIAN MANIFOLD

Definition 8.1 (Mihai and Rosca [14]). If dp is the soldering form (or a line element) of an LP-Sasakian manifold $\bar{M}$ and C is a vector field such that

$$\bar{\nabla} C = f dp + \xi \wedge C = f dp + \alpha \otimes \xi - \eta \otimes C,$$

where $f \in C^{\infty}\bar{M}$ and $\alpha = \flat(C)$is the dual 1-form of C, then C is defined as a *structure conformal vector field* ($\mathcal{L}_C g = \rho g; \rho = 2f$). The vector C is *null vector* if $g(C, C) = 0$.

The existence of the conformal vector field C defined above is determined by the following theorems:

Theorem 8.2 (Mihai and Rosca [14]). *Let $\bar{M}(\phi, \xi, \eta, g)$ be an LP-Sasakian manifold and let C be a null structure conformal vector field on $\bar{M}$. The existence of C is determined by an exterior differential system in involution and any manifold $\bar{M}$ which carries such a C is foliated by a 3-dimensional totally geodesic and of scalar curvature (-1) submanifolds, tangent to C, ϕC and ξ.*

Theorem 8.3 (Mihai and Rosca [14]). *Let C be a null structure conformal vector field on an LP-Sasakian manifold $\bar{M}(\phi, \xi, \eta, g)$ and ρ (resp. α) be the conformal scalar associated with C (resp. the dual form of C). Then one has the following properties:*

(i) *ρ is an eigenfunction of Δ and an isoparametric function;*
(ii) *if $\mathcal{L}_C^{*}\eta$ denotes the formal adjoint of $\mathcal{L}_C\eta$, then one has the relation $\mathcal{L}_C^{*}\eta = (n-1)\alpha$;*
(iii) *the two form $\alpha \wedge \eta$ is harmonic;*
(iv) *if $Z, Z' \in T\bar{M}$, then*

$$\left(\mathcal{L}_C \bar{\nabla}\right)(Z, Z') = -\rho\eta(Z) Z' - \rho\eta(Z') Z + -\rho g(Z, Z')\xi;$$

(v) *C defines an infinitesimal conformal transformation of η, of the adjoint $*\eta$ of η and of all the functions $g(C, Z)$.*

9. ξ-NULL GEODESIC GRADIENT VECTOR FIELDS ON AN LP-SASAKIAN MANIFOLD

Definition 9.1 (Matsumoto, Mihai and Rosca [13]). Let $\bar{M}(\phi, \xi, \eta, g)$ be an *LP*-Sasakian manifold and let $\bar{\nabla}$, dp and U be the Levi-Civita covariant differential operator with respect to g, the soldering form (or a line element) and a real null vector field on $\bar{M}$, respectively. If U satisfies

$$\bar{\nabla} U = \lambda dp + \eta \otimes U + u \otimes \xi,$$

where λ (resp. $u = b(U)$ is the associated scalar field (resp. the dual form of U), then U is said to be a *ξ-null geodesic gradient vector field.*

The existence of the vector field U has been ensured by the following theorem:

Theorem 9.2 (Matsumoto, Mihai and Rosca [13]). *Let $\bar{M}(\phi, \xi, \eta, g)$ be an LP-Sasakian manifold and let U be ξ-null geodesic gradient vector field on $\bar{M}$. The existence of U is determined by an exterior differential system in involution and any M which carries such a null vector field U is the local Riemannian product $M = M_U \times M_U^{\perp}$ such that*

(i) *M_U is totally geodesic surface of scalar curvature -1 tangential to U and ξ;*
(ii) *$M_U^{\perp}$ is a totally umbilical 2-codimensional submanifold having U as normal null section. Furthermore*
(iii) *U is an exterior concurrent vector field;*
(iv) *the conformal scalar λ associated with U is an isoparametric function and satisfies*

$$Ric\,(\phi U) + \lambda^2 = 0;$$

(v) *U defines an infinitesimal contact transformation on M and ϕU admits infinitesimal transformations of generators ξ.*

Also the necessary and sufficient condition in order that the vector field U defines an infinitesimal conformal transformation in an *LSP*-Sasakian manifold is determined by the following theorem:

Theorem 9.3 (Matsumoto, Mihai and Rosca [13]). *Let $\bar{M}(\phi, \xi, \eta, g)$ be an LSP-Sasakian manifold having Φ as almost cosymplectic form and let ψ be the Lefebvre form associated with the semi-cosymplectic structure defined by the pair (Φ, η). Suppose that $\bar{M}$ carries a ξ-null geodesic gradient vector field U. Then the necessary* *and sufficient condition in order that U defines an infinitesimal transformation of Φ, that is, $\mathcal{L}_U\Phi = r\Phi$, is that the conformal scalar r be defined by*

$$r = -2\eta(U) + \text{constant};$$

and in this case U defines also an infinitesimal conformal transformation of ψ, that is, $\mathcal{L}_U\psi = r\psi$.

If in addition of U, $\bar{M}$ carries a null structure conformal vector field C, then $\bar{M}$ is the local Riemannian product $\bar{M} = M_C \times M_C^{\perp}$ such that

(i) M_C is a 3-dimensional submanifold of scalar curvature -1 and it is totally geodesic and tangent to C, U and ξ;

(ii) $M_C^{\perp}$ is totally umbilical 3-codimensional submanifold.

Furthermore, the conformal scalar ρ and λ corresponding to C and U respectively, satisfy $\rho\lambda =$ constant and

(a) C defines an infinitesimal conformal transformation of the dual form of U;

(b) the Lie derivative with respect to U of the dual form of C is $d\eta$-exact.

Pandey and Ojha [17] introduced and studied D-conformal transformation in an LP-contact manifold. The conditions for an LP-contact to be an LP-cosymplectic manifold are also obtained. LP-Sasakian manifolds are also studied by Pokhariyal [19]. In 1999, the authors of [5] studied the LP-Sasakian manifold and obtained the following results:

Proposition 9.4 (De, Matsumoto and Shaikh [5]). *Each LP-Sasakian space form is of curvature* 1.

Theorem 9.5 (De, Matsumoto and Shaikh [5]). *Each LP-Sasakian space form is a trivial LSP-Sasakian space form.*

Theorem 9.6 (De, Matsumoto and Shaikh [5]). *A conformally flat LP-Sasakian manifold is trivial LSP-Sasakian. Especially, if the scalar curvature r satisfies $r = n(n-1)$, then the manifold is a trivial LSP-Sasakian space form.*

Again the nature of Weyl-semi-symmetric LP-Sasakian manifolds is obtained by the following theorem:

Theorem 9.7 (De, Matsumoto and Shaikh [5]). *An n-dimensional $(n > 3)$ LP-Sasakian manifold satisfying $R(X,Y)\cdot C = 0$ is a trivial LSP-Sasakian space form, where C is the conformal curvature tensor.*

From the above theorem the following corollary follows:

Corollary 9.8 (De, Matsumoto and Shaikh [5]). *An n-dimensional ($n > 3$) conformally symmetric LP-Sasakian manifold is LSP-Sasakian.*

Theorem 9.9 (De, Matsumoto and Shaikh [5]). *An n-dimensional ($n > 3$) conformally recurrent LP-Sasakian manifold is a trivial LSP-Sasakian space form.*

10. LP-Sasakian manifolds with η-parallel Ricci tensor

Definition 10.1. The Ricci tensor S of an LP-Sasakian manifold $\bar{M}$is called *η-parallel* if it satisfies

$$\left(\bar{\nabla}_X S\right)(\phi Y, \phi Z) = 0$$

for all vector fields X, Y and Z.

The notion of *Ricci-η-parallelity* for Sasakian manifolds was introduced by Kon [8]. A necessary and sufficient condition for an LP-Sasakian manifold to have η-parallel Ricci tensor is obtained in Mihai, Shaikh and De [15] which is as follows:

Proposition 10.2 (Mihai, Shaikh and De [15]). *An LP-Sasakian manifold $\bar{M}(\phi, \eta, \xi, g)$ has η-parallel Ricci tensor if and only if*

$$\begin{aligned}&\left(\bar{\nabla}_X S\right)(Y, Z)\\ &\quad = S(X, \phi Z)\eta(Y) + S(X, \phi Y)\eta(Z) - (n-1)\left(\Phi\left(X, Y\right)\eta(Z) + \Phi\left(X, Z\right)\eta(Y)\right)\end{aligned}$$

for all $X, Y, Z \in T\bar{M}$.

Taking an orthonormal frame field and then by contraction the authors [15] proved the following propostion:

Proposition 10.3 (Mihai, Shaikh and De [15]). *Let $\bar{M}(\phi, \eta, \xi, g)$ be an n-dimensional ($n > 3$) LP-Sasakian manifold with η-parallel Ricci tensor. Then we have the following property:*

(a) *the scalar curvature r of $\bar{M}$ is constant;*
(b) *the square of the length $|Q|^2$ of the Ricci operator Q given by $g(QX, Y) = S(X, Y)$ of $\bar{M}$ is constant.*

11. η-EINSTEIN LP-SASAKIAN MANIFOLDS

Definition 11.1 (Mihai, Shaikh and De [15]). An LP-Sasakian manifold $\bar{M}$ is said to be *η-Einstein* if its Ricci tensor S is of the form

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions on $\bar{M}$.

In [15] the Ricci tensor of an LP-Sasakian manifold is obtained as

$$S(Y,Z) = \left(\frac{r}{n-1} - 1\right) g(Y,Z) + \left(\frac{r}{n-1} - n\right) \eta(Y)\eta(Z)$$

and the following theorem is proved.

Theorem 11.2 (Mihai, Shaikh and De [15]). *Let $\bar{M}(\phi,\eta,\xi,g)$ be an n-dimensional $(n > 3)$ η-Einstein LP-Sasakian manifold which is not an Einstein one. Then the scalar curvature r of $\bar{M}$ is a constant if and only if the timelike vector field ξ is harmonic.*

If an LP-Sasakian manifold with η-parallel Ricci tensor admitting a non-null concircular vector field, then the following result is obtained.

Theorem 11.3 (Mihai, Shaikh and De [15]). *Let $\bar{M}(\phi,\eta,\xi,g)$ be an n-dimensional $(n > 3)$ η-Einstein LP-Sasakian manifold with η-parallel Ricci tensor admitting a non-null concircular vector field. Then any one of the following conditions hold in $\bar{M}$:*

(a) *the concircular vector field reduces to a parallel vector field;*
(b) *the scalar curvature of the manifold is given by $r = n\,(n-1)$;*
(c) *the timelike vector field ξ is harmonic.*

12. SOME TRANSFORMATION IN LP-SASAKIAN MANIFOLDS

We now consider a transformation μ which transforms an LP-Sasakian structure (ϕ,η,ξ,g) into another LP-Sasakian structure $(\bar{\phi},\bar{\eta},\bar{\xi},\bar{g})$. We denote by the notation "bar" the geometric objects which are transformed by the transformation μ.

Considering this transformation the following result is obtained.

Theorem 12.1 (Mihai, Shaikh and De [15]). *In an LP-Sasakian manifold $\bar{M}(\phi,\eta,\xi, g)$ the transformation μ which leaves the curvature tensor invariant and $\eta(\bar{\xi}) \neq 0$ is an isometry.*

Again in an almost paracontact Riemannian manifold, if an infinitesimal transformation V satisfies

$$(\mathcal{L}_V\eta)(X) = \sigma\eta(X)$$

for a scalar function σ, then we call it an infinitesimal paracontact transformation. In particular, if σ vanishes identically, then it is called an infinitesimal strict paracontact transformation [10].

The nature of an infinitesimal paracontact transformation in an LP-Sasakian manifold is determined by the following theorem:

Theorem 12.2 (Mihai, Shaikh and De [15]). *In an LP-Sasakian manifold, the infinitesimal paracontact transformation which leaves a Ricci tensor invariant is an infinitesimal strict paracontact transformation.*

13. 3-DIMENSIONAL LP-SASAKIAN MANIFOLDS

Next, Shaikh and De [27] studied the 3-dimensional LP-Sasakian manifolds and obtained several characteristic results in this manifold which can be stated as follows:

Theorem 13.1. *A 3-dimensional LP-Sasakian manifold satisfying the condition $R(X,Y)\cdot S = 0$ is a space form.*

Definition 13.2. An LP-Sasakian manifold is said to be *locally ϕ-symmetric* if

$$\phi^2\left(\bar{\nabla}_W R\right)(X,Y,Z) = 0$$

for all vector fields W, X, Y, Z orthogonal to ξ.

The necessary and sufficient condition for an LP-Sasakian manifold to be locally ϕ-symmetric is obtained by the following theorems (cf. Shaikh and De [27]):

Theorem 13.3. *A 3-dimensional LP-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature r is constant.*

Theorem 13.4. *If a 3-dimensional LP-Sasakian manifold satisfies the condition $R(X,Y)\cdot S = 0$, then the manifold is locally ϕ-symmetric.*

Theorem 13.5. *If a 3-dimensional LP-Sasakian manifold with η-parallel Ricci tensor is locally ϕ-symmetric.*

Theorem 13.6. *If a 3-dimensional LP-Sasakian manifold with η-parallel Ricci tensor is a space form.*

Theorem 13.7. *If a 3-dimensional LP-Sasakian manifold satisfies the condition*

$$\left(\bar{\nabla}_X S\right)(Y,Z)+\left(\bar{\nabla}_Y S\right)(X,Z)+\left(\bar{\nabla}_Z S\right)(X,Y)=0,$$

then the manifold is a space form and hence is locally ϕ-symmetric.

Theorem 13.8. *If a 3-dimensional LP-Sasakian manifold admits a non-null concircular vector field then the manifold is a space form.*

14. Submanifolds

Let M be a submanifold of an LAP-manifold $\bar{M}(\phi,\xi,\eta,g)$. Let the induced metric on M also be denoted by g. Then Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y=\nabla_X Y+h(X,Y),\qquad X,Y\in TM,$$

$$\bar{\nabla}_X N=-A_N X+\nabla^{\perp}_X N,\qquad N\in T^{\perp}M,$$

where ∇ is the induced connection on M, h is the second fundamental form of the immersion, and $-A_N X$ and $\nabla^{\perp}_X N$ are the tangential and normal parts of $\bar{\nabla}_X N$. From these two equations one gets

$$g(h(X,Y),N)=g(A_N X,Y).$$

Moreover, we have

$$\begin{aligned}&(\bar{\nabla}_X\phi)Y\\&\quad=((\nabla_X P)Y-A_{FY}X-th(X,Y))+((\nabla_X F)Y+h(X,PY)-fh(X,Y)),\end{aligned}$$

$$\begin{aligned}&(\bar{\nabla}_X\phi)N\\&\quad=((\nabla_X t)N-A_{fN}X-PA_N X)+((\nabla_X f)N+h(X,tN)-FA_N X),\end{aligned}$$

where

$$\begin{aligned}\phi X &\equiv FX\in T^{\perp}M;\quad PX\in TM,\quad FX\in T^{\perp}M,\\ \phi N &\equiv tN+fN;\qquad tN\in TM,\quad fN\in T^{\perp}M,\\ (\nabla_X P)Y &\equiv \nabla_X PY-P\nabla_X Y,\\ (\nabla_X F)Y &\equiv \nabla^{\perp}_X FY-F\nabla_X Y,\\ (\nabla_X t)N &\equiv \nabla_X tN-t\nabla^{\perp}_X N,\\ (\nabla_X f)N &\equiv \nabla^{\perp}_X fN-f\nabla^{\perp}_X N.\end{aligned}$$

Let $\xi \in TM$. We write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Then we get

$$P\xi = 0 = F\xi, \qquad \eta \circ P = 0 = \eta \circ F,$$

$$P^2 + tF = I + \eta \otimes \xi, \qquad FP + fF = 0,$$

$$f^2 + Ft = I, \qquad tf + Pt = 0,$$

$$\ker(P) = \ker(P^2) = \ker(tF - I - \eta \otimes \xi),$$

$$\ker(F) = \ker(tF) = \ker(P^2 - I - \eta \otimes \xi),$$

$$\ker(t) = \ker(Ft) = \ker(f^2 - I),$$

$$\ker(f) = \ker(f^2) = \ker(Ft + I)$$

$$\ker(P|_{\{\xi\}^{\perp}}) = \ker(P^2|_{\{\xi\}^{\perp}}) = \ker(tF|_{\{\xi\}^{\perp}} - I),$$

$$\ker(F|_{\{\xi\}^{\perp}}) = \ker(tF|_{\{\xi\}^{\perp}}) = \ker(P^2|_{\{\xi\}^{\perp}} - I)_x.$$

The following two propositions are for submanifolds of *LP*-cosymplectic manifolds, tangent to ξ (cf. Tripathi[30]).

Proposition 14.1 (Tripathi [30]). *For a submanifold M of an LP-cosymplectic manifold such that $\xi \in TM$, we have*

$$\nabla_X \xi = 0, \quad h(X, \xi) = 0, \quad A_N X \in \{\xi\}^{\perp}, \quad A_N \xi = 0.$$

Proposition 14.2 (Tripathi [30]). *For a submanifold M of an LP-cosymplectic manifold such that $\xi \in TM$, we have*

$$(\nabla_X P)Y - A_{FY}X - th(X,Y) = 0,$$

$$(\nabla_X F)Y + h(X, PY) - fh(X,Y) = 0,$$

$$(\nabla_X t)N - A_{fN}X - PA_N X = 0,$$

$$(\nabla_X f)N + h(X, tN) - FA_N X = 0.$$

Consequently,

$$(\nabla_X P)\xi = 0, \qquad (\nabla_X F)\xi = 0,$$

$$\nabla_\xi P = 0, \qquad \nabla_\xi F = 0, \qquad \nabla_\xi t = 0, \qquad \nabla_\xi f = 0.$$

15. DIFFERENT TYPES OF SUBMANIFOLDS

A submanifold M of a Lorentzian almost paracontact manifold $\bar{M}$ with $\xi \in TM$ becomes an *almost semi-invariant submanifold* (cf. Kalpana and Singh [7]) of $\bar{M}$ if TM can be decomposed as a direct sum of mutually orthogonal differentiable distributions

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D} \oplus \{\xi\},$$

where

$$\mathcal{D}^1 = \ker(F|_{\{\xi\}^\perp}) = \{X \in \{\xi\}^\perp : \|X\| = \|PX\|\} = TM \cap \phi(TM),$$

$$\mathcal{D}^0 = \ker(P|_{\{\xi\}^\perp}) = \{X \in \{\xi\}^\perp : \|X\| = \|FX\|\} = TM \cap \phi(T^\perp M).$$

Here, the distribution $\mathcal{D}^1$ is invariant, the distribution $\mathcal{D}^0$ is anti-invariant and the distribution $\mathcal{D}$ is neither invariant nor anti-invariant by ϕ. Moreover, we have

$$T^\perp M = \bar{\mathcal{D}}^1 \oplus \bar{\mathcal{D}}^0 \oplus \bar{\mathcal{D}},$$

where

$$\bar{\mathcal{D}}^1 = \ker(t) = T^\perp M \cap \phi(T^\perp M),\ \bar{\mathcal{D}}^0 = \ker(f) = T^\perp M \cap \phi(TM),$$

$$F\mathcal{D}^0 = \bar{\mathcal{D}}^0,\ F\mathcal{D} = \bar{\mathcal{D}},\ t\bar{\mathcal{D}}^0 = \mathcal{D}^0,\ t\bar{\mathcal{D}} = \mathcal{D}.$$

For $X \in TM$ we can write

$$X = U^1X + U^0X + UX - \eta(X)\xi, \tag{29}$$

where U^1, U^0 and U are projection operators of TM on $\mathcal{D}^1$, $\mathcal{D}^0$ and $\mathcal{D}$ respectively.

A submanifold M of a Lorentzian almost paracontact manifold $\bar{M}$ is an *invariant* (resp. *anti-invariant*) *submanifold* of $\bar{M}$ if $\phi(TM) \subset TM$ (resp. $\phi(TM) \subset T^\perp M$). An almost semi-invariant submanifold of a Lorentzian almost paracontact manifold is a *semi-invariant submanifold* if $\mathcal{D} = \{0\}$. A semi-invariant submanifold of a Lorentzian almost paracontact manifold becomes an invariant submanifold (resp. anti-invariant submanifold) if $\mathcal{D}^0 = \{0\}$ (resp. $\mathcal{D}^1 = \{0\}$). An almost semi-invariant submanifold is *proper* if none of the distributions $\mathcal{D}^1$, $\mathcal{D}^0$ and $\mathcal{D}$ is zero. A semi-invariant submanifold is *proper* if $\mathcal{D}^0 \neq \{0\} \neq \mathcal{D}^1$.

16. Submanifolds of Lorentzian s-paracontact manifolds

Analogous to the definition of special paracontact Riemannian manifolds, in [32] the following definition is given.

Definition 16.1 (Tripathi and Shukla [32]). An LAP-manifold is said to be a *Lorentz-ian s-paracontact manifold* if $\phi X = \bar{\nabla}_X \xi$ or $\Phi(X, Y) = (\bar{\nabla}_X \eta)Y$.

An LP-Sasakian manifold is always a Lorentzian s-paracontact manifold. Tripathi and Shukla [32] also prove the following theorem.

Theorem 16.2. *For a submanifold M of a Lorentzian s-paracontact manifold, it follows that*

$$\begin{aligned} \phi X &= \nabla_X \xi + h(X, \xi), && \xi \in TM, \\ \phi X &= -A_\xi X + \nabla^\perp_X \xi, && \xi \in T^\perp M, \\ \eta(A_N X) &= 0, && \xi \in T^\perp M, \\ \eta(A_N X) &= g(\phi X, N), && \xi \in TM \end{aligned}$$

for all $X \in TM$ and $N \in T^\perp M$. Moreover, let ξ be tangential to M. Then M is an invariant submanifold if and only if $h(X, \xi) = 0$, and M is an anti-invariant submanifold if and only if $\nabla_X \xi = 0$.

The following theorem is for totally umbilical submanifolds.

Theorem 16.3 (Tripathi and Shukla [32]). *If M is a totally umbilical submanifold of a Lorentzian s-paracontact manifold such that ξ is tangential to M, then*

(a) *M is necessarily minimal and consequently totally geodesic, and*
(b) *M is an invariant submanifold and $\nabla_X \xi \neq 0$.*

The characterization of an anti-invariant submanifold is as follows.

Theorem 16.4 (Tripathi and Shukla [32]). *A submanifold M of a Lorentzian s-paracontact manifold such that ξ is normal to M is an anti-invariant submanifold if and only if $A_\xi X = 0$. Consequently, if M is totally geodesic then it is anti-invariant.*

Since LP-Sasakian and LSP-Sasakian manifolds are always Lorentzian s-paracontact manifolds, therefore the results of this section are valid for the submanifolds of LP-Sasakian and LSP-Sasakian manifolds.

17. Integrability conditions

Let M be a semi-invariant submanifold of an LAP-manifold $\bar{M}$, and let $[\phi, \phi]$ be the Nijenhuis tensor of ϕ. Let superscripts T and $\perp$ in a term denote its tangential and normal parts respectively. Then necessary and sufficient condition for the integrability of the distributions $\mathcal{D}^1 \oplus \{\xi\}$ and $\mathcal{D}^0 \oplus \{\xi\}$ on a semi-invariant submanifold of an LAP-manifold are given in the following two theorems.

Theorem 17.1 (Tripathi [29]). *Let M be a semi-invariant submanifold of an LAP-manifold. Then the following three statements are equivalent:*

(a) *the distribution $\mathcal{D}^1 \oplus \{\xi\}$ is integrable,*
(b) $([\phi, \phi](X, Y))^T = [P, P](X, Y), \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}$,
(c) $([\phi, \phi](X, Y))^\perp = 0, \quad U^0[P, P](X, Y) = 0, \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}$.

Theorem 17.2 (Tripathi [29]). *The distribution $\mathcal{D}^0 \oplus \{\xi\}$ on a semi-invariant submanifold M of an LAP-manifold is integrable if and only if*

$$[P, P](X, Y) = 0, \qquad X, Y \in \mathcal{D}^0 \oplus \{\xi\}.$$

The distribution $\{\xi\}^\perp$ in a Lorentzian almost paracontact manifold is called the *paracontact distribution* [32]. In [32], the following theorem is proved.

Theorem 17.3. *On a Lorentzian s-paracontact manifold $\bar{M}$ the paracontact distribution $\{\xi\}^\perp$ is integrable.*

This theorem implies the following theorem.

Theorem 17.4 (Tripathi and Shukla [32]). *Let M be a submanifold of a Lorentzian s-paracontact manifold such that ξ is tangential to M. Then the paracontact distribution $\{\xi\}^\perp$ on M is integrable. Consequently, if M is semi-invariant then the distribution $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable; and if M is almost semi-invariant then the distribution $\mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}$ is integrable.*

Following theorems are for integrability conditions of semi-invariant submanifolds of LP-cosymplectic manifolds [30].

Theorem 17.5 (Tripathi [30]). *Let M be a semi-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. Then $\mathcal{D} \in \{\mathcal{D}^1, \mathcal{D}^0\}$ is integrable if and only if $\mathcal{D} \oplus \{\xi\}$ is integrable.*

Theorem 17.6 (Tripathi [30]). *Let M be a semi-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. Then the following statements are equivalent:*

(a) $\mathcal{D}^0$ is integrable,
(b) $\mathcal{D}^0 \oplus \{\xi\}$ is integrable,
(c) $A_{FX}Y = 0, \qquad X, Y \in \mathcal{D}^0$,
(d) $h(Y, Z) \in \bar{\mathcal{D}}^1, \qquad X \in \mathcal{D}^0,\ Z \in TM$.

Theorem 17.7 (Tripathi [30]). *Let M be a semi-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. Then the following statements are equivalent:*

(a) $\mathcal{D}^1$ is integrable,
(b) $\mathcal{D}^1 \oplus \{\xi\}$ is integrable,
(c) $h(X, PY) = h(PX, Y), \qquad X, Y \in \mathcal{D}^1$,
(d) $g(h(X, PY), FZ) = g(h(PX, Y), FZ), \qquad X, Y \in \mathcal{D}^1,\ Z \in TM$.

18. Totally umbilical and totally geodesic submanifolds

Let $\mathcal{D}$ be a distribution on a submanifold M of an LAP-manifold. Then M is $\mathcal{D}$-totally umbilical if $h(X,Y) = g(X,Y)\mathbf{K}$ for some $\mathbf{K} \in T^{\perp}M$. In [30], the following results are proved.

Theorem 18.1. *Let $\mathcal{D}$ be a distribution on a submanifold M of an LP-cosymplectic manifold such that $\xi \in \mathcal{D}$. If M is $\mathcal{D}$-totally umbilical then M is $\mathcal{D}$-totally geodesic.*

Theorem 18.2. *Each totally umbilical submanifold M of an LP-cosymplectic manifold such that $\xi \in TM$, is totally geodesic.*

Theorem 18.3. *Each totally umbilical semi-invariant submanifold of an LP-cosymplectic manifold is totally geodesic.*

Theorem 18.4. *If M is a totally umbilical semi-invariant submanifold of an LP-cosymplectic manifold, then $\mathcal{D}^0$ and $\mathcal{D}^1$ are integrable.*

Theorem 18.5. *Let M be a submanifold of an LP-cosymplectic manifold such that $\xi \in TM$. Then*

(a) $\{\xi\}$ and $\{\xi\}^{\perp}$ are parallel,
(b) $\{\xi\}$ and $\{\xi\}^{\perp}$ are integrable and their leaves are totally geodesic in M,
(c) M is locally product of leaves of $\{\xi\}$ and $\{\xi\}^{\perp}$,

(d) M is $\left(\{\xi\}, \{\xi\}^{\perp}\right)$-mixed totally geodesic.

Theorem 18.6. *Let M be a semi-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. If M is $(\mathcal{D}^0, \mathcal{D}^1)$-mixed totally geodesic then $\mathcal{D}^0$ is integrable.*

Theorem 18.7. *Let M be a semi-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. If M is $\mathcal{D}^1$-totally geodesic then $\mathcal{D}^1$ is integrable.*

19. Non-existence

19.1. **Non-existence of an anti-invariant distribution.** A distribution $\mathcal{A}$ on a submanifold of an LAP-manifold is an anti-invariant distribution if $\phi(\mathcal{A}) \subseteq T^{\perp}M$. Let M be a submanifold of a Lorentzian s-paracontact manifold $\bar{M}$ with $\xi \in TM$. Then, we get (cf. Tripathi [32])

$$\eta(A_N X) = g(FX, N).$$

Moreover, if $\bar{M}$ is an LP-Sasakian manifold, then

$$(\nabla_X P)Y - A_{FY}X - th(X,Y) = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

Let $X \in \mathcal{A}$ and $Y \in TM$. Then

$$\begin{aligned} g(A_{FX}X, Y) &= g(h(Y,X), FX) \\ &= g(th(Y,X), X) \\ &= g(\nabla_Y PX - P\nabla_Y X - A_{FX}Y - g(\phi Y, \phi X)\xi - \eta(X)\phi^2 Y, X) \\ &= -g(\nabla_Y X, PX) - g(A_{FX}Y, X) \\ &= -g(A_{FX}X, Y), \end{aligned}$$

which implies that $A_{FX}X = 0$, $X \in \mathcal{A}$ and consequently

$$0 = \eta(A_{FX}X) = g(FX, FX) = g(\phi X, \phi X) = g(X, X),$$

that is, $\mathcal{A} = \{0\}$. Thus we obtain the following theorem (cf. [32]).

Theorem 19.1. *Let M be a submanifold of an LP-Sasakian or LSP-Sasakian manifold $\bar{M}$ with $\xi \in TM$. Then there does not exist any anti-invariant distribution $\mathcal{A}$ such that $\mathcal{A} \perp \{\xi\}$.*

It is known that each submanifold of a contact metric manifold (and hence Sasakian manifold) normal to the structure vector field is anti-invariant (cf. Tripathi and Shukla [31]). Contrast to this result, in view of the definitions of CR

(cf. De and Sengupta [3], Prasad [21]), semi-invariant (cf. Kalpana and Guha [6]) and almost semi-invariant (cf. Kalpana and Singh [7]) submanifolds of Lorentzian almost paracontact manifolds and in view of Theorem 19.1 we have the following theorem (see Tripathi and Shukla [32], also see De and Shaikh [4]).

Theorem 19.2. *An LP-Sasakian or LSP-Sasakian manifold does not admit any proper CR, semi-invariant or almost semi-invariant submanifold. In fact, in these cases the anti-invariant distribution $\mathcal{D}^0$ becomes $\{0\}$.*

19.2. **Non-existence of proper mixed foliated semi-invariant submanifolds.** In [9], a semi-invariant submanifold is said to be mixed foliated if $\mathcal{D}^1 \oplus \{\xi\}$ is integrable and $h(Z+\xi, X) = 0$ for all $Z \in \mathcal{D}^1$ and $X \in \mathcal{D}^0$. Tripathi [28], has shown that a Sasakian manifold does not admit any proper mixed foliated semi-invariant submanifold.

For a submanifold M of a Lorentzian s-paracontact manifold, it follows that (cf. Tripathi and Shukla [32])

$$\phi X = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi), \qquad \xi, X \in TM.$$

If M is semi-invariant, then for $X \in \mathcal{D}^0$ we get $\nabla_X \xi = 0$ and $\phi X = h(X, \xi)$. Moreover, if M is assumed to be mixed foliated, then for $X \in \mathcal{D}^0$ we get $\phi X = 0$, that is $\mathcal{D}^0 = \{0\}$. Thus we have the following theorem.

Theorem 19.3. *A Lorentzian s-paracontact manifold can not admit any proper mixed foliated semi-invariant submanifold.*

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