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ON SEMI-INVARIANT SUBMANIFOLDS OF LORENTZIAN ALMOST PARACONTACT MANIFOLDS

MUKUT MANI TRIPATHI

ABSTRACT. Semi-invariant submanifolds of Lorentzian almost paracontact manifolds are studied. Integrability of certain distributions on the submanifold are investigated. It has been proved that a LP-Sasakian manifold does not admit a proper semi-invariant submanifold.

1. INTRODUCTION

Matsumoto [7] introduced the notion of a Lorentzian almost paracontact manifold. Submanifolds of a Lorentzian almost paracontact manifold have been studied in Prasad and Ojha [11]. In the present paper we study semi-invariant submanifolds of Lorentzian almost paracontact manifolds. The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3 some necessary and sufficient conditions for integrability of certain distributions on semi-invariant submanifolds are obtained. In the last section (Section 4), it has been shown that a LP-Sasakian manifold does not admit a proper semi-invariant submanifold.

2. PRELIMINARIES

Let $\bar{M}$ be a Lorentzian almost paracontact manifold (cf. [7], [8]) with a Lorentzian almost paracontact structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a (time-like) vector field, η is a 1-form and g is a Lorentzian metric on $\bar{M}$ such that

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

$$\Phi(X, Y) \equiv g(\phi X, Y) = g(X, \phi Y) = \Phi(Y, X), \quad g(X, \xi) = \eta(X) \tag{3}$$

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for all $X, Y \in T\bar{M}$.

A Lorentzian almost paracontact manifold is called (cf. Matsumoto [7]):

Lorentzian paracontact manifold if

$$\Phi(X,Y) = \frac{1}{2}\left((\bar{\nabla}_X \eta)Y + (\bar{\nabla}_Y \eta)X\right), \tag{4}$$

Lorentzian para-Sasakian (in brief, *LP-Sasakian*) *manifold* if

$$(\bar{\nabla}_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X, \tag{5}$$

Lorentzian special para-Sasakian (in brief, *LSP-Sasakian*)*manifold* if

$$\Phi(X,Y) = eg(\phi X, \phi Y), \quad e^2 = 1. \tag{6}$$

Here $\bar{\nabla}$ is the covariant differentiation with respect to g.

Let M be a submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g). Let the induced metric on M also be denoted by g. Then Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad X, Y \in TM, \tag{7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \qquad N \in T^{\perp}M, \tag{8}$$

where ∇ is the induced connection on M, h is the second fundamental form of the immersion, and $-A_N X$ and $\nabla^{\perp}_X N$ (resp.) are the tangential and normal (resp.) parts of $\bar{\nabla}_X N$. From (7) and (8) one gets

$$g(h(X,Y), N) = g(A_N X, Y). \tag{9}$$

Moreover, we have

$$\begin{aligned}&(\bar{\nabla}_X \phi)Y \\ &= ((\nabla_X P)Y - A_{FY}X - th(X,Y)) + ((\nabla_X F)Y + h(X, PY) - fh(X,Y)),\end{aligned} \tag{10}$$

$$\begin{aligned}&(\bar{\nabla}_X \phi)N \\ &= ((\nabla_X t)N - A_{fN}X - PA_N X) + ((\nabla_X f)N + h(X, tN) - FA_N X),\end{aligned} \tag{11}$$

where

$$\phi X \equiv PX + FX; \qquad PX \in TM,\ FX \in T^{\perp}M, \tag{12}$$

$$\phi N \equiv tN + fN; \qquad tN \in TM,\ fN \in T^{\perp}M, \tag{13}$$

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y, \tag{14}$$

$$(\nabla_X F)Y \equiv \nabla^{\perp}_X FY - F\nabla_X Y, \tag{15}$$

$$(\nabla_X t)N \equiv \nabla_X tN - t\nabla_X^{\perp} N, \tag{16}$$

$$(\nabla_X f)N \equiv \nabla_X^{\perp} fN - f\nabla_X^{\perp} N. \tag{17}$$

Let $\xi \in TM$. We write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Then we get

$$P\xi = 0 = F\xi, \qquad \eta \circ P = 0 = \eta \circ F, \tag{18}$$

$$P^2 + tF = I + \eta \otimes \xi, \qquad FP + fF = 0, \tag{19}$$

$$f^2 + Ft = I, \qquad tf + Pt = 0, \tag{20}$$

$$\ker(P) = \ker(P^2) = \ker(tF - I - \eta \otimes \xi), \tag{21}$$

$$\ker(F) = \ker(tF) = \ker(P^2 - I - \eta \otimes \xi), \tag{22}$$

$$\ker(t) = \ker(Ft) = \ker(f^2 - I), \tag{23}$$

$$\ker(f) = \ker(f^2) = \ker(Ft + I) \tag{24}$$

$$\ker(P|_{\{\xi\}^{\perp}}) = \ker(P^2|_{\{\xi\}^{\perp}}) = \ker(tF|_{\{\xi\}^{\perp}} - I), \tag{25}$$

$$\ker(F|_{\{\xi\}^{\perp}}) = \ker(tF|_{\{\xi\}^{\perp}}) = \ker(P^2|_{\{\xi\}^{\perp}} - I). \tag{26}$$

A submanifold M of a Lorentzian almost paracontact manifold $\bar{M}$ with $\xi \in TM$ is called a *semi-invariant submanifold* of $\bar{M}$ if TM can be decomposed as a direct sum of mutually orthogonal differentiable distributions :

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\},$$

where

$$\mathcal{D}^1 = \ker(F|_{\{\xi\}^{\perp}}) = \{X \in \{\xi\}^{\perp} : \|X\| = \|PX\|\} = TM \cap \phi(TM),$$
$$\mathcal{D}^0 = \ker(P|_{\{\xi\}^{\perp}}) = \{X \in \{\xi\}^{\perp} : \|X\| = \|FX\|\} = TM \cap \phi(T^{\perp}M).$$

Moreover, we have

$$T^{\perp}M = \bar{\mathcal{D}}^1 \oplus \bar{\mathcal{D}}^0$$

where

$$\bar{\mathcal{D}}^1 = \ker(t) = T^{\perp}M \cap \phi(T^{\perp}M), \ \bar{\mathcal{D}}^0 = \ker(f) = T^{\perp}M \cap \phi(TM), \ F\mathcal{D}^0 = \bar{\mathcal{D}}^0,$$

and $t\bar{\mathcal{D}}^0 = \mathcal{D}^0$. For $X \in TM$ we can write

$$X = U^1 X + U^0 X - \eta(X)\xi \tag{27}$$

where U^1 and U^0 are projection operators of TM on $\mathcal{D}^1$ and $\mathcal{D}^0$ respectively.

A semi-invariant submanifold of a Lorentzian almost paracontact manifold is a *invariant submanifold* (resp. *anti-invariant submanifold*) if $\mathcal{D}^0 = \{0\}$ (resp. $\mathcal{D}^1 = \{0\}$). A semi-invariant submanifold is *proper* if $\mathcal{D}^0 \neq \{0\} \neq \mathcal{D}^1$.

3. Integrability conditions

Let M be a semi-invariant submanifold of a Lorentzian almost paracontact manifold $\bar{M}$. The Nijenhuis tensor $[\phi, \phi]$ of ϕ is given by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \tag{28}$$

for $X, Y \in T\bar{M}$. Using (1), (27), (12) and $\ker(F) = \mathcal{D}^1 \oplus \{\xi\}$, for $X, Y \in \mathcal{D}^1 \oplus \{\xi\}$ we get

$$[\phi,\phi](X,Y) = [P,P](X,Y) + U^0[X,Y] - F([PX,Y] + [X,PY]). \tag{29}$$

Let superscripts T and $\perp$ in a term denote its tangential and normal parts respectively. From (29) we can state the following.

Proposition 3.1. *If M is a semi-invariant submanifold of a Lorentzian almost paracontact manifold, then for $X, Y \in \mathcal{D}^1 \oplus \{\xi\}$ we get*

$$([\phi,\phi](X,Y))^T = [P,P](X,Y) + U^0[X,Y], \tag{30}$$

$$([\phi,\phi](X,Y))^\perp = -F([PX,Y] + [X,PY]). \tag{31}$$

Consequently, for $X \in \mathcal{D}^1 \oplus \{\xi\}$ we have

$$([\phi,\phi](X,\xi))^T = [P,P](X,\xi) + U^0[X,\xi], \tag{32}$$

$$([\phi,\phi](X,\xi))^\perp = -F[PX,\xi]. \tag{33}$$

In the following theorem, we find some necessary and sufficient condition for the integrability of the distribution $\mathcal{D}^1 \oplus \{\xi\}$ on a semi-invariant submanifold of a Lorentzian almost paracontact manifold.

Theorm 3.2. *Let M be a semi-invariant submanifold of a Lorentzian almost paracontact manifold. Then the following three statements are equivalent:*

- **(a)** *The distribution $\mathcal{D}^1 \oplus \{\xi\}$ is integrable.*
- **(b)** $([\phi,\phi](X,Y))^T = [P,P](X,Y), \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}$.
- **(c)** $([\phi,\phi](X,Y))^\perp = 0, \quad U^0[P,P](X,Y) = 0, \quad X, Y \in \mathcal{D}^1 \oplus \{\xi\}$.

Proof. The distribution $\mathcal{D}^1 \oplus \{\xi\}$ is integrable if and only if

$$U^0[X,Y] = 0 \quad \text{for} X,Y \in \mathcal{D}^1 \oplus \{\xi\}.$$

In view of (30) it follows that **(a)**$\Leftrightarrow$**(b)**.

Next, since

$$[P,P](X,Y) = U^1[X,Y] + [PX,PY] - P[PX,Y] - P[X,PY], \quad X,Y \in \mathcal{D}^1 \oplus \{\xi\},$$

operating by U^0 to this equation, in view of integrability of $\mathcal{D}^1 \oplus \{\xi\}$, we get

$$U^0[P,P](X,Y) = 0, \quad X,Y \in \mathcal{D}^1 \oplus \{\xi\}.$$

Taking account of (31), the integrability of $\mathcal{D}^1 \oplus \{\xi\}$ indicates

$$([\phi,\phi](X,Y))^{\perp} = 0, \quad X,Y \in \mathcal{D}^1 \oplus \{\xi\}.$$

Thus **(a)** $\Rightarrow$ **(c)**.

Conversely, let **(c)** be true. Then by (31) and $FX = \phi U^0 X$, we get

$$\phi U^0 \left([PX,Y] + [X,PY]\right) = 0, \qquad X,Y \in \mathcal{D}^1 \oplus \{\xi\}.$$

which operated by ϕ yields

$$U^0 \left([PX,Y] + [X,PY]\right) = 0, \qquad X,Y \in \mathcal{D}^1 \oplus \{\xi\}. \tag{34}$$

Next from $U^0[P,P](X,Y) = 0$, for $X,Y \in \mathcal{D}^1 \oplus \{\xi\}$, we get

$$U^0[PX,PY] = 0. \tag{35}$$

Now for $X,Y \in \mathcal{D}^1$ from (35) we have $U^0[X,Y] = 0$. In view of (34), we also have

$$0 = U^0 \left([PX,P\xi] + [P^2X,\xi]\right) = U^0[X,\xi], \qquad X \in \mathcal{D}^1.$$

Thus taking account of

$$U^0[X,Y] = 0 = U^0[X,\xi], \qquad X \in \mathcal{D}^1,$$

we get

$$U^0[X,Y] = 0 \qquad X,Y \in \mathcal{D}^1 \oplus \{\xi\},$$

which makes **(c)** $\Rightarrow$**(a)**. □

Theorm 3.3. *The distribution $\mathcal{D}^0 \oplus \{\xi\}$ on a semi-invariant submanifold M of a Lorentzian almost paracontact manifold $\bar{M}$ is integrable if and only if*

$$[P,P](X,Y) = 0, \qquad X,Y \in \mathcal{D}^0 \oplus \{\xi\}.$$

Proof. In view of $\ker(P) = \mathcal{D}^0 \oplus \{\xi\}$ and

$$[P,P](X,Y) = [PX,PY] + P^2[X,Y] - P[PX,Y] - P[X,PY], \qquad X,Y \in TM,$$

the proof follows immediately. □

4. NONEXISTENCE OF PROPER SEMI-INVARIANT SUBMANIFOLDS

From the definition of *LP*-Sasakian manifold, we get

$$\phi X = \bar{\nabla}_X \xi, \qquad X,Y \in T\bar{M}. \tag{36}$$

We call a Lorentzian almost paracontact manifold $\bar{M}$, a *Lorentzian special paracontact manifold* if it satisfies (36). Obviously, a *LP*-Sasakian manifold is a Lorentzian special paracontact manifold. Now, we prove the following theorem.

Theorm 4.1. *On a Lorentzian special paracontact manifold $\bar{M}$ the distribution $\mathcal{T}$ determined by η is integrable.*

Proof. Let $X,Y \in \mathcal{T}$. Then $\eta(X) = 0 = \eta(Y)$ and consequently, in view of (2), from (36) and (3) it follows that $\eta[X,Y] = 0, for\, X,Y \in \mathcal{T}$. □

This theorem implies the following theorem.

Theorm 4.2. *Let M be a semi-invariant submanifold of a Lorentzian special paracontact manifold. Then the distribution $\mathcal{D}^1 \oplus \mathcal{D}^0$ is integrable.*

Let M be a submanifold of a Lorentzian special paracontact manifold $\bar{M}$ with $\xi \in TM$. Then, in view of (36), from (7) and (12) we get

$$PX = \nabla_X \xi, \qquad FX = h(X,\xi) \ (\Leftrightarrow tN = A_N \xi).$$

Consequently, we get

$$\eta(A_N X) = g(FX, N). \tag{37}$$

Moreover, if $\bar{M}$ is *LP*-Sasakian, then in view of (5) and (10) we get

$$(\nabla_X P)Y - A_{FY}X - th(X,Y) = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X. \tag{38}$$

Finally, we prove the following theorem.

Theorm 4.3. *A LP-Sasakian manifold does not admit a proper semi-invariant submanifold.*

Proof. We shall prove that $\mathcal{D}^0 = \{0\}$. Let $X \in \mathcal{D}^0$ and $Y \in TM$. We get

$$\begin{aligned} g(A_{FX}X, Y) &= g(h(Y, X), FX) = g(th(Y, X), X) \\ &= g(\nabla_Y PX - P\nabla_Y X - A_{FX}Y - g(\phi Y, \phi X)\xi - \eta(X)\phi^2 Y, X) \\ &= -g(\nabla_Y X, PX) - g(A_{FX}Y, X) = -g(A_{FX}X, Y), \end{aligned}$$

which implies that

$$A_{FX}X = 0, \qquad X \in \mathcal{D}^0$$

and consequently

$$0 = \eta(A_{FX}X) = g(FX, FX) = g(\phi X, \phi X) = g(X, X),$$

that is, $\mathcal{D}^0 = \{0\}$. □

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DEPARTMENT OF MATHEMATICS AND ASTRONOMY, LUCKNOW UNIVERSITY, LUCKNOW - 226 007, INDIA
E-mail address: mmt66@satyam.net.in