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SCORE SEQUENCES OF HYPERTOURNAMENT MATRICES

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ABSTRACT. A k-hypertournament is a complete k-hypergraph with all k-edges endowed with orientations, i.e., orderings of the vertices in the edges. The incidence matrix associated with a k-hypertournament is called a k-hypertournament matrix, where each row stands for a vertex of the hypertournament. Some properties of the hypertournament matrices are investigated.

The sequences of the numbers of 1's and -1's of rows of a k-hypertournament matrix are respectively called the score sequence (resp. losing score sequence) of the matrix and so of the corresponding hypertournament. A necessary and sufficient condition for a sequence to be the score sequence (resp. the losing score sequence) of a k-hypertournament is proved.

1. INTRODUCTION

A tournament is a complete directed graph. It is well known (cf. Reid [6]) that every tournament contains a hamilton path and that a tournament is strongly connected if and only if every vertex is contained in cycles of all possible lengths. Together with these facts, one of the fundamental and well-known facts on tournaments is Landau's theorem. When we call the outdegree of a vertex of a tournament the *score* of the vertex, Landau's theorem shows a condition for the existence of a tournament whose vertices have the scores as given ones.

Hypergraphs are a generalization of graphs. While edges of a graph are pairs of vertices of the graph, edges of a hypergraph are subsets of the vertex set, consisting of at least two vertices. We call hypergraphs only having edges consisting of k vertices k-hypergraphs. A *k-hypertournament* is a complete oriented k-hypergraph.

Instead of scores of vertices in a tournament, Zhou, Yao and Zhang [7] considered *scores* and *losing scores* of vertices in a k-hypertournament, and derived a result

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analogous to Landau's theorem. Landau's theorem has attracted quite a bit of attention providing a dozen of different proofs. Included among these proofs are Fulkerson's, Ryser's, Brauer, Gentry and Shaw's, Landau's, and Bang-Sharp's. The proof of Bang and Sharp [2] might be more noteworthy than others, as Erdös called it *the proof in the Book.*

In this paper, we prove the result of Zhou, Yau and Zhang [7] in the same way as Bang and Sharp's proof. Also, we define k-hypertournament matrices as the incidence matrices of k-hypertournaments and obtain some basic properties of such matrices.

2. HYPERTOURNAMENT MATRICES

Let V be the set of n vertices $v_1, v_2, \cdots, v_n$. For fixed $k, 2 \le k \le n-1$, a subset of k vertices is called a k-edge. The set of the vertices with k-edges defines a k-hypergraph (cf. Berge [3]). Especially, the k-hypergraph containing all of the k-edges on V is said to be complete.

A k-edge endowed with an orientation is called a *k-arc*. A *k-hypertournament* H is defined as the pair of the vertex set V and a set A_k of k-arcs on V, where there is exactly one k-arc for every possible k-edge, i.e., $H = (V, A_k)$. So, a k-hypertournament is the complete k-hypergraph with all k-edges given orientations, i.e., orderings of k vertices in k-edges. Since a k-arc consists of k vertices, the number of k-arcs in A_k is $\binom{n}{k}$, and so it is written as $A_k = \{e_1, e_2, \cdots, e_{\binom{n}{k}}\}$. Now we define an $n \times \binom{n}{k}$ matrix $M = (m_{ij})$ by

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is in } e_j \text{ and } v_i \text{ is not the last element of } e_j, \\ -1 & \text{if } v_i \text{ is in } e_j \text{ and } v_i \text{ is the last element of } e_j, \\ 0 & \text{if } v_i \text{ is not in } e_j. \end{cases}$$

The matrix M is the incidence matrix of a k-hypertournament H, and is called a *k-hypertournament matrix.* Each row and column of M corresponds to a vertex and a k-arc of H, respectively. Note that our k-hypertournament matrices are defined in transposed form of the incidence matrices introduced in Bang and Sharp [4].

In a k-hypertournament on the vertices $v_1, \cdots, v_n$, the *score* s_i of a vertex v_i is defined to be the number of k-arcs which contain v_i not as the last element and the *losing score* r_i of v_i to be the number of k-arcs containing v_i as the last element. The incidence matrix of a k-hypertournament defined in this way, though not telling the orientation of k-arcs, distinguishes the last element of each arc and provides the information on the score and the losing score of every vertex. Note that a k-hypertournament matrix corresponds to $\binom{n}{k}(k-1)!$ k-hypertournaments. The following are some properties of k-hypertournament matrices.

Let H be a k-hypertournament on vertices $v_1, \cdots, v_n$ and M the corresponding k-hypertournament matrix.

(1) The matrix M is a $(1,0,-1)$-matrix, and each row and column respectively corresponds to each vertex and k-arc of H.

(2) Since each column of M stands for a k-arc, it contains exactly $k-1$ 1's, one -1, and $n-k$ 0's.

(3) Let s_i be the score of vertex v_i of H. Then s_i is the number of 1's in row i of M. Similarly, let r_i be the losing score of v_i. Then it is the number of -1's in the ith row of M. Also, for each $i \in \{1, \cdots, n\}$, $s_i + r_i = \binom{n-1}{k-1}$ is the number of k-arcs containing v_i, that is, the number of nonzeroes in the ith row.

(4) $\sum_{i=1}^{n} r_i$ is the sum of the losing scores of all vertices. This is equal to the total number of -1's in the matrix M, which is equal to the number of columns $\binom{n}{k}$ of M since each column contains one and only one -1.

(5) The score sum $\sum_{i=1}^{n} s_i$ is the total number of 1's in the matrix M, which is $(k-1)$ times the number of columns of M, and so $(k-1)\binom{n}{k}$. And $\sum_{i=1}^{n} r_i + \sum_{i=1}^{n} s_i = k\binom{n}{k}$ standing for the number of nonzero entries of M.

(6) The column sum vector of M is given as the transpose of

$$(1,1,\cdots,1)M = (k-2, k-2, \cdots, k-2),$$

i.e., $M^T\mathbf{1} = (k-2)\,\mathbf{1}$, where $\mathbf{1} = (1,1,\cdots,1)^T$.

(7) $M\mathbf{1} = \begin{bmatrix} s_1 - r_1 \\ s_2 - r_2 \\ \vdots \\ s_n - r_n \end{bmatrix}$ is the row sum vector of M.

3. Score Sequences

In this section, we employ the idea of Bang and Sharp [2] in their proof of Landau's theorem to prove a necessary and sufficient condition for a nonnegative sequence to be a score or a losing score sequence for a k-hypertournament matrix, and hence for a k-hypertournament.

Proposition 1 (Landau's Theorem [6]). *Given a sequence of nonnegative integers, $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$, there exists a tournament matrix M such that $s = (s_1, s_2, \cdots, s_n)^T = M\mathbf{1}$ if and only if*

$$\sum_{i=1}^{l} s_i \geq \binom{l}{2} \quad \text{for } l = 1, \cdots, n,$$

and the equality holds when $l = n$.

Bang and Sharp [2] used Hall's theorem about systems of distinct representatives for a collection of sets. Given a collection of sets $A_1, \cdots, A_r$, a system of distinct representatives of the collection is defined as a system of distinct elements $a_1, \cdots, a_r \in \bigcup_{i=1}^{r} A_i$ such that $a_i \in A_i$ for all $i = 1, \cdots, r$.

Lemma 2 (Hall's Theorem [6]). *The set $A_1, \cdots, A_r$ possess a system of distinct representatives if and only if, for each $m \leq r$,*

$$\left| A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_m} \right| \geq m$$

for any $\{i_1, i_2, \cdots, i_m\} \subset \{1, 2, \cdots, r\}$.

Theorm 3. *A nondecreasing sequence of nonnegative integers $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ is the losing score sequence of a k-hypertournament H if and only if it satisfies*

$$\sum_{i=1}^{l} r_i \geq \binom{l}{k} \quad \text{for } \ l = 1, 2, \cdots, n,$$

and the equality holds when $l = n$.

Proof. We follow the proof of Bang and Sharp [2] (see also Ree and Koh [6]) for score sequences of tournaments.

For the necessity, if $l < k$ then clearly $\sum_{i=1}^{l} r_i \geq \binom{l}{k} = 0$. Assume that $l \geq k$. Let $v_1, v_2, \cdots, v_l$ be the vertices with losing scores $r_1, r_2, \cdots, r_l$. Then the induced k-hypertournament H_l on the l vertices $v_1, v_2, \cdots, v_l$ is contained in H whose vertex set is $\{v_1, \cdots, v_n\}$. Each vertex v_i for $1 \leq i \leq l$ is possibly the loser, or the last element, of some arcs containing some of the vertices v_j's for $l+1 \leq j \leq n$. That is, for $1 \leq i \leq l$, the losing score of v_i in H_l is less than or equal to r_i. Since the losing score sum of H_l is $\binom{l}{k}$ by property 4, $\sum_{i=1}^{l} r_i \geq \binom{l}{k}$ and the losing score sum of H is $\sum_{i=1}^{n} r_i = \binom{n}{k}$.

For the sufficiency, let $X_1, X_2, \cdots, X_n$ be pairwise disjoint sets with $|X_i| = r_i$ for $1 \le i \le n$. Consider these n sets as the vertices of a complete k-hypergraph on n vertices.

Define the orientation of each k-edge $\{X_{i_1}, X_{i_2}, \cdots, X_{i_k}\}$ as follows. Form the $\binom{n}{k}$ set

$$F = \{\, X_{i_1} \cup \cdots \cup X_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n \,\}.$$

For $1 \le l \le \binom{n}{k}$, consider the union of any l members of F. Let I denote the set of distinct subscripts of the X_i's that make up these l members of F. Note that the index set I can be made from at most $\binom{|I|}{k}$ distinct members of F, i.e., $l \le \binom{|I|}{k}$.

From the assumption that r_i's are nondecreasing and $\sum_{i=1}^{|I|} r_i \ge \binom{|I|}{k}$, the following inequalities are derived.

$$|\bigcup_{i\in I} X_i| = \sum_{i\in I} |X_i| = \sum_{i\in I} r_i \ge \sum_{i=1}^{|I|} r_i \ge \binom{|I|}{k} \ge l.$$

So by Hall's Theorem, F has $\binom{n}{k}$ distinct representatives from the union of the members of F so that each set of F contains one of the representatives.

Orient a k-edge $\{X_{i_1}, \cdots, X_{i_k}\}$ to form a k-arc so that X_{i_j} is the last element in this arc if and only if the representative of the member $X_{i_1} \cup \cdots \cup X_{i_k}$ in F is in X_{i_j}. Since both of the number of representatives and $|X_1 \cup \cdots \cup X_n| = \sum_{i=1}^n |X_i| = \sum_{i=1}^n r_i$ are $\binom{n}{k}$, each element of $X_1 \cup \cdots \cup X_n$ appears exactly once as a representative, i.e., the losing score of X_i is $|X_i| = r_i$. Hence, we obtain a k-hypertournament with losing score sequence $0 \le r_1 \le r_2 \le \cdots \le r_n$. □

Corollary 4. *A nonincreasing sequence of nonnegative integers $s_1 \ge s_2 \ge \cdots \ge s_n \ge 0$ is a score sequence of a k-hypertournament H if and only if it satisfies*

$$\sum_{i=1}^{l} s_i \le l\binom{n-1}{k-1} - \binom{l}{k} \quad \text{for } l = 1, 2, \cdots, n,$$

and the equality holds when $l = n$.

If we arrange the score sequence in nondecreasing order, then we obtain the same inequalities for s_i as in Zhou, Yao and Zhang [7]:

$$\sum_{i=1}^{l} s_i \ge l\binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k} \quad \text{for } l = 1, 2, \cdots, n,$$

and the equality holds when $l = n$.

Corollary 5. *Sequences $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$ and $r_1 \geq r_2 \geq \cdots \geq r_n \geq 0$ are the score and losing score sequences of a k-hypertournament if and only if they satisfy*

$$s_i + r_i = \binom{n-1}{k-1}, \quad \sum_{i=1}^{l} s_i \geq l\binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k}$$

$$\text{and} \quad \sum_{i=1}^{l} r_i \leq \binom{n}{k} - \binom{n-l}{k}$$

for $1 \leq l \leq n$, and the equalities hold when $l = n$.

Proof. Let $\tilde{s}_j = s_{n-j+1}$ and $\tilde{r}_j = r_{n-j+1}$. Then the sequences $\tilde{s}_j$ and $\tilde{r}_j$ satisfy the conditions in Theorem 3 and Corollary 4. So using $k\binom{n}{k} = n\binom{n-1}{k-1}$, we have

$$\begin{aligned}
\sum_{j=1}^{l} s_j &= \sum_{j=1}^{l} \tilde{s}_{n-j+1} \\
&= (k-1)\binom{n}{k} - \sum_{i=1}^{n-l} \tilde{s}_i \\
&\geq n\binom{n-1}{k-1} - \binom{n}{k} - (n-l)\binom{n-1}{k-1} + \binom{n-l}{k} \\
&= l\binom{n-1}{k-1} + \binom{n-l}{k} - \binom{n}{k}
\end{aligned}$$

and

$$\sum_{j=1}^{l} r_j = \sum_{j=1}^{l} \tilde{r}_{n-j+1} = \binom{n}{k} - \sum_{i=1}^{n-l} \tilde{r}_i \leq \binom{n}{k} - \binom{n-l}{k},$$

where $1 \leq l \leq n$, and equalities hold if $l = n$. □

Given a nonincreasing sequence of integers, we also get a condition for the existence of a k-hypertournament matrix having the given sequence as the elements of its row sum vector.

Corollary 6. *A nonincreasing sequence of integers $\{t_i \mid i = 1, 2, \cdots, n\}$ is the elements of the row sum vector of a k-hypertournament matrix M on n vertices if and only if t_i has the same parity as that of $\binom{n-1}{k-1}$ for all $i = 1, 2, \cdots, n$, and*

$$\sum_{i=1}^{l} t_i \leq l\binom{n-1}{k-1} - 2\binom{l}{k} \quad \text{for } l = 1, 2, \cdots, n,$$

and the equality holds when $l = n$, i.e.,

$$\sum_{i=1}^{n} t_i = n\binom{n-1}{k-1} - 2\binom{n}{k} = (k-2)\binom{n}{k}.$$

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