J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 8, Number 1 (May 2001), Pages 33–46
ISSN 1226-0657

GEOMETRY OF COISOTROPIC SUBMANIFOLDS

DAE HO JIN

ABSTRACT. The purpose of this paper is to study totally umbilical coisotropic submanifold (M, g, SM) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$.

1. INTRODUCTION

The theory of submanifolds of a Riemannian or semi-Riemannian manifold is one of the most important topics of differential geometry. In case $\bar{g}$ is degenerate on the tangent bundle TM of M we say that M is a *lightlike (degenerate, null) submanifold* of $\bar{M}$. While the geometry of semi-Riemannian submanifolds is fully developed, its counter part of lightlike submanifolds is relatively new and in a developing stage (see Duggal-Bejancu [1], Duggal-Jin [2]).

According to the behavior of the induced tensor field g on the submanifold M of $\bar{g}$ and the rank of the radical distribution, we have four typical classes of submanifolds; that is, since $\bar{g}$ is degenerate, for each tangent space T_xM we consider

$$T_xM^{\perp} = \{u \in T_x\bar{M} : \bar{g}(u, v) = 0, \ \forall v \in T_xM\},$$

which is a degenerate subspace of $T_x\bar{M}$. Since M is lightlike, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspace but no longer complementary. In this case the dimension of $\operatorname{Rad} T_xM = T_xM \cap T_xM^{\perp}$ depends on the point $x \in M$. The submanifold M of $\bar{M}$ is said to be *r-lightlike (r-degenerate, r-null) submanifold* if the mapping

$$\operatorname{Rad} TM : x \in M \longrightarrow \operatorname{Rad} T_xM$$

defines a smooth distribution on M of rank $r > 0$. Then we call $\operatorname{Rad} TM$ the *radical (lightlike, null) distribution* on M.

Received by the editors January 5, 2001.
2000 *Mathematics Subject Classification.* 53C40.
Key words and phrases. umbilical coisotropic submanifold.

The followings are four possible cases with respect to the dimension m and codimension n of M and rank r of $\mathrm{Rad}\,TM$ (see Duggal-Bejancu [1]):

(I) *r-lightlike submanifold*: $0 < r < \min\{m, n\}$,
(II) *coisotropic submanifold*: $1 < r = n < m$,
(III) *isotropic submanifold*: $1 < r = m < n$,
(IV) *totally lightlike submanifold*: $1 < r = m = n$.

Recently Duggal and Jin [2] studied the geometry of totally umbilical lightlike submanifolds of a semi-Riemannian manifold M and found the conditions for the induced connection on M to be a metric connection and its induced Ricci tensor of M to be symmetric.

The purpose of the present paper is to study the geometry of totally umbilical coisotropic submanifolds in a semi-Riemannian manifold. We characterize M embedded in $\bar{M}$ of constant curvature, and find the conditions for the induced connection on M to be a metric connection and its induced Ricci tensor of M to be symmetric.

2. Coisotropic Submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$; $1 \leq q \leq m+n-1$ and (M, g) be a coisotropic submanifold of dimension m of $\bar{M}$. Then we have $\mathrm{Rad}\,TM = TM^{\perp}$. Let SM be the complementary vector subbundle to $\mathrm{Rad}\,TM$ in TM which is called the *screen distribution*, that is, we have the following decomposition

$$TM = \mathrm{Rad}\,TM \perp SM, \tag{1}$$

where $\perp$ means the orthogonal direct sum. From now on we denote a coisotropic submanifold by (M, g, SM). Throughout this paper we denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over C. We use the same notation for any other vector bundle. In the sequel we use the range of indices: $i, j, k, \cdots \in \{1, \cdots, n\}$.

Theorem 2.1 (Duggal-Bejancu [1]). *Let (M, g, SM) be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $\mathrm{tr}(TM)$ of $\mathrm{Rad}\,TM$ in $SM^{\perp}$ such that, for any basis $\{\xi_i\}$ of $\Gamma(\mathrm{tr}(TM)|_{\mathcal{U}})$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist smooth sections $\{N_i\}$ of $SM^{\perp}|_{\mathcal{U}}$*

satisfying:

$$\bar{g}(N_i,\, \xi_j) \;=\; \delta_{ij}, \tag{2}$$

$$\bar{g}(N_i,\, N_j) \;=\; 0. \tag{3}$$

The vector bundle $\mathrm{tr}(TM)$ called the *transversal vector bundle* of M, we obtain

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM) = (\mathrm{Rad}\, TM \oplus \mathrm{tr}(TM)) \perp SM. \tag{4}$$

Using the above decomposition we have a local quasi-orthonormal field of frames of $\bar{M}$ along M:

$$\{\xi_1, \cdots, \xi_n,\; N_1, \cdots, N_n,\; F_{n+1}, \cdots, F_m\} \tag{5}$$

where $\{\xi_1, \cdots, \xi_n\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}\, TM)$, $\{N_1, \cdots, N_n\}$ a lightlike basis of $\Gamma(\mathrm{tr}(TM))$, and $\{F_{r+1}, \cdots, F_m\}$ an orthogonal basis of $\Gamma((SM)|_{\mathcal{U}})$.

We have also that $\mathrm{Rad}\, TM \oplus \mathrm{tr}(TM)$ is non-degenerate. Since

$$\{\xi_1, \cdots, \xi_n,\; N_1, \cdots, N_n\}$$

is a null basis of $\Gamma(SM^{\perp})$, the restriction of $\bar{g}$ on the screen distribution SM has the index $q - n$.

Now, we define locally the differential 1-forms $\{\eta_i\}$ on $\Gamma(TM)$ by

$$\eta_i(X) \;=\; \bar{g}(X,\, N_i), \qquad \forall\, X \in \Gamma(TM). \tag{6}$$

It follow that $\{\eta_i\}$ are dual to the basis $\{\xi_i\}$ of $\Gamma(TM^{\perp})$. Denote by P the projection of TM on the screen distribution SM with respect to the decomposition (1) and for any $X \in \Gamma(TM)$ obtain

$$X = PX + \eta_i(X)\, \xi_i. \tag{7}$$

Suppose $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ and according to (4) we put

$$\bar{\nabla}_X Y = \;\nabla_X Y + h\,(X,\, Y), \tag{8}$$

$$\bar{\nabla}_X N = \;-\, A_N X + \nabla^{\perp}_X N \tag{9}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(\mathrm{tr}(TM))$, where $\nabla_X Y$ and $A_N X$ belong to $\Gamma(TM)$, while $h(X,\, Y)$ and $\nabla^{\perp}_X N$ belong to $\Gamma(\mathrm{tr}(TM))$.

It is easy to check that ∇ is a torsion-free linear connection on M, h is a $\Gamma(\mathrm{tr}(TM))$-valued symmetric $F(M)$-bilinear form on $\Gamma(TM)$, A_N is a linear operator on $\Gamma(TM)$ and $\nabla^{\perp}$ linear connection on the transversal vector bundle $\mathrm{tr}(TM)$. According to the general theory of submanifolds we call (8) and (9) the *Gauss formulae* and the *Weingarten equation* respectively. Since $\{\xi_i,\, N_i\}$ is locally pairs of lightlike sections on $\mathcal{U} \subset M$, we define symmetric $F(M)$-bilinear forms h_i and 1-forms τ_{ij}, respectively, on $\mathcal{U}$ by

$$\begin{cases} h_i(X, Y) &= \bar{g}(h(X, Y), \xi_i), \\ \tau_{ij}(X) &= \bar{g}(\nabla^{\perp}_X N_i, \xi_j), \end{cases} \tag{10}$$

for any $X, Y \in \Gamma(TM)$. It follow that

$$\begin{cases} h(X, Y) &= h_i(X, Y) N_i, \\ \nabla^{\perp}_X N_i &= \tau_{ij}(X) N_j. \end{cases} \tag{11}$$

Hence, on $\mathcal{U}$, the expressions (8) and (9) become

$$\bar{\nabla}_X Y = \nabla_X Y + h_i(X, Y) N_i, \tag{12}$$

$$\bar{\nabla}_X N_i = - A_{N_i} X + \tau_{ij}(X) N_j, \tag{13}$$

for any $X, Y \in \Gamma(TM)$. We call h and h_i $(i = 1, \cdots, n)$ the *second fundamental form* and the *local second fundamental forms* of M with respect to $\mathrm{tr}(TM)$ respectively and A_{N_i} $(i = 1, \cdots, n)$ the *shape operators* of M.

Further, taking into account that $\bar{\nabla}$ is a metric connection and by using (12) we obtain

$$(\nabla_X g)(Y, Z) = h_i(X, Y)\eta_i(Z) + h_i(X, Z)\eta_i(Y) \tag{14}$$

for any $X, Y, Z \in \Gamma(TM)$. Thus, in general, the induced connection ∇ is linear but not a metric (Levi-Civita) connection. However, because $\mathrm{tr}(TM)$ is a totally lightlike vector bundle it follows that $\nabla^{\perp}$ is a metric connection.

Next, since $\bar{\nabla}$ is a metric connection and $\{\xi_i\}$ are lightlike orthogonal vector field, we derive

$$\bar{g}(\bar{\nabla}_X \xi_i, \xi_j) + \bar{g}(\xi_i, \bar{\nabla}_X \xi_j) = 0,$$

for any $X \in \Gamma(TM)$, which imply

$$h_i(X, \xi_j) + h_j(X, \xi_i) = 0, \tag{15}$$

$$h_i(X, \xi_i) = 0, \tag{16}$$

for any $X \in \Gamma(TM)$. Replacing X by ξ_i in (15) and by using (16) with $X = \xi_j$, we obtain

$$h_j(\xi_i, \xi_i) = 0. \tag{17}$$

Thus the local second fundamental forms of a coisotropic submanifold vanish identically on $\Gamma(\mathrm{Rad}\, TM) \times \Gamma(\mathrm{Rad}\, TM)$. By using (9) we obtain

$$\bar{g}(A_{N_i} X, N_j) + \bar{g}(A_{N_j} X, N_i) = 0 \tag{18}$$

and

$$\bar{g}\,(A_{N_i}X,\, N_i) \,=\, 0 \tag{19}$$

for any $X \in \Gamma(TM)$.

Next, consider the decomposition (1) and obtain

$$\nabla_X PY \,=\, \nabla^*_X PY + h^*\,(X,\, PY), \tag{20}$$

$$\nabla_X \xi \,=\, -\,A_\xi X + \nabla^{*\perp}_X \xi \tag{21}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}\, TM)$, where $\nabla^*_X PY$ and $A_\xi X$ belong to $\Gamma(SM)$, while $h^*\,(X,\, PY)$ and $\nabla^{*\perp}_X \xi$ belong to $\Gamma(\mathrm{Rad}\, TM)$. It follows that ∇^* and $\nabla^{*\perp}$ are linear connection on the screen and radical distribution respectively, A_ξ is a linear operator on $\Gamma(TM)$, h^* is bilinear form on $\Gamma(TM) \times \Gamma(SM)$. We call (20) and (21) the *Gauss formulae* and the *Weingarten equation* respectively for the screen distribution SM. It is easy to check that the linear connection ∇^* is a metric connection on SM. Locally, we define on $\mathcal{U}$

$$h^*_i\,(X,\, PY) \,=\, \bar{g}\,(\,h^*(X,\, PY),\, N_i\,),$$

$$\rho_{ij}\,(X) \,=\, \bar{g}\,(\nabla^{*\perp}_X \xi_i,\, N_j\,),$$

for any $X,\, Y, \in \Gamma(TM)$. It follow that

$$h^*(X,\, PY) \,=\, h^*_i\,(X,\, PY)\,\xi_i,$$

$$\nabla^{*\perp}_X \xi_i \,=\, \rho_{ij}\,(X)\,\xi_j,$$

for any $X, Y \in \Gamma(TM)$. Hence, on $\mathcal{U}$, locally (20) and (21) become

$$\nabla_X PY \,=\, \nabla^*_X PY + h^*_i(X,\, PY)\,\xi_i, \tag{22}$$

$$\nabla_X \xi_i \,=\, -\,A_{\xi_i} X + \rho_{ij}\,(X)\,\xi_j, \tag{23}$$

We call h^* and h^*_i the *second fundamental form* and the *local second fundamental forms* of SM with respect to $\mathrm{Rad}\, TM$ and A_{ξ_i} the *shape operators* of the screen distribution SM. The geometric objects from Gauss and Weingarten equations (12) and (13) on one side and (22) and (23) on the other side are related by

$$h^*_i\,(X,\, PY) \,=\, g(A_{N_i}X,\, PY), \tag{24}$$

$$h_i\,(X,\, PY) \,=\, g(A_{\xi_i}X,\, PY), \tag{25}$$

$$A_{\xi_i}\xi_i \,=\, 0, \quad g\,(A_{\xi_i}X,\, N_j) \,=\, 0, \tag{26}$$

$$\rho_{ij}\,(X) \,=\, -\,\tau_{ji}\,(X),$$

for any $X, Y \in \Gamma(TM)$. Hence (23) becomes

$$\nabla_X \xi_i = - A_{\xi_i} X - \tau_{ji}(X)\, \xi_j. \tag{27}$$

The equations in (26) say that ξ_i are eigenvectors of A_{ξ_i} corresponding to the zero eigenvalues and A_{ξ_i} are $\Gamma(SM)$-valued linear operators.

Since ∇ is a torsion-free linear connection, by using (22) we obtain

$$\nabla^*_{PX} PY - \nabla^*_{PY} PX - [PX, PY] = \{ h^*_i(PY, PX) - h^*_i(PX, PY) \} \xi_i. \tag{28}$$

From the equations (6) and (28) we obtain

$$\eta_i([PX, PY]) = h^*_i(PX, PY) - h^*_i(PY, PX).$$

From the last equation and (24), since according to (7) the components of any $X \in \Gamma(TM)$ with respect to ξ_i are $\eta_i(X)$, we obtain

Theorem 2.2. *Let (M, g, SM) be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*

(1) *SM is integrable.*
(2) *h^* is symmetric on $\Gamma(SM)$.*
(3) *A_N is self-adjoint on $\Gamma(SM)$ with respect to g.*
(4) *∇^* is torsion-free linear connection.*

A coisotropic submanifold (M, g, SM) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *totally geodesic* if its second fundamental tensor h vanishes, i.e.,

$$h(X, Y) = 0 \quad \text{for any } X, Y \in \Gamma(TM).$$

By direct calculation it is easy to see that M is totally geodesic if and only if the local second fundamental tensors h_i all vanish on M, i.e.,

$$h_i(X, Y) = 0 \quad \text{for any } X, Y \in \Gamma(TM).$$

Next by using (14)–(17) and (24)–(26) we obtain the following theorem.

Theorem 2.3 (Duggal-Bejancu [1]). *Let (M, g, SM) be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:*

(1) *M is totally geodesic.*
(2) *The induced linear connection ∇ on M is metric connection.*
(3) *The second fundamental forms h_i of M vanish on M.*
(4) *A^*_ξ vanish on $\Gamma(TM)$ for any $\xi \in \Gamma(\mathrm{Rad}\, TM)$.*
(5) $\mathrm{Rad}\, TM$ *is a Killing distribution.*
(6) $\mathrm{Rad}\, TM$ *is a parallel distribution with respect to ∇.*

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇ respectively. Then by straightforward calculations and using (12) and (13), we obtain the following equations (29)–(30) for any $X, Y, Z \in \Gamma(TM)$.

$$\begin{aligned}\bar{R}(X, Y)Z &\\ = R(X, Y)Z + h_i(X, Z)A_{N_i}Y - h_i(Y, Z)A_{N_i}X &\\ + \{(\nabla_X h_i)(Y, Z) - (\nabla_Y h_i)(X, Z) + \tau_{ji}(X)h_j(Y, Z) - \tau_{ji}(Y)h_j(X, Z)\}N_i, \quad (29)\end{aligned}$$

$$\begin{aligned}\bar{R}(X, Y)N_i &\\ = -\nabla_X(A_{N_i}Y) + \nabla_Y(A_{N_i}X) + A_{N_i}[X, Y] + \tau_{ij}(X)A_{N_j}Y - \tau_{ij}(Y)A_{N_j}X &\\ + \{h_j(Y, A_{N_i}X) - h_j(X, A_{N_i}Y) + 2\,d\tau_{ij}(X, Y) + \tau_{ik}(Y)\tau_{kj}(X) &\\ - \tau_{ik}(X)\tau_{kj}(Y)\}N_j \quad (30)\end{aligned}$$

Consider the Riemannian curvature of type (0, 4) of $\bar{\nabla}$ and by using (29)–(30) and the definition of curvature tensors, we derive the following structure equations (31)–(34).

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)PZ, PW) &\\ = g(R(X, Y)PZ, PW) + h_i(X, PZ)h_i^*(Y, PW) - h_i(Y, PZ)h_i^*(X, PW), \quad (31)\end{aligned}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)PZ, \xi_i) &\\ = g(R(X, Y)PZ, \xi_i) + h_j^*(X, PZ)h_j(Y, \xi_i) - h_j^*(Y, PZ)h_j(X, \xi_i) &\\ = (\nabla_X h_i)(Y, PZ) - (\nabla_Y h_i)(X, PZ) &\\ + \tau_{ji}(X)h_j(Y, PZ) - \tau_{ji}(Y)h_j(X, PZ), \quad (32)\end{aligned}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)PZ, N_i) &\\ = \bar{g}(R(X, Y)PZ, N_i) + h_j(X, PZ)g(A_{N_j}Y, N_i) - h_j(Y, PZ)g(A_{N_j}X, N_i) &\\ = g(\nabla_X(A_{N_i}Y) - \nabla_Y(A_{N_i}X) - A_{N_i}[X, Y], PZ) &\\ + \tau_{ij}(Y)h_j^*(X, PZ) - \tau_{ij}(X)h_j^*(Y, PZ), \quad (33)\end{aligned}$$

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)\xi_i, N_j) &\\ = \bar{g}(R(X, Y)\xi_i, N_j) + h_k(X, \xi_i)g(A_{N_k}Y, N_j) - h_k(Y, \xi_i)g(A_{N_k}X, N_j) &\\ = h_i(X, A_{N_j}Y) - h_i(Y, A_{N_j}X) - 2\,d\tau_{ji}(X, Y) &\\ + \tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X). \quad (34)\end{aligned}$$

Denote by R^* the curvature tensors of ∇^*. Then by the same calculations of (29)–(34), and using (22), (23) and (25), we obtain the following equations (35)–(40).

$$\begin{aligned}
&R(X, Y)PZ \\
&= R^*(X, Y)PZ + h_i^*(X, PZ)A_{\xi_i}Y - h_i^*(Y, PZ)A_{\xi_i}X \\
&\quad + \{(\nabla_X^* h_i^*)(Y, PZ) - (\nabla_Y^* h_i^*)(X, PZ) \\
&\qquad + h_j^*(X, PZ)\tau_{ij}(Y) - h_j^*(Y, PZ)\tau_{ij}(X)\}\xi_i, \qquad (35)
\end{aligned}$$

$$\begin{aligned}
&R(X, Y)\xi_i \\
&= -\nabla_X^*(A_{\xi_i}Y) + \nabla_Y^*(A_{\xi_i}X) + A_{\xi_i}[X, Y] + \tau_{ji}(Y)A_{\xi_j}X - \tau_{ji}(X)A_{\xi_j}Y \\
&\quad + \{h_j^*(Y, A_{\xi_i}X) - h_j^*(X, A_{\xi_i}Y) - 2d\tau_{ji}(X, Y) \\
&\qquad + \tau_{jk}(X)\tau_{ki}(Y) - \tau_{ki}(X)\tau_{jk}(Y)\}\xi_j, \qquad (36)
\end{aligned}$$

$$\begin{aligned}
&g(R(X, Y)PZ, PW) \\
&= g(R^*(X,Y)PZ, PW) + h_i^*(X, PZ)h_i(Y, PW) - h_i^*(Y, PZ)h_i(X, PW), \qquad (37)
\end{aligned}$$

$$\begin{aligned}
&\bar{g}(R(X, Y)PZ, N_i) \\
&= (\nabla_X^* h_i^*)(Y, PZ)) - (\nabla_Y^* h_i^*)(X, PZ)) + h_j^*(X, PZ)\tau_{ij}(Y) - h_j^*(Y, PZ)\tau_{ij}(X) \\
&= g(\nabla_X(A_{N_i}Y) - \nabla_Y(A_{N_i}X) - A_{N_i}[X, Y], PZ) + h_j(Y, PZ)g(A_{N_j}X, N_i) \\
&\quad - h_j(X, PZ)g(A_{N_j}Y, N_i) + h_j^*(X, PZ)\tau_{ij}(Y) - h_j^*(Y, PZ)\tau_{ij}(X), \qquad (38)
\end{aligned}$$

$$\begin{aligned}
&g(R(X, Y)PZ, \xi_i) \\
&= g(\nabla_X^*(A_{\xi_i}Y) - \nabla_Y^*(A_{\xi_i}X) - A_{\xi_i}[X, Y], PZ) + \tau_{ji}(X)h_j(Y, PZ) \\
&\qquad - \tau_{ji}(Y)h_j(X, PZ) \\
&= (\nabla_X h_i)(Y, PZ) - (\nabla_Y h_i)(X, PZ) + h_j(X, \xi_i)h_j^*(Y, PZ) \\
&\quad - h_j(Y, \xi_i)h_j^*(X, PZ) + \tau_{ji}(X)h_j(Y, PZ) - \tau_{ji}(Y)h_j(X, PZ), \qquad (39)
\end{aligned}$$

$$\begin{aligned}
&\bar{g}(R(X, Y)\xi_i, N_j) \\
&= h_j^*(Y, A_{\xi_i}X) - h_j^*(X, A_{\xi_i}Y) - 2d\tau_{ji}(X, Y) + \tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X) \\
&= h_i(Y, A_{N_j}X) - h_i(X, A_{N_j}Y) - 2d\tau_{ji}(X, Y) + h_k(Y, \xi_i)g(A_{N_k}X, N_j) \\
&\quad - h_k(X, \xi_i)g(A_{N_k}Y, N_j) + \tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X). \qquad (40)
\end{aligned}$$

Now suppose that M is totally geodesic, then it follows from the Theorem 2.3 that h_i and A_{ξ_i} both vanish on M due to (25) and (26).

The *type number* $t(u)$ of M at a point u is the rank of the shape operator A_N at u. By the relation (19) it follows that $t(u) \leq m - n$. Using this in the equations (29) and (31) we obtain the following theorem.

Theorem 2.4. *Let* (M, g, SM) *be a coisotropic submanifold of* $(\bar{M}, \bar{g})$ *such that* $t(u) = m-n$. *Then* M *is totally geodesic in* $\bar{M}$ *if and only if the induced connection on* M *have the same curvature tensor as the Levi-Civita connection on* $\bar{M}$, *i.e., we have*

$$\bar{R}(X, Y)Z = R(X, Y)Z, \qquad \forall\, X, Y, Z \in \Gamma(TM).$$

Proof. Suppose the equation in theorem is satisfied. Using the local expression of the equation (31) by the local quasi-orthonormal field of frames $\{\xi_i, F_\alpha\}$ we obtain

$$A^\alpha_\beta\, B_{\Gamma\delta} = A^\alpha_\gamma\, B_{\beta\,\delta}, \qquad A^\alpha_i\, B_{\Gamma\delta} = 0$$

where A^α_β and $B_{\Gamma\delta}$ are the local components of A_{N_i} and h_i respectively. This implies $B_{\Gamma\delta} = 0$ because of $t(u) = m - n$. Thus M is totally geodesic. □

3. Totally Umbilical Submanifolds

Let (M, g, SM) be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. M is said to be *totally umbilical* in $\bar{M}$ if there is a smooth affine normal vector field $\mathcal{H} \in \Gamma(\mathrm{tr}(TM))$ on M, called *the transversal curvature vector field* of M, such that

$$h\,(X, Y) = \mathcal{H}\,\bar{g}\,(X, Y)$$

for all $X, Y \in \Gamma(TM)$ (see O'Neill [3]). By straightforward calculation and using (11) it is easy to see that M is totally umbilical, if and only if, locally, on each coordinate neighborhood $\mathcal{U}$ there exist smooth functions $H_i \in F(\mathrm{tr}(TM))$ such that

$$h_i\,(X, Y) = H_i\,\bar{g}\,(X, Y), \tag{41}$$

for any $X, Y \in \Gamma(TM)$. Above definition does not depend on the screen distribution of M. On the other hand, from (25) and by using the fact SM is non-degenerate distribution, we conclude that M is totally umbilical, if and only if, on each $\mathcal{U}$ there exist H_i such that

$$A_{\xi_i}X = H_i\,PX, \quad \forall\, X \in \Gamma(TM). \tag{42}$$

Note that in case M is totally umbilical, we have

$$h_i(X,\, \xi_j) = 0, \quad A_{\xi_i}\xi_j = 0, \quad \forall X \in \Gamma(TM). \tag{43}$$

Using the equations (29) and (32) we have the following theorem.

Theorem 3.1. *Let $(M,\, g,\, SM)$ be a totally umbilical coisotropic submanifold of a semi-Riemannian manifold of constant curvature $(\bar{M}(\bar{c}),\, \bar{g})$. Then, the functions H_i from (41) satisfy the following partial differential equations*

$$\xi_j(H_i) - H_j H_i + H_k \tau_{ki}(\xi_j) = 0, \tag{44}$$

and the curvature tensor R of M is given by

$$R(X,Y)Z$$
$$= \bar{c}\{g(Y,Z)X - g(X,Z)Y\} + H_i\{g(Y,Z)A_{N_i}X - g(X,Z)A_{N_i}Y\},$$

for any $X,\, Y \in \Gamma(TM)$. Moreover,

$$PX(H_i) + H_k \tau_{ki}(PX) = 0. \tag{45}$$

Proof. Taking account of (41) in (32), and using the fact that $\bar{M}$ is a space of constant curvature obtain

$$\{X(H_i) - H_i H_j \eta_j(X) + H_k \tau_{ki}(X)\}\, g(Y,\, PZ)$$
$$- \{Y(H_i) - H_i H_j \eta_j(Y) + H_k \tau_{ki}(Y)\}\, g(X,\, PZ) = 0, \tag{46}$$

for any $X,\, Y,\, Z \in \Gamma(TM)$. Take $X = \xi_k$ and $Z = Y \in \Gamma(SM)$ such that $g(Y,\, Y) \neq 0$ on $\mathcal{U}$ and use (6) one obtains the first equation in theorem. Next, the second equation in theorem follows from (29) taking into account that $\bar{M}$ is a space of constant curvature and by using the first equation in theorem. Finally, take $X = PX$ and $Y = PY$ in (46) and by using (6) and taking account that SM is non-degenerate one obtain

$$\{PX(H_i) + H_k \tau_{ki}(PX)\}\, PY = \{PY(H_i) + H_k \tau_{ki}(PY)\}\, PX.$$

Now suppose there exists a vector field $X_o \in \Gamma(TM)$ such that

$$PX_o(H_i) + H_k \tau_{ki}(PX_o) \neq 0 \quad \text{at each point } u \in M.$$

Then from the last equation it follow that all vectors from the fiber $(SM)_u$ are collinear with $(PX_o)_u$. This is a contradiction as $\dim(SM)_u = m - n$. Hence the last equation in theorem is true at any point of $\mathcal{U}$, which completes the proof. □

From (34) and the fact that $\bar{M}$ is a space of constant curvature we obtain

$$H_i \{g(X, A_{N_j}Y) - g(Y, A_{N_j}X)\}$$
$$= 2\, d\tau_{ji}(X, Y) - \tau_{jk}(X)\tau_{ki}(Y) + \tau_{ki}(X)\tau_{jk}(Y).$$

In case $H_i \neq 0$ on $\mathcal{U}$ we say that M is *proper totally umbilical.* From the last equation we obtain the following theorem.

Theorem 3.2. *Let (M, g, SM) be a proper totally umbilical coisotropic submanifold of a semi-Riemannian manifold of constant curvature $(\bar{M}(\bar{c}), \bar{g})$. Then the screen distribution SM is integrable, if and only if, each 1-forms τ_{ij} induced by SM satisfy $d(\mathrm{Tr}(\tau_{ij})) = 0$, where $\mathrm{Tr}(\tau_{ij})$ is the trace of the matrix (τ_{ij}).*

Next, the screen distribution SM is said to be *totally umbilical* in M if there is a smooth vector field $\mathcal{K} \in \Gamma(\mathrm{Rad}\,TM)$ on M, such that, $h^*(X, PY) = \mathcal{K}\, g(X, PY)$ for all $X, Y \in \Gamma(TM)$. By straightforward calculation it is easy to see that SM is totally umbilical, if and only if, on any coordinate neighborhood $\mathcal{U} \subset M$, there exists a smooth functions K_i such that

$$h_i^*(X, PY) = K_i\, g(X, PY) \tag{47}$$

for any $X, Y \in \Gamma(TM)$. It follows that h_i^* are symmetric on $\Gamma(SM)$ and hence according to Theorem 2.2, the screen space SM is integrable. In case $K_i = 0$ (resp. $K_i \neq 0$) on $\mathcal{U}$ we say that SM is *totally geodesic* (resp. *proper totally umbilical*). From (19), (24) and (47) we obtain

$$P(A_{N_i}X) = K_i\, PX, \quad h_i^*(\xi_j, PX) = 0 \tag{48}$$

for any $X \in \Gamma(TM)$. Note that in case SM is totally umbilical, we have from (7) and (48)

$$A_{N_i}X = K_i\, PX + \eta_j(A_{N_i}X)\xi_j, \tag{49}$$

and from (18) we have

$$\eta_i(A_{N_j}X) = -\eta_j(A_{N_i}X).$$

Lemma 3.1. *Let (M, g, SM) be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$ of constant curvature $\bar{c}$, such that the screen distribution SM is totally umbilical. If M is also totally umbilical, then the mean curvature vectors K_i of SM are a solution of the following partial differential equations*

$$X(K_i) - K_j\,\tau_{ij}(X) - K_i\,H_j\,\eta_j(X) - H_j\,\eta_j(A_{N_i}X) - \bar{c}\,\eta_i(X) = 0. \tag{50}$$

Proof. Taking account of (47) and (49) into (33) and (38), and using (6), (25), (28) and the fact that $\bar{M}$ is a space of constant curvature we obtain

$$\{X(K_i) - K_j\,\tau_{ij}(X) - K_i\,H_j\,\eta_j(X) - H_j\,\eta_j(A_{N_i}X) - \bar{c}\,\eta_i(X)\}\,g\,(Y,\,PU)$$
$$= \{Y(K_i) - K_j\,\tau_{ij}(Y) - K_i\,H_j\,\eta_j(Y) - H_j\,\eta_j(A_{N_i}Y) - \bar{c}\,\eta_i(Y)\}\,g\,(X,\,PU).$$

Thus by the method of Theorem 3.1 we have the equation (50). □

From the equations in Lemma 3.1 we have the following theorem.

Theorem 3.3. *Let $(M,\,g,\,SM)$ be a proper totally umbilical coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}\,(\bar{c}),\,\bar{g})$ of constant curvature $\bar{c}$. If the screen distribution SM is totally geodesic, then $\bar{c} = 0$, i.e., the ambient semi-Riemannian manifold $\bar{M}$ is semi-Euclidean space.*

Corollary 3.4. *Under the hypothesis of Lemma 3.1, the induced connection ∇ on M is metric connection if and only if the mean curvature vectors K_i of SM are a solution of the following partial differential equations*

$$X(K_i) - K_j\,\tau_{ij}(X) - \bar{c}\,\eta_i(X) = 0.$$

4. Induced Ricci Tensor

From (12) it follows that h_i are symmetric bilinear forms on $\Gamma(TM)$ and they do not depend on the screen distribution. In fact, we have

$$\bar{g}\,(\bar{\nabla}_X Y,\,\xi_i) = h_i\,(X,\,Y),$$

for any $X,\,Y \in \Gamma(TM)$. However, we note that h_i and τ_{ij} depend on the section $\xi_i \in \Gamma(\mathrm{Rad}\,TM)$. Indeed, in case we take $\xi_i^* = \alpha_{ij}\,\xi_j$, where α_{ij} are smooth functions with $\Delta = \det(\alpha_{ij}) \neq 0$. It follows that $N_i^* = \frac{1}{\Delta}\,A_{ij}\,N_j$, where A_{ij} is the ij-cofactor of the determinant of Δ. Hence by straightforward calculation we obtain $B_i^* = \alpha_{ij}\,h_j$ and τ_{ij}^* are denoted by affine combinations of τ_{ij} with coefficients α_{ij}, A_{ij} and $X(A_{ij})$. Moreover, we have

$$\mathrm{Tr}(\tau_{ij})\,(X) = \mathrm{Tr}(\tau_{ij}^*)\,(X) + X\,(\log\Delta),$$

for any $X \in \Gamma(TM)$. Thus we obtain the following theorem.

Theorem 4.1. *Let $(M,\,g,\,SM)$ be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M},\,\bar{g})$. Suppose $\mathrm{Tr}(\tau_{ij})$ and $\mathrm{Tr}(\tau_{ij}^*)$ are 1-forms on $\mathcal{U}$ with respect to ξ_i and ξ_i^* respectively. Then $d\,(\mathrm{Tr}(\tau_{ij}^*)) = d\,(\mathrm{Tr}(\tau_{ij}))$ on $\mathcal{U}$.*

We find the local expression of structure equations of a coisotropic submanifold. Consider the frames field $\{F_1, \cdots, F_{m-n}, \xi_1, \cdots, \xi_n, N_1, \cdots, N_n\}$ on $\bar{M}$. In the sequel we use the range of indices:

$$\alpha, \beta, \gamma, \cdots \in \{1, 2, \cdots, m-n\}; \; A, B, C, D, \cdots \in \{1, 2, \cdots, m\}.$$

Denote by $\{X_A\}$ the frame fields $\{F_\alpha, \xi_i\}$ on M, i.e.,

$$X_1 = F_1, \cdots, X_{m-n} = F_{m-n}; \quad X_{m-n+1} = \xi_1, \cdots, X_m = \xi_n.$$

Then consider the local components of curvature tensors $\bar{R}$ and R as follows:

$$\begin{aligned} \bar{R}_{ABCD} &= \bar{g}\,(\bar{R}\,(X_D, X_C)X_B, X_A), & R_{ABCD} &= g\,(R\,(X_D, X_C)X_B, X_A), \\ \bar{R}_{iBCD} &= \bar{g}\,(\bar{R}\,(X_D, X_C)X_B, N_i), & R_{iBCD} &= \bar{g}\,(R\,(X_D, X_C)X_B, N_i), \\ \bar{R}_{ijCD} &= \bar{g}\,(\bar{R}\,(X_D, X_C)N_j, N_i), & R_{ijCD} &= \bar{g}\,(R\,(X_D, X_C)N_j, N_i). \end{aligned}$$

We are now concerned with local expression of Ricci-tensor of a coisotropic submanifold M of an $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ of. By using the frames field on M we obtain the following local expression for the Ricci tensor

$$Ric\,(X, Y) = g^{\alpha\beta}\, g\,(R(X, F_\alpha)Y, F_\beta) + \bar{g}\,(R(X, \xi_i)Y, N_i).$$

By using the symmetries of curvature tensor and the first Bianchi identity and taking into account (31) and (40) we obtain

$$\begin{aligned} &Ric\,(X, Y) - Ric\,(Y, X) \\ &\quad = g^{\alpha\beta}\,\{h_i^*(X, X_\alpha)\, h_i(Y, X_\beta) - h_i(X, X_\alpha)\, h_i^*(Y, X_\beta)\} \\ &\qquad\qquad + h_i^*\,(Y, A_{\xi_i}X) - h_i^*\,(X, A_{\xi_i}Y) - 2\,d\,(\mathrm{Tr}(\tau_{ij}))\,(X, Y). \end{aligned}$$

Replacing X and Y by X_A and X_B respectively and by using (24) and (25) we have

$$R_{AB} - R_{BA} = 2\,d\,(\mathrm{Tr}(\tau_{ij}))\,(X_A, X_B)$$

where we put $R_{AB} = Ric\,(X_B, X_A)$. Therefore, by using Theorem 4.1 we conclude

Theorem 4.2. *Let (M, g, SM) be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the Ricci tensor of the induced connection ∇ on M is symmetric if and only if each 1-form $\mathrm{Tr}(\tau_{ij})$ induced by SM is closed, i.e., $d\,(\mathrm{Tr}(\tau_{ij})) = 0$ on any $\mathcal{U} \subset M$.*

From the Theorems 3.2 and 4.2 we obtain the following theorem.

Theorem 4.3. *Let* (M, g, SM) *be a proper totally umbilical coisotropic submanifold of a semi-Riemannian manifold of constant curvature* $(\bar{M}\,(\bar{c}),\, \bar{g})$. *The screen distribution* SM *is integrable, if and only if, the Ricci tensor of the induced connection* ∇ *on* M *is symmetric.*

Corollary 4.4. *Under the hypothesis of Theorem 4.3, if the screen distribution* SM *is totally umbilical, then the Ricci tensor of the induced connection* ∇ *on* M *is symmetric.*

REFERENCES

1. Duggal, K. L. and Bejancu, A.: *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and its Applications, 364. Kluwer Acad. Publishers, Dordrecht, 1996. MR **97e**:53121
2. Duggal, K. L. and Jin, D. H.: Totally Umbilical Lightlike Submanifolds. To appear in *Geom. Dedicata.*
3. O'Neill, B.: *Semi-Riemannian Geometry*, With Applications to Relativity, Pure and Applied Mathematics, 103. Academic Press, New York, 1983. MR **85f**:53002

DEPARTMENT OF MATHEMATICS, DONGGUK UNIVERSITY, GYEONGJU, GYEONGBUK 780-714, KOREA

E-mail address: jindh@dongguk.ac.kr