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THE GLOBAL EXISTENCE OF SOLUTIONS OF A REACTION-DIFFUSION EQUATION

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ABSTRACT. We establish the global existence of nonnegative solutions to some reaction-diffusion equation for exponential nonlinearity for small initial data.

1. INTRODUCTION

We consider a reaction diffusion system

$$\text{(1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u + k u \phi(v) = 0 & \text{on } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v - u \phi(v) = 0 & \text{on } \mathbb{R}^+ \times \Omega \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ of class C^1, d_j $(j = 1, 2)$ and k are positive constants, and ϕ is a nonnegative function of class $C_1(\mathbb{R}^+)$. We also consider the homogeneous boundary conditions

$$\text{(2)} \qquad \begin{cases} a_1 \dfrac{\partial u}{\partial n} + (1 - a_1) u = 0 & \text{on } \mathbb{R}^+ \times \Gamma \\ a_2 \dfrac{\partial v}{\partial n} + (1 - a_2) v = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases}$$

where $\Gamma = \partial\Omega$, and $a_1(x)$ and $a_2(x)$ are nonnegative functions of class $C^1(\Gamma)$ with $a_j(x) \leq 1$.

We study the question of the existence of global solutions of problem (1)–(2) in the class $C(\bar{\Omega})$ with initial data

$$\text{(3)} \qquad u(0, x) = u_0, \quad v(0, x) = v_0$$

where u_0, v_0 are in $L^\infty(\Omega)$ and u_0, $v_0 \geq 0$. This problem was initially raised by Martin and has been studied successively by many authors (cf. Alikakor [1], Barabanova [2], Conway and Smoller [5], Haraux and Youkana [7], Masuda [8],

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Youkana [9, Chapitre 3]) for various types of nonlinear $\phi(v)$. Recently, Barabanova [2] proved the existence of global solutions for the nonlinear function

$$\phi(v) = e^{\alpha v}, \quad \alpha > 0,$$

in any dimension n with arbitrary v_0 and u_0 satisfying

$$||u_0||_{L^\infty(\Omega)} < \frac{8d_1 d_2}{\alpha n(d_1 - d_2)^2}. \tag{4}$$

If $|d_1 - d_2|$ is small enough, the equations (1)–(2) with (3) have the global solutions for relatively large initial data. However, if αn or $|d_1 - d_2|$ is large then it is true only for small data.

In this paper we find suitable range for $\gamma = \frac{d_1}{d_2}$ and $k > 0$ for the same result with any initial data $||u_0||_\infty < 1$.

2. Statement and proof of the main result

Throughout the paper we denote by $||\cdot||_p$ the norm in $L^p(\Omega)$, $1 \le p < \infty$. The study of local existence and uniqueness of solutions to the initial-boundary value problem for (1)–(2) with (3) in the framework of $C(\bar{\Omega})$ or $L^p(\Omega)$ is classical. In particular, for nonnegative u_0 and v_0, there exists a local nonnegative solution (u, v) of class $C(\bar{\Omega})$ of (1)–(2) with (3) on $(0, T)$, where T is the eventual blowing-up time in $L^\infty(\Omega)$ (cf. Cazenave and Haraux [3, 4]). Also it is evident that u satisfies the maximum principle, i.e., $||u(t)||_\infty \le ||u_0||_\infty$. Finally we note that as a consequence of the results of [6], to prove that the solutions of (1)–(2) with (3) and (4) are global, it is sufficient to derive a uniform estimate of $||u\phi(u)||_p$ on $(0, T)$ for some $p > \frac{n}{2}$. Since $||u(t)||_\infty$ is bounded by $||u_0||_\infty$, it is, therefore, good enough to obtain a bound in $||\phi(u)||_p$ on $(0, T)$ which does not depend on t.

The main result can be stated as follows.

Theorem 1. *Assume that $\phi(v) = e^{\alpha v}$ and $k \ge \alpha n$. Then if γ satisfies $3 - 2\sqrt{2} \le \gamma \le 3 + 2\sqrt{2}$ the solutions of (1)–(2) with (3) for nonnegative initial data u_0, v_0 in $L^\infty(\Omega)$ satisfying $||u_0||_\infty < 1$ are global and uniformly bounded on $(0, \infty) \times \Omega$.*

To prove this theorem we need the following proposition.

Proposition 2. *Let (u, v) be a solution of (1)–(2) with (3) for arbitrary v_0 and u_0 satisfying $||u_\infty|| < 1$. Let*

$$g(u) = \frac{1}{1-u}.$$

Then, for p satisfying $n > p > \frac{n}{2}$, the integral

$$\int_\Omega g(u)e^{\alpha p v}\,dx \tag{5}$$

is nonincreasing on $(0,T)$.

Proof. By a standard argument, we get the following inequality for any solution (u,v) of (1)–(2) with (3).

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{1}{1-u}e^{\alpha p v}dx\right)\\
&\leq -\int_\Omega e^{\alpha p v}\left(\frac{2}{(1-u)^3}d_1|\nabla u|^2 + \frac{\alpha p}{(1-u)^2}(d_1+d_2)\nabla u\cdot\nabla v + \frac{\alpha^2p^2}{1-u}d_2|\nabla v|^2\right)dx\\
&\qquad + \int_\Omega e^{\alpha p v}u\phi(v)\left(\frac{\alpha p}{1-u} - \frac{k}{(1-u)^2}\right)dx.
\end{aligned}$$

The integrand of the first integral is nonnegative definite if

$$\frac{\alpha^2p^2}{(1-u)^4}(d_1+d_2)^2 \leq 8d_1d_2\frac{\alpha^2p^2}{(1-u)^4}$$

and actually it is nonnegative definite if $\gamma = \frac{d_1}{d_2}$ satisfies

$$3-2\sqrt{2} \leq \gamma \leq 3+2\sqrt{2}.$$

Also since

$$\frac{\alpha p}{1-u} - \frac{k}{(1-u)^2} < \frac{-\alpha(n-p)-\alpha p u}{(1-u)^2} < 0,$$

we obtain the desired result. □

Now using this result, we prove the main theorem.

Proof of **Theorem 1**. Let p be as in Proposition 2. The boundedness of $||u_0||_\infty$ implies

$$||u\phi(v)||_p \leq ||u_0||_\infty||\phi(v)||_p.$$

Since $g(u) \geq 1$, we have

$$(||\phi(v)||_p)^p = \int_\Omega |\phi(v)|^p\,dx \leq \int_\Omega e^{\alpha p v}\,dx \leq \int_\Omega g(u)e^{\alpha p v}.$$

By Proposition 2, we obtain

$$(||\phi(v)||_p)^p \leq \int_\Omega g(u)e^{\alpha p v_0}\,dx \leq |\Omega|g(||u_0||_\infty)e^{\alpha p||v_0||_\infty}.$$

Hence $u\phi(v)$ is uniformly bounded in $L^p(\Omega)$ for all $t \in [0,T]$. By the results of Haraux and Kirane [6], we conclude that the solutions of the initial-boundary problem (1)–(2) with (3) are global and uniformly bounded on $(0,\infty)\times\Omega$. □

3. BEHAVIOR AS $t \to \infty$

Masuda [8], for nonnegative global solution (u,v) of (1)–(2) with (3), has proved that there exist two nonnegative constants u^*, v^* such that

$$||u-u^*|| \to 0, \quad ||v-v^*|| \to 0,$$

and $u^*\phi(v^*) = 0$. It is obvious that $u^* = v^* = 0$ when both u and v have non-Neumann boundary conditions. Therefore we have the following asymptotic behavior of solutions with $\phi(v) = e^{\alpha v}$.

Theorem 3. *Let (u,v) be any nonnegative global solution of* (1)–(2) *with* (3) *for a_1, $a_2 \neq 1$. Let λ_i be the first eigenvalue of $-\Delta$ in Ω with boundary condition* (2) *defined by a_i, $i = 1, 2$. Then there exist constants K_1, K_2, $K_3 > 0$ such that*

$$||u(t)||_\infty \leq K_1 e^{-(\lambda_1 d_1 + k)t}, \tag{6}$$

$$||v(t)||_\infty \leq K_2 e^{-\min\{\lambda_1 d_1 + k, \lambda_2 d_2\}t} \quad \textit{if} \quad \lambda_1 d_1 + k \neq \lambda_2 d_2, \tag{7}$$

$$||v(t)||_\infty \leq (K_2 + K_3 t) e^{-\lambda_2 d_2 t} \quad \textit{if} \quad \lambda_1 d_1 + k = \lambda_2 d_2. \tag{8}$$

Proof. The solution of the equation

$$\frac{\partial u}{\partial t} - d_1\Delta u + ku = 0 \tag{9}$$

with the boundary (2) with initial data (3) is a supersolution for

$$\frac{\partial u}{\partial t} - d_1\Delta u + ku\phi^{\alpha v} = 0$$

with the same boundary condition and initial data. It is known (cf. Cazenave and Haraux [3, 4]) that the solution of the linear equation (9) has asymptotic $e^{-(\lambda_1 d_1 + k)t}$ so that we have proven (6). Also, using the boundedness of $||v(t)||_\infty$ and applying the expression of the solution of the system (1) in terms of the semigroup by $\frac{\partial}{\partial t} - d_2\Delta$, as in Haraux and Kirane [6], we obtain the inequality

$$||v(t)||_\infty \leq M_1 e^{-\lambda_2 d_2 t} + M_2 e^{-\lambda_2 d_2 t} + \int_0^t e^{(-(\lambda_1 d_1 + k) + \lambda_2 d_2)\sigma} d\sigma,$$

where $M_1, M_2 > 0$. Then computing this integral depending on the relation between $\lambda_1 d_1 + k$ and $\lambda_2 d_2$, we obtain the desired results (7) and (8). We refer Barabanova [2] for more detailed proof. □

Remark. The study related to this problem has been made in the point of the growth rate of the nonlinear function $\phi(v)$. However, when $\phi(v) = e^{\alpha v}$ then some restrictions on u_0 appear. Youkana [9] proved this case in small dimensions $n = 1$ and $n = 2$. Barabanova [2] extended this result for any dimension n. For his result, he also used the semigroup method and Lyapunov function techniques. Moreover, he observed that the problem is still open when $\phi(v)$ has faster growth than $e^{\alpha v}$ or when the initial data $u_0 \geq 0$ in L^∞ is given arbitrarily.

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