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THE EXISTENCE OF SOLUTIONS OF PSEUDO-LAPLACIAN EQUATIONS

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ABSTRACT. This paper gives the sufficient conditions for the existence of positive solution of a quasilinear elliptic equation with homogeneous Dirichlet boundary condition.

1. INTRODUCTION

In this paper, we consider a class of quasilinear elliptic problems of the form

$$-\Delta_p u = f(x, u) \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty.$$

The problems of type (1) have been studied by many authors (cf. Díaz [5], Brézis and Oswald [3], Huang [8], Kim [9, 10, 11]). Huang [8] has investigated the positive solution of pseudo-Laplacian equation involving critical Sobolev exponents using the concentration compactness of Lions [12, 13]. And the author has studied the existence of multiple positive solutions (cf. [9]) and positive solutions (cf. [10]) for pseudo-Laplacian equations with critical (Sobolev) exponents in $f(x, u)$.

In this paper we investigate the existence of solutions under the conditions (P) and (H1)–(H3) of $f(x, u)$ given below. Since $\Delta_p u$ is Laplacian equation for $p = 2$ (cf. Brézis and Oswald [3]), we will extend to the more general case of pseudo-Laplacian equations. However, since the pseudo-Laplacian equations are degenerated elliptic, the solutions of such equations are generally only the weak solutions. Tolkdorf [14] has shown that the bounded solutions of above equation belong to $C^{1,\alpha}(\overline{\Omega})$ for some α $(0 < \alpha < 1)$ under a suitable growth condition of f and not always belong to

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$C^2(\Omega)$. Our proof of existence relies on a minimization technique used by many authors (cf. Amann [1]; Benguria, Brézis and Lieb [2]; de Figueiredo and Gossez [4]; Fučík and Kufner [7])

In this paper we give the sufficient conditions for the existence of the positive solution of a pseudo-Laplacian equation with a homogeneous Dirichlet boundary condition:

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and the function $f(x,u) : \Omega \times [0,\infty) \to \mathbb{R}$ satisfies the following conditions:

(H1) For almost all $x \in \Omega$, the function $u \longmapsto f(x,u)$ is continuous on $[0,\infty)$ and for each $\delta > 0$, there is a constant $C_\delta \geq 0$ such that

$$f(x,u) \geq -C_\delta u^{p-1} \text{ for almost all } x \in \Omega \text{ and for all } u \in [0,\delta].$$

(H2) For each $u \geq 0$, the function $x \longmapsto f(x,u)$ belongs to $L^\infty(\Omega)$.

(H3) There is a constant $C > 0$ such that

$$f(x,u) \leq C(u^{p^*-1} + 1) \text{ for almost all } x \in \Omega \text{ and for all } u \geq 0,$$

where $1 \leq p^* \leq \frac{Np}{N-p}$ if $1 < p < N$ and $1 \leq p^* < \infty$ if $p \geq N$.

We introduce the measurable function

$$a_0(x) = \liminf_{u \downarrow 0} \frac{f(x,u)}{u^{p-1}}$$

so that $-\infty < a_0(x) \leq +\infty$.

In Díaz and Saa [6], a solution of (P) was shown to exist at most one solution by assuming (H1), (H2) and (H3) with $p^* = p$.

Moreover, it is known (see Díaz and Saa [6, Theorem 2]) that, if a solution of (P) exists and the function $u \longmapsto f(x,u)/u^{p-1}$ is decreasing on $(0,\infty)$, then

$$\lambda_1(-\Delta_p u - a_0(x)|u|^{p-2}u) < 0 \tag{2}$$

where $\lambda_1(-\Delta_p u - a(x)|u|^{p-2}u)$ denotes the first eigenvalue of

$$-\Delta_p u - a(x)|u|^{p-2}u$$

with zero Dirichlet condition on $\partial\Omega$. By the strong maximum principle (cf. Vázquez [15]), we know that u is a positive solution in Ω. In [6], the condition (H3) is assumed only for index $p-1$. In the present paper, we shall generalize the growth condition on $f(x,u)$.

But in our case it is necessary to have an additional assumption to ensure the existence of a solution. Let us assume that

$$\limsup_{u\uparrow\infty} \frac{pF(x,u)}{u^p} \leq \lambda_1(-\Delta_p) \tag{3}$$

uniformly for almost all $x \in \Omega$ where $F(x,u) = \int_0^u f(x,t)dt$, and $\lambda_1(-\Delta_p)$ denotes the first eigenvalue of $-\Delta_p u$, and that

$$\limsup_{u\uparrow\infty} \frac{pF(x,u)}{u^p} < \lambda_1(-\Delta_p) \tag{4}$$

on a subset of Ω of positive measure.

From (H1), there is a constant C such that $a_0(x) \geq -C$ and from (3) and (4), we may assume that there is a function $\alpha(x) \in L^\infty(\Omega)$ such that

$$\limsup_{u\uparrow\infty} \frac{pF(x,u)}{u^p} \leq \alpha(x) \leq \lambda_1(-\Delta_p) \tag{5}$$

uniformly for almost all $x \in \Omega$.

Then, under the above conditions, there exists a weak solution of (P). The following remark shows that there exist such a function $f(x,u)$ satisfying the above conditions (H1)–(H3).

Remark 1. If

$$f(x,u) = e^{-|x|u^p} u^{p^*-1} \tag{6}$$

and

$$\begin{aligned} F(x,u) &= \int_0^u f(x,t)dt = \int_0^u e^{-|x|t^p} t^{p^*-1} dt \\ &= \frac{1}{p}\int_0^{u^p} e^{-|x|s} s^{\frac{p^*}{p}-1} ds \\ &= \frac{-1}{p|x|}\frac{u^{p^*-2}}{e^{|x|u^p}} + O\Big(\frac{u^{p^*-4}}{|x|^2 e^{|x|u^p}}\Big), \end{aligned}$$

then we have

$$\lim_{u\uparrow\infty} \frac{pF(x,u)}{u^p} = 0 \leq \lambda_1(-\Delta_p).$$

Thus we can take the function f of type (6) satisfying both the condition (H3) and the assumption (4). Then the function f also satisfy both conditions (H1) and (H2).

2. EXISTENCE THEOREM

To prove the main theorem (Theorem 3 below) we need the following lemma.

Lemma 2. *Assume that the inequalities* (4) *and* (5) *hold. Then there is* $\delta > 0$ *such that, for every* $u \in W_0^{1,p}(\Omega)$,

$$\psi(u) = \frac{1}{p}\int_\Omega \left[\,|\nabla u|^p - \alpha(x)u^p\right] \geq \delta \int_\Omega |\nabla u|^p.$$

Proof. It follows from Poincaré's inequality that

$$\psi(u) \geq \frac{1}{p}\int_\Omega \left[\,|\nabla u|^p - \lambda_1(-\Delta_p)u^p\right] \geq 0.$$

If $\psi(u) = 0$, then $\int |\nabla u|^p = \int \lambda_1(-\Delta_p)u^p$ and thus

$$0 = \psi(u) = \frac{1}{p}\int_\Omega (\lambda_1(-\Delta_p) - \alpha(x))u^p.$$

Since $\lambda_1(-\Delta_p) > \alpha(x)$ on a subset of Ω of positive measure, $u = 0$ on a subset of Ω of positive measure. By the unique continuation property, we obtain $u \equiv 0$. Assume now that the conclusion is false. Then there is a sequence (u_n) in $W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u_n|^p = 1,$$

where u_n converge weakly to u (*in notation*, $u_n \rightharpoonup u$) in $W_0^{1,p}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$ and $0 \leq \psi(u_n) \to 0$ as $n \to \infty$. We obtain

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \geq \int_\Omega |\nabla u|^p$$

and

$$\int_\Omega |\nabla u_n|^p \to \int_\Omega \alpha(x)u^p.$$

Hence $0 \leq \psi(u) \leq 0$, i.e., $\psi(u) = 0$. Thus $u \equiv 0$. But $1 = \int_\Omega |\nabla u_n|^p \to 0$ which is impossible. □

Theorem 3. *Under the conditions* (H1), (H2) *and* (H3), *with inequalities* (2) *and* (4), *there exists a weak solution* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of* (P).

Proof. We consider the functional $E : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{\infty\}$ defined by

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x,u) \quad \text{for all } \ u \in W_0^{1,p}(\Omega),$$

where $F(x,u) = \int_0^u f(x,t)dt$ and $f(x,u)$ is extended to be $f(x,0)$ for $u \leq 0$. Note that $E(u)$ is well-defined, since

$$F(x,u) \leq C(\frac{1}{p^*}|u|^{p^*} + |u|) \text{ for all } x \in \Omega \text{ and for all } u \in \mathbb{R}.$$

To attain the infimum of $E(u)$, we must prove the following properties (a) and (b), and prove its infimum $\not\equiv 0$ by checking the following property (c):

(a) E is coercive on $W_0^{1,p}(\Omega)$,

(b) E is lower semicontinuous for the weak $W_0^{1,p}(\Omega)$ topology, and

(c) There is some $\phi \in W_0^{1,p}(\Omega)$ such that $E(\phi) < 0$.

We will prove (a). From the condition (H3) and the expression (5), there is a function $\beta(x) \in L^1(\Omega)$ such that

$$F(x,u) \leq (\alpha(x) + \lambda_1(-\Delta_p)\delta)\frac{u^p}{p} + \beta(x).$$

Thus

$$\begin{aligned}
E(u) &= \int_\Omega \left[\frac{1}{p}|\nabla u|^p - F(x,u)\right] \\
&\geq \int_\Omega \left[\frac{1}{p}|\nabla u|^p - \alpha(x)\frac{u^p}{p} - \frac{\lambda_1(-\Delta_p)\delta}{p}u^p - \beta(x)\right] \\
&= \psi(u) - \int_\Omega \frac{\lambda_1(-\Delta_p)\delta}{p}u^p - \int_\Omega \beta(x) \\
&\geq \delta \int_\Omega |\nabla u|^p - \frac{\lambda_1(-\Delta_p)\delta}{p}\int_\Omega u^p - \int_\Omega \beta(x) \\
&= \delta \int_\Omega \left[|\nabla u|^p - \frac{\lambda_1(-\Delta_p)}{p}u^p\right] - \int_\Omega \beta(x) \\
&\geq \delta \int_\Omega \left[|\nabla u|^p - \frac{|\nabla u|^p}{p}\right] - \int_\Omega \beta(x) \\
&= \frac{\delta}{p}\int_\Omega |\nabla u|^p - \int_\Omega \beta(x).
\end{aligned}$$

Thus $E(u)$ is coercive on $W_0^{1,p}(\Omega)$ under the norm $\|u\|_{W_0^{1,p}(\Omega)} = \left[\int_\Omega |\nabla u|^p\right]^{1/p}$. This proves (a).

We will prove (b). Let $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. By Sobolev's imbedding theorem, passing to a subsequence if necessary, we may suppose that $u_n \to u$ in $L^{p^*}(\Omega)$, $u_n(x) \to u(x)$ for almost all $x \in \Omega$ and $|u_n(x)| \leq h(x)$ for some $h \in L^{p^*}(\Omega)$. Then it follows from the condition (H3) that

$$|F(x,u_n(x))| \leq C(h(x)^{p^*} + h(x)).$$

Since the right side of the above inequality is in $L^1(\Omega)$, we have, by Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\int_\Omega F(x,u_n)=\int_\Omega F(x,u).$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} E(u_n) &= \liminf_{n\to\infty}\left(\int_\Omega \frac{1}{p}|\nabla u_n|^p - F(x,u_n)\right)\\ &\geq \liminf_{n\to\infty}\int_\Omega \frac{1}{p}|\nabla u_n|^p - \lim_{n\to\infty}\int_\Omega F(x,u_n)\\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x,u)\\ &= E(u).\end{aligned}$$

This proves (b).

We will prove (c). We fix any $\phi \in W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla\phi|^p - \int_{[\phi\neq 0]} a_0\phi^p < 0.$$

Such ϕ always exists by expression (1). We may always assume that $\phi > 0$ and that $\phi \in L^\infty(\Omega)$. Otherwise, we replace ϕ by $|\phi|$ and truncate ϕ. We note that

$$\liminf_{u\downarrow 0}\frac{F(x,u)}{u^p} \geq \frac{1}{p}a_0(x)$$

and thus

$$\liminf_{\varepsilon\downarrow 0}\frac{F(x,\varepsilon\phi)}{\varepsilon^p} \geq \frac{1}{p}a_0(x)\phi^p(x) \quad \text{for almost all} \quad x\in[\phi\neq 0].$$

On the other hand, we deduce from the condition (H1) that

$$\frac{F(x,\varepsilon\phi)}{\varepsilon^p} \geq -C\phi^p \geq -C.$$

Therefore, by Fatou's lemma, it follows

$$\liminf_{\varepsilon\downarrow 0}\int_{[\phi\neq 0]}\frac{F(x,\varepsilon\phi)}{\varepsilon^p} \geq \frac{1}{p}\int_{[\phi\neq 0]} a_0\phi^p.$$

Thus we have

$$\liminf_{\varepsilon\downarrow 0}\int_\Omega\frac{F(x,\varepsilon\phi)}{\varepsilon^p} \geq \frac{1}{p}\int_{[\phi\neq 0]} a_0\phi^p.$$

Hence we obtain

$$\frac{1}{p}\int_\Omega |\nabla\phi|^p - \int_\Omega \frac{F(x,\varepsilon\phi)}{\varepsilon^p} < 0$$

for $\varepsilon > 0$ small enough. This proves (c).

Using properties (a),(b) and (c) we see that

$$\inf_{u \in W_0^{1,p}(\Omega)} E(u)$$

is achieved by some $u \not\equiv 0$. We may assume that $u \geq 0$. Otherwise we replace u by u^+ and use the fact that $F(x,u) \leq F(x,u^+)$, from $F(x,u) = f(x,0)u \leq 0$ for $u \leq 0$. Then we know that $E(u)$ is of class C^1. Thus there exists a weak solution u of (P).

If we knew in addition that $u \in L^\infty(\Omega)$, we would conclude that u is a solution of (P). To show that $u \in L^\infty(\Omega)$, we introduce a truncated problem. We set, for each integer $k > 0$,

$$\begin{cases} f^k(x,u) = \max\{f(x,u), -ku^p\} & \text{if } u \geq 0 \\ f^k(x,u) = f^k(x,0) = f(x,0) & \text{if } u \leq 0 \end{cases}$$

and

$$a_0^k(x) = \liminf_{u \downarrow 0} \frac{f^k(x,u)}{u^{p-1}}.$$

Now, conditions (H1), (H2) and (H3) hold for $f^k(x,u)$. Since $f \leq f^k$ and $a_0(x) \leq a_0^k(x)$,

$$\lambda_1(-\Delta_p u - a_0^k(x)|u|^{p-2}u) \leq \lambda_1(-\Delta_p u - a_0(x)|u|^{p-2}u) < 0$$

holds. From this, the assumption (2) holds for $a_0^k(x)$. Moreover, the assumption (4) holds for $f^k(x,u)$ provided that k is large enough. Set

$$E_k(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F^k(x,u)$$

for all $u \in W_0^{1,p}(\Omega)$. It follows from the previous argument that

$$\inf_{u \in W_0^{1,p}(\Omega)} E_k(u)$$

is achieved by some u_k. Moreover, u_k satisfies

$$\begin{cases} -\Delta_p u_k = f^k(x,u_k) & \text{in } \Omega \\ u_k \geq 0,\ u_k \not\equiv 0 & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exist constants D_k, C_k such that

$$-D_k(|u|^p + 1) \leq f^k(x,u) \leq C_k(|u|^{p^*-1} + 1).$$

Therefore $E_k(u)$ is of class C^1 and by a standard bootstrap argument, $u_k \in L^\infty(\Omega)$. Set $v = \min\{u, u_k\}$. We claim that

$$E(v) \leq E(u).$$

This shows that $u \in L^\infty(\Omega)$. Indeed, we have

$$\frac{1}{p}\int_\Omega |\nabla u_k|^p - \int_\Omega F^k(x,u_k) \le \frac{1}{p}\int_\Omega |\nabla \phi|^p - \int_\Omega F^k(x,\phi)$$

for all $\phi \in W_0^{1,p}(\Omega)$. Choosing $\phi = \max\{u, u_k\}$, we obtain

$$\begin{aligned}\frac{1}{p}\int_{[u_k\ge u]} |\nabla u_k|^p - \int_{[u_k\ge u]} F^k(x,u_k) + \frac{1}{p}\int_{[u_k<u]} |\nabla u_k|^p - \int_{[u_k<u]} F^k(x,u_k)\\ \le \frac{1}{p}\int_{[u_k\ge u]} |\nabla u_k|^p - \int_{[u_k\ge u]} F^k(x,u_k) + \frac{1}{p}\int_{[u_k<u]} |\nabla u|^p - \int_{[u_k<u]} F^k(x,u).\end{aligned}$$

Thus we find

$$\frac{1}{p}\int_{[u_k<u]} |\nabla u_k|^p - \int_{[u_k<u]} F^k(x,u_k) \le \frac{1}{p}\int_{[u_k<u]} |\nabla u|^p - \int_{[u_k<u]} F^k(x,u).$$

On the other hand, we have

$$\begin{aligned}E(v) - E(u) &= \int_{[u_k<u]} \left\{\frac{1}{p}|\nabla u_k|^p - \frac{1}{p}|\nabla u|^p - F(x,u_k) + F(x,u)\right\}\\ &\le \int_{[u_k<u]} F^k(x,u_k) - F^k(x,u) - F(x,u_k) + F(x,u)\\ &= \int_{[u_k<u]} \left[\int_{u_k}^{u} f(x,t) - f^k(x,t)\right] dt \le 0.\end{aligned}$$

Thus $E(v) \le E(u)$.

From this, we know $v = u$, $u \le u_k$. Therefore $u \in L^\infty(\Omega)$. □

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