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STABLE RANK OF TWISTED CROSSED PRODUCTS OF C^*-ALGEBRAS BY ABELIAN GROUPS

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ABSTRACT. We estimate the stable rank of twisted crossed products of C^*-algebras by topological Abelian groups. As an application we estimate the stable rank of twisted crossed products of C^*-algebras by solvable Lie groups. In particular, we obtain the stable rank estimate of twisted group C^*-algebras of solvable Lie groups by the (reduced) dimension and (generalized) rank of groups.

INTRODUCTION

The stable rank of C^*-algebras was introduced by Rieffel [Rf1] as a noncommutative analogue of the covering dimension of spaces, and some fundamental formulas of the stable rank, connected stable rank and general stable rank were obtained (*cf.* Rieffel [Rf1, Rf2]; Nistor [Ns1, Ns2]). The most interesting results among them would be the stable rank and connected stable rank estimates of crossed products of C^*-algebras by the integers, that is, the formulas (*cf.* Rieffel [Rf1, Theorem 7.1 and Corollary 8.6]).

(F): for any C^*-algebra $\mathfrak{A}$,

$$\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{A}) + 1 \text{ and } \mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{A}) + 1,$$

where α is an action of $\mathbb{Z}$ by automorphisms of $\mathfrak{A}$.

The first one is useful to estimate the stable rank of irrational rotation algebras as an application, and the second one is applicable to the cancellation property of finitely generated projective modules over those crossed products by the integers. See also Blackadar [Bl, §6.5] for a relation between the stable rank and the cancellation property. On the other hand, a general theory of twisted crossed products of C^*-algebras by locally compact groups has been studied by Packer & Raeburn [PR1,

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PR2]. In particular, the covariant representation theory of these twisted crossed products and its related properties have been considered in details. Also, see Packer & Raeburn [PR3] for the structure of twisted group C^*-algebras.

Our aim of this paper is to generalize the Rieffel's stable rank estimate of crossed products by the integers to the case of twisted crossed products by topological Abelian groups. The main results concern the cases by either compact connected Abelian groups or the real numbers (*cf.* Theorems 1 and 3). For the proof we follow the same argument as used by Rieffel for the stable rank estimate of crossed products by the integers. However, it requires more careful framework for generating elements of twisted crossed products by those groups.

Moreover, the case of twisted crossed products of nonunital C^*-algebras by those groups should be treated independently and is more complicated than the unital case (*cf.* Theorem 4).

As an application we obtain the stable rank and connected stable rank estimates of twisted crossed products by solvable Lie groups. In particular, we estimate the stable ranks of twisted group C^*-algebras of solvable Lie groups by the (reduced) dimension and (generalized) rank of groups. For the proofs we use the Packer and Raeburn's decomposition theorem of twisted crossed products Packer & Raeburn [PR1, Theorem 4.1]. Also, see Sheu [Sh], Sudo & Takai [ST1, ST2] and Sudo [Sd1, Sd2, Sd3, Sd4, Sd5, Sd6, Sd7, Sd8] for some related works on the stable rank of group C^*-algebras. In particular, the stable rank estimates of crossed products by either the real numbers or the tori and its applications were considered in Sudo [Sd8] with the same spirit as the above case of twisted crossed products.

NOTATIONS AND DEFINITIONS

For a C^*-algebra $\mathfrak{A}$ (or its unitization $\mathfrak{A}^+$), its (topological) stable rank, connected stable rank and general stable rank are denoted by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ and $\mathrm{gsr}(\mathfrak{A})$ respectively (*cf.* Rieffel [Rf1]; Nistor [Ns1]). By definition, these stable ranks take values in $\{1, 2, \ldots, \infty\}$, and $\mathrm{sr}(\mathfrak{A}) \leq n$ if and only if the open subspace $L_n(\mathfrak{A})$ of $\mathfrak{A}^n$ is dense in $\mathfrak{A}^n$, where $(a_j)_{j=1}^n \in L_n(\mathfrak{A})$ if and only if $\sum_{j=1}^n a_j^* a_j$ (or $\sum_{j=1}^n b_j a_j$ for some $(b_j)_{j=1}^n \in \mathfrak{A}^n$) is invertible in $\mathfrak{A}$, and $\mathrm{csr}(\mathfrak{A}) \leq n$ if and only if the connected component $GL_m(\mathfrak{A})_0$ of $GL_m(\mathfrak{A})$ with the unit acts transitively on $L_m(\mathfrak{A})$ for any $m \geq n$, or equivalently $L_m(\mathfrak{A})$ is path-connected for any $m \geq n$, and $\mathrm{gsr}(\mathfrak{A}) \leq n$

if and only if $GL_m(\mathfrak{A})$ acts transitively on $L_m(\mathfrak{A})$ for any $m \geq n$. In general, $\mathrm{gsr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ (*cf.* Rieffel [Rf1, Corollary 4.10 and p. 328]).

Let $C_0(X)$ denote the C^*-algebra of all continuous functions vanishing at infinity on a locally compact Hausdorff space X. Set $C(X) = C_0(X)$ when X is compact.

Let $\mathfrak{A} \rtimes_{\alpha,u} G$ denote the twisted crossed product of a (separable) C^*-algebra $\mathfrak{A}$ by a locally compact group G with (α, u) a twisted action, where α is an action of automorphisms of $\mathfrak{A}$ by G and u is a Borel map from the direct product $G \times G$ to the unitary group $\mathcal{U}M(\mathfrak{A})$ of the multiplier algebra $M(\mathfrak{A})$ of $\mathfrak{A}$ such that (in particular)

$$\begin{cases} \alpha_e = \mathrm{id}, \quad u(e,g) = u(g,e) = 1 & \text{for } g \in G \text{ and } e \text{ the unit of } G, \\ \alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh} & \text{for } g, h \in G. \end{cases}$$

A covariant representation of a twisted dynamical system $(\mathfrak{A}, G, \alpha, u)$ (or $\mathfrak{A} \rtimes_{\alpha,u} G$) is a pair (π, U) (or its integrated form $\pi \times U$) with π a non-degenerate representation of $\mathfrak{A}$ on a Hilbert space H and U a Borel measurable map from G to the unitary group on H such that

$$U_g U_h = \pi(u(g,h)) U_{gh}, \quad \pi(\alpha_g(a)) = U_g \pi(a) U_g^* \quad \text{for } g, h \in G \text{ and } a \in \mathfrak{A}.$$

See Packer & Raeburn [PR1, PR2] and Busby & Smith [BS] for more details. Refer to Blackadar [Bl] and Pedersen [Pd] for general references. In what follows G will be a Lie group.

THE MAIN RESULTS

First of all, we consider the case of twisted crossed products by compact connected commutative Lie groups, *i. e.*, the l-torus $\mathbb{T}^l$.

Theorem 1. *Let $\mathfrak{A}$ be a unital C^*-algebra, $G = \mathbb{T}^l$, and $\mathfrak{A} \rtimes_{\alpha,u} G$ the twisted crossed product of $\mathfrak{A}$ by G with (α, u) a twisted action. Then*

$$\mathrm{sr}(\mathfrak{A} \rtimes_{\alpha,u} G) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

Proof. By assumption of $\mathfrak{A}$ being unital we have $u(l,m) \in \mathfrak{A} = M(\mathfrak{A})$ for any $l, m \in G$.

Let (π, U) be a covariant representation of the twisted dynamical system $(\mathfrak{A}, G, \alpha, u)$, and $\pi \times U$ its associated integrated representation of $\mathfrak{A} \rtimes_{\alpha,u} G$ defined by

$$(\pi \times U)(x) = \int_G \pi(x(t)) U_t dt \quad \text{for } x \in L^1(G, \mathfrak{A}),$$

where $L^1(G, \mathfrak{A})$ is the dense space (or $*$-algebra) of all $\mathfrak{A}$-valued integrable functions on G (with the convolution product associated with the twisted action (α, u)). We may assume that $\pi \times U$ is universal. In fact, (π, U) will be defined to be the direct sum representation of all covariant representations of the system, or we may assume that (π, U) is the regular representation induced by a faithful representation of $\mathfrak{A}$ since $G = \mathbb{T}^l$ is amenable, (*cf.* Packer & Raeburn [PR1, Theorem 3.11]). Then $\mathfrak{A} \rtimes_{\alpha,u} G$ is identified with the C^*-algebra generated by the set $\{af \mid a \in \mathfrak{A}, f \in L^1(G)\}$, where $L^1(G)$ is the space (or $*$-algebra) of all integrable measurable functions on G (with the usual convolution product). Note that $(\pi \times U)(af) = \int_G \pi(a) f(t) U_t dt = \pi(a)(\int_G f(t) U_t dt)$. Moreover, the set of linear spans:

$$\operatorname{span}\{af \mid a \in \mathfrak{A}, f \in L^1(G)\}$$

is dense in $\mathfrak{A} \rtimes_{\alpha,u} G$ (*cf.* Packer & Raeburn [PR1, Definition 2.4]; [PR2, Corollary 1.5]). In fact, note that for $b \in \mathfrak{A}$ and $g \in L^1(G)$,

$$\left(\int_G g(t) U_t dt\right) \pi(b) = \int_G g(t) U_t \pi(b) U_t^* U_t dt = \int_G g(t) \pi(\alpha_t(b)) U_t dt.$$

Since the integral $\int_G g(t)\pi(\alpha_t(b))U_t dt$ is regarded as the integrable function from $t \in G$ to $g(t)\pi(\alpha_t(b))U_t \in \mathfrak{A}$, it is approximated by finite sums of elements of the form af for $a \in \mathfrak{A}$ and $f \in L^1(G)$. Also, note that for $x, y \in L^1(G, \mathfrak{A})$,

$$\begin{aligned}
(\pi \times U)(x)(\pi \times U)(y) &= \int_G \pi(x(t)) U_t dt \int_G \pi(y(s)) U_s ds \\
&= \iint_{G \times G} \pi(x(t)) \pi(\alpha_t((y(s))) U_t U_s dt ds \\
&= \iint_{G \times G} \pi(x(t)) \pi(\alpha_t(y(s))) \pi(u(t,s)) U_{t+s} dt ds \\
&= \int_G \left(\int_G \pi((x(t)\alpha_t(y(s-t))u(t,s-t)) dt\right) U_s ds \\
&= \int_G \pi((x *_{\alpha,u} y)(s)) U_s ds = (\pi \times U)(x *_{\alpha,u} y),
\end{aligned}$$

where $x *_{\alpha,u} y$ defined as above is the convolution product associated with (α, u), and this product is also approximated by finite sums of the form ch for $c \in \mathfrak{A}$ and $h \in L^1(G)$.

Now note that $L^1(G)$ is contained in $L^1(G, \mathfrak{A})$ since $\mathfrak{A}$ is unital. Thus, $L^1(G)$ is contained in $\mathfrak{A} \rtimes_{\alpha,u} G$. Also, since $G = \mathbb{T}^l$ has only trivial multipliers (*cf.* Baggett & Kleppner [BK, Lemma 3.2]), the convolution product of $L^1(G, \mathfrak{A})$ associated with (α, u) must be the usual one on the restriction to $L^1(G)$. Hence the C^*-completion

$C^*(\mathbb{T}^l) \cong \mathbb{C} \rtimes \mathbb{T}^l$ of $L^1(\mathbb{T}^l)$ is also contained in $\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^l$. Therefore, the unitization $C^*(\mathbb{T}^l)^+$ is contained in the unitization $(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^l)^+$. By the Fourier transform, we have $C^*(\mathbb{T}^l)^+ \cong C_0(\mathbb{Z}^l)^+ \cong C((\mathbb{Z}^l)^+)$, where $(\mathbb{Z}^l)^+$ is the one-point compactification of $\mathbb{Z}^l$. Moreover, $(\mathbb{Z}^l)^+$ is identified with a closed subset of $\mathbb{T}$. Thus, $C((\mathbb{Z}^l)^+)$ is generated by a single unitary, say W. Combining this observation with the above argument, we have that $(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^l)^+$ is generated by the linear span:

$$\mathrm{span}\{aW^k \,|\, a \in \mathfrak{A}, k \in \mathbb{Z}\}.$$

Set $\mathfrak{B} = (\mathfrak{A} \rtimes_{\alpha,u} G)^+$ and $\mathfrak{D} = \{aW^k \,|\, a \in \mathfrak{A}, k \in \mathbb{Z}\}$. Now suppose that $\mathrm{sr}(\mathfrak{A}) \leq n-1$. We consider the following operations on $\mathfrak{D}^n$:

(Op_1): The left multiplication on $\mathfrak{D}^n$ by elementary matrices over $\mathfrak{D}^n$.

(Op_2): The right multiplication on $\mathfrak{D}^n$ by $(W^{k_j})_{j=1}^n$ for $k_j \in \mathbb{Z}$.

Note that $\mathfrak{D}^n$ is stable under these operations.

For any $(d_j)_{j=1}^n \in \mathfrak{D}^n$ with $d_j = \sum_{s=1}^{m_j} a_{j,s} W^{k_{j,s}}$ for $a_{j,s} \in \mathfrak{A}$ and $k_{j,s} < k_{j,s+1}$ $(1 \leq s \leq m_j - 1)$, define the length

$$L(d_j) \text{ of } d_j \text{ to be } k_{j,m_j} - k_{j,1} + 1,$$

and define the length

$$L((d_j)_{j=1}^n) \text{ of } (d_j)_{j=1}^n \text{ to be } \sum_{j=1}^n L(d_j).$$

Now take any element of $\mathfrak{B}^n$. Then it is approximated closely by an element $(d_j)_{j=1}^n$ of $\mathfrak{D}^n$. Take a small open neighborhood $\mathcal{V}$ of $(d_j)_{j=1}^n$. Then consider the above two operations (Op_1) and (Op_2) on $\mathcal{V} \cap \mathfrak{D}^n$. Then there exists an element of $\mathfrak{D}^n$ whose length is the smallest among

$$\{L((d'_j)_{j=1}^n) \,|\, (d'_j)_{j=1}^n \text{ transferred from } (d_j)_{j=1}^n \in \mathcal{V} \cap \mathfrak{D}^n \text{ by } (\mathrm{Op}_1), (\mathrm{Op}_2)\}.$$

Assume that $(d_j)_{j=1}^n$ is such an element.

Then suppose that $d_j \neq 0$ for any $1 \leq j \leq n$. Then we show a contradiction in the following. By permutation by (Op_1), we may assume that $L(d_n)$ is the maximum among $\{L(d_j)\}_{j=1}^n$. Let

$$d_j = \sum_{s=1}^{m_j} a_{j,s} W^{k_{j,s}} (1 \leq j \leq n) \text{ and } k_{j,s} < k_{j,s+1} (1 \leq s \leq m_j - 1).$$

By (Op_2) as $(d_j)_{j=1}^n \mapsto (d_j W^{k_{l,m_l} - k_{j,m_j}})_{j=1}^n$, the highest term of $d_j W^{k_{l,m_l} - k_{j,m_j}}$ with respect to the power of W is $a_{j,m_j} W^{k_{l,m_l}}$ $(1 \leq j \leq n-1)$. Since $\mathrm{sr}(\mathfrak{A}) \leq n-1$, by replacing $(a_{j,m_j})_{j=1}^{n-1}$ with its near element in $\mathfrak{A}^{n-1}$, there exists $(b_j)_{j=1}^{n-1} \in \mathfrak{A}^{n-1}$ such that $\sum_{j=1}^{n-1} b_j a_{j,m_j} = a_{n,m_n}$. By (Op_1) as the following multiplication

$$\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ -b_1 & \cdots & -b_{n-1} & 1 \end{pmatrix} \begin{pmatrix} d_1 W^{k_{n,m_n} - k_{1,m_1}} \\ \vdots \\ d_{n-1} W^{k_{n,m_n} - k_{n-1,m_{l-1}}} \\ d_n \end{pmatrix} = \begin{pmatrix} d_1' \\ \vdots \\ d_{n-1}' \\ d_n' \end{pmatrix},$$

we have $L(d_j) = L(d_j')$ for $1 \le j \le n-1$, but $L(d_n) > L(d_n')$. This is the contradiction.

From the above observation, we may take $(d_j)_{j=1}^n \in \mathfrak{D}^n$ with $d_1 = 0$. Then we may let $d_1 = \varepsilon 1$ for $\varepsilon > 0$ small enough. By (Op_1) by subtraction matrices over $\mathfrak{D}$, it is able to map $(d_j)_{j=1}^n$ to $(1, 0, \ldots, 0) \in L_n(\mathfrak{B})$. Note that $L_n(\mathfrak{B})$ is stable under (Op_1), and this local density of $L_n(\mathfrak{B})$ is always true.

As for (Op_2), we consider the case of $W^{k_1} = W$ and $W^{k_j} = 1$ for $2 \le j \le n$. The general case is reduced to this case by multiplications and permutations. Note that $d_1 W = W(W^* d_1 W)$, and the map $\mathfrak{B}^n \ni (d_j)_{j=1}^n \mapsto (W^* d_1 W, d_2, \ldots, d_n)$ is an inner automorphism of $\mathfrak{B}^n$ so that it is a homeomorphism (which need not preserve $L_n(\mathfrak{B})$ by the local density of $L_n(\mathfrak{B})$ as given above). And the (Op_2) by $(W, 1, \ldots, 1)$ preserves $L_n(\mathfrak{B})$.

It is deduced from the above argument that $L_n(\mathfrak{B})$ is dense in $\mathfrak{B}^n$. Hence we obtain $\mathrm{sr}(\mathfrak{B}) = \mathrm{sr}(\mathfrak{A} \rtimes_{\alpha,u} G) \le n$. □

Remark. Our above argument for the proof also works for compact Abelian groups K with only trivial multipliers or the cohomology group $H^2(K, \mathbb{T})$ trivial, so that their twisted group C^*-algebras are commutative (*cf.* Packer [Pk, Example 1.2]). When both of (α, u) are trivial, we have

$$\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^n \cong \mathfrak{A} \otimes C^*(\mathbb{T}^n), \text{ and } C^*(\mathbb{T}^n) \cong C_0(\mathbb{Z}^n).$$

Since $\mathbb{Z}^n$ is discrete, $\mathrm{sr}(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^n) = \mathrm{sr}(\mathfrak{A})$. When u is trivial and $\mathfrak{A} = \mathfrak{B} \rtimes_\beta \mathbb{Z}$ and α is the dual action of β, we have $\mathfrak{A} \rtimes_\alpha \mathbb{T} \cong \mathfrak{B} \otimes \mathbb{K}$ by Takai duality (*cf.* Takai [Tk]; Blackadar [Bl]; Pedersen [Pd]), where $\mathbb{K}$ is the C^*-algebra of all compact operators. By Rieffel [Rf1, Theorems 3.6 and Theorem 6.4], we obtain $\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{T}) = \min\{\mathrm{sr}(\mathfrak{B}), 2\}$. In particular, if $\mathfrak{A}$ is a simple crossed product of the form $C(X) \rtimes_\alpha \mathbb{Z}$ for X a compact space with $\dim X \ge 2$ so that $\mathrm{sr}(C(X)) \ge 2$ by Rieffel [Rf1, Proposition 1.7], and $\mathrm{sr}(\mathfrak{A}) = 1$ (for example, the noncommutative tori of the form $C(\mathbb{T}^n) \rtimes_\Theta \mathbb{Z}$ with Θ the multi-rotation action on $\mathbb{T}^n$ by $(e^{2\pi i \theta_j})_{j=1}^n$ for θ_j irrational), then

$$\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{T}) = 2 = \mathrm{sr}(\mathfrak{A}) + 1.$$

By using the same argument of Rieffel [Rf1, Corollary 8.6] we have

Corollary 2. *Under the same situation as Theorem 1, we obtain*

$$\mathrm{csr}(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^l) \leq \mathrm{sr}(\mathfrak{A}) + 1.$$

Proof. The proof of Theorem 1 implies that the set of all elements of the form Ee_1 for E an elementary matrix over $\mathfrak{D}$ and $e_1 = (1, 0, \dots, 0)$ is dense in $\mathfrak{B}^n$. Thus, the set of all elements of the form Le_1 for $L \in GL_n(\mathfrak{B})_0$ the connected component of $GL_n(\mathfrak{B})$ with the identity matrix is dense in $\mathfrak{B}^n$. Hence $L_n(\mathfrak{B})$ is connected. □

Remark. When both of (α, u) are trivial, we have $\mathrm{csr}(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^n) = \mathrm{csr}(\mathfrak{A})$. Since $\mathrm{gsr}(\cdot) \leq \mathrm{csr}(\cdot)$ by definition, we have $\mathrm{gsr}(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{T}^n) \leq \mathrm{sr}(\mathfrak{A}) + 1$.

By the same way as the proof of Theorem 1 and Corollary 2, we obtain

Theorem 3. *Let $\mathfrak{A}$ be a unital C^*-algebra. Then*

$$\begin{cases} \mathrm{sr}(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{R}) \leq \mathrm{sr}(\mathfrak{A}) + 1, \\ \mathrm{csr}(\mathfrak{A} \rtimes_{\alpha,u} \mathbb{R}) \leq \mathrm{sr}(\mathfrak{A}) + 1. \end{cases}$$

Proof. Note that $\mathfrak{A} \rtimes_{\alpha,u} \mathbb{R}$ is generated by elements of the form af for $a \in \mathfrak{A}$ and $f \in L^1(\mathbb{R})$. Also, the cohomology group $H^2(\mathbb{R}, \mathbb{T})$ is trivial, and $C^*(\mathbb{R})^+ \cong C_0(\mathbb{R})^+ \cong C(\mathbb{T})$ by the Fourier transform. □

Remark. When both of (α, u) is trivial, $\mathfrak{A} \rtimes_{\alpha,u} \mathbb{R} \cong \mathfrak{A} \otimes C_0(\mathbb{R})$. By Rieffel [Rf1, Corollary 7.2], we know that $\mathrm{sr}(\mathfrak{A} \otimes C_0(\mathbb{R})) \leq \mathrm{sr}(\mathfrak{A}) + 1$ since $\mathfrak{A} \otimes C_0(\mathbb{R})$ is regarded as a closed ideal of $\mathfrak{A} \otimes C(\mathbb{T})$. By the same way as Remark of Theorem 1, one can obtain

$$\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{R}) = \mathrm{sr}(\mathfrak{A}) + 1$$

for certain C^*-algebras $\mathfrak{A} = \mathfrak{B} \rtimes_\beta \mathbb{R}$ with α the dual action of β. In particular, we may take the foliation C^*-algebras of the form $C(\mathbb{T}^{n+1}) \rtimes_\beta \mathbb{R}$ with β free and its transversal $\mathbb{T}^n$ so that

$$C(\mathbb{T}^{n+1}) \rtimes_\beta \mathbb{R} \cong (C(\mathbb{T}^n) \rtimes_{\beta'} \mathbb{Z}) \otimes \mathbb{K}$$

where β' is the restriction to the transversal $\mathbb{T}^n$. It is known by Raeburn & Rosenberg [RR, Theorem 4.1] that the cohomology group $H^2(\mathbb{R}, C(X, \mathbb{T}))$ for X a compact metrizable space is trivial, where $C(X, \mathbb{T})$ is the space of all continuous $\mathbb{T}$-valued functions on X, which is also the unitary group of $M(C_0(X))$ (*cf.* Packer [Pk, Definitions in §1]). However, more general results about the cohomology vanishing of $H^2(\mathbb{R}, \mathcal{U}M(\mathfrak{A}))$ for $\mathfrak{A}$ a C^*-algebra would be unknown.

By the similar way as the proof of Theorem 1, we obtain in the nonunital case.

Theorem 4. *Let $\mathfrak{A}$ be a nonunital C^*-algebra, G either $\mathbb{T}^n$ or $\mathbb{R}$, and $\mathfrak{A} \rtimes_{\alpha,u} G$ the twisted crossed product of $\mathfrak{A}$ by G with (α, u) a twisted action. Then*

$$\mathrm{sr}(\mathfrak{A} \rtimes_{\alpha,u} G) \leq \mathrm{sr}(C^*(\mathfrak{A}, u(G,G))) + 1,$$

where $C^*(\mathfrak{A}, u(G,G))$ means the C^*-algebra generated by $\mathfrak{A}$ and all $u(s,t)$ for $s, t \in G$. Moreover, we obtain

$$\begin{aligned} \max\left\{\mathrm{sr}(\mathfrak{A}), \mathrm{sr}\left(C^*(u(G,G))\right)\right\} &\leq \mathrm{sr}\left(C^*(\mathfrak{A}, u(G,G))\right) \\ &\leq \max\left\{\mathrm{sr}(\mathfrak{A}), \mathrm{sr}\left(C^*(u(G,G))\right), \mathrm{csr}\left(C^*(u(G,G))\right)\right\}, \end{aligned}$$

where $C^*(u(G,G))$ means the C^*-algebra generated by $u(G,G)$.

Proof. The line of the proof is the same as Theorem 1. Note that $\mathfrak{A} \rtimes_{\alpha,u} G$ is identified with the C^*-algebra generated by the set

$$\{(\pi \times U)(af), (\pi \times U)(u(s,t)f) \mid a \in \mathfrak{A}, s, t \in G, f \in L^1(G)\},$$

where $\pi(u(s,t))$ means the image of $u(s,t) \in \mathcal{U}M(\mathfrak{A})$ under the canonical extension of π on $\mathfrak{A}$ to $M(\mathfrak{A})$. Moreover, the set of linear spans:

$$\mathrm{span}\{(\pi \times U)(af), (\pi \times U)(\prod_{j=1}^{k} u(s_j, t_j)f) \mid a \in \mathfrak{A}, s_j, t_j \in \mathbb{Z}, f \in L^1(G)\}$$

is dense in $\mathfrak{A} \rtimes_{\alpha,u} G$, where each $u(s_j, t_j)$ may be replaced with its adjoint $u(s_j, t_j)^*$. In fact, note that $au(s,t), u(s,t)a \in \mathfrak{A}$ for any $a \in \mathfrak{A}$ and $s, t \in G$ since $\mathfrak{A}$ is a closed ideal of $M(\mathfrak{A})$. Also, $C^*(\mathfrak{A}, u(G,G))$ may be identified with $C^*(\pi(\mathfrak{A}), \pi(u(G,G)))$ generated by $\pi(\mathfrak{A})$ and $\pi(u(G,G))$.

Moreover, observe that for $f \in L^1(G)$ and $l, m \in G$,

$$\begin{aligned}
\int_G f(t)U_t dt \pi(u(l,m)) &= \int_G f(t)U_t dt U_l U_m U_{l+m}^* \\
&= \int_G f(t)U_t U_l U_m U_{l+m}^* dt \\
&= \int_G f(t)\pi(u(t,l))U_{t+l}U_m U_{l+m}^* dt \\
&= \int_G f(t)\pi(u(t,l))\pi(u(t+l,m)) \\
&= \int_G f(t)\pi(u(t,l))\pi(u(t+l,m))\pi(u(t,l+m))U_t U_{l+m} U_{l+m}^* dt \\
&= \int_G f(t)\pi(u(t,l))\pi(u(t+l,m))\pi(u(t,l+m))U_t dt.
\end{aligned}$$

Therefore, this integral is approximated by finite sums of elements of the form

$$(\pi \times U)(\prod_{j=1}^{k} u(s_j, t_j)g) \quad \text{for } g \in L^1(G) \text{ and } s_j, t_j \in G.$$

Similarly, note that for $a \in \mathfrak{A}$,

$$\begin{aligned}
\pi(u(l,m)) \int_G f(t)U_t dt \pi(a) &= \pi(u(l,m)) \int_G f(t)U_t \pi(a) U_t^* U_t dt \\
&= \int_G f(t)\pi(u(l,m))\pi(\alpha_t(a))U_t dt,
\end{aligned}$$

which is approximated by finite sums of elements of the form $(\pi \times U)(bg)$ for $b \in \mathfrak{A}$ and $g \in L^1(G)$.

Next suppose that $\mathrm{sr}(C^*(\mathfrak{A}, u(G,G))) \le n-1$ as $\mathrm{sr}(\mathfrak{A}) \le n-1$ in the proof of Theorem 1 and follow the same argument as given there.

Finally, note that the following sequence is exact:

$$0 \to \mathfrak{A} \to C^*(\mathfrak{A}, u(G,G)) \to C^*(u(G,G)) \to 0,$$

since $\mathfrak{A}$ is a closed ideal of $C^*(\mathfrak{A}, u(G,G))$. Then use [Rf1, Theorems 4.3, 4.4 and 4.11] which imply that for any exact sequence: $0 \to \mathfrak{J} \to \mathfrak{B} \to \mathfrak{B}/\mathfrak{J} \to 0$ of C^*-algebras,

$$\max\{\mathrm{sr}(\mathfrak{J}), \mathrm{sr}(\mathfrak{B}/\mathfrak{J})\} \le \mathrm{sr}(\mathfrak{B}) \le \max\{\mathrm{sr}(\mathfrak{J}), \mathrm{sr}(\mathfrak{B}/\mathfrak{J}), \mathrm{csr}(\mathfrak{B}/\mathfrak{J})\}. \qquad \square$$

Remark. When $u(n,m) \in \mathbb{T}$ for any $n, m \in G$, we have $C^*(u(G,G)) \cong \mathbb{C}$. Thus we obtain

$$\mathrm{sr}(C^*(\mathfrak{A}, u(G,G))) = \mathrm{sr}(\mathfrak{A} + \mathbb{C}1) = \mathrm{sr}(\mathfrak{A}).$$

The framework for generating elements in the proofs of Theorems 1 and 4 will be useful in another situation somewhere else. The structure of the algebra $C^*(u(G,G))$ in a general situation might be interesting or rather complicated. Note that $\pi(u(s,t)) = U_sU_tU_{s+t}^{-1}$ so that $C^*(u(G,G))$ is a C^*-subalgebra of the algebra generated by U_s for $s \in G$.

By the same was as Corollary 2 we obtain

Corollary 5. *Under the same situation as Theorem* 4 *we have*

$$\mathrm{csr}(\mathfrak{A} \rtimes_{\alpha,u} G) \le \mathrm{sr}(C^*(\mathfrak{A}, u(G,G))) + 1.$$

For the convenience, we now check the following which would be known to specialists:

Proposition 6. Let $\mathfrak{A}$ be a C^*-algebra, and $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ the crossed product of $\mathfrak{A}$ by $\mathbb{Z}$. Then, for any twisted action (α, u),

$$\mathfrak{A} \rtimes_{\alpha,u} \mathbb{Z} \cong \mathfrak{A} \rtimes_\alpha \mathbb{Z}.$$

Moreover, we obtain $\mathfrak{A} \rtimes_{\alpha,u} \mathbb{Z}_n \cong \mathfrak{A} \rtimes_\alpha \mathbb{Z}_n$, where $\mathbb{Z}_n$ means the finite cyclic group of the order n.

Proof. Let $E = \mathcal{U}M(\mathfrak{A}) \rtimes \mathbb{Z}$ be a semi-direct product with the product defined by

$$(u,m)(v,n) = (u\alpha_m(v)u(m,n), m+n)$$

for $(u,m),(v,n) \in E$. That the product is associative follows from

$$\begin{aligned}(u,m)(v,n)(w,l) &= (u\alpha_m(v)u(m,n), m+n)(w,l)\\ &= (u\alpha_m(v)u(m,n)\alpha_{m+n}(w)u(m+n,l), m+n+l)\end{aligned}$$

for $(w,l) \in E$, and on the other hand,

$$\begin{aligned}(u,m)(v,n)(w,l) &= (u,m)(v\alpha_n(w)u(n,l), n+l)\\ &= (u\alpha_m(v\alpha_n(w)u(n,l))u(m,n+l), m+n+l)\\ &= (u\alpha_m(v)\alpha_{m+n}(w)\alpha_m(u(n,l))u(m,n+l), m+n+l).\end{aligned}$$

Moreover, we have

$$\begin{aligned}u(m,n)\alpha_{m+n}(w) &= u(m,n)\alpha_{m+n}(w)u(m,n)^*u(m,n)\\ &= \mathrm{Ad}\,u(m,n)\alpha_{m+n}(w)u(m,n)\\ &= \alpha_m \circ \alpha_n(w)u(m,n) = \alpha_{m+n}(w)u(m,n),\\ \alpha_m(u(n,l))u(m,n+l) &= u(m,n)u(m+n,l).\end{aligned}$$

Therefore, E is a group with $\mathbb{Z}$ a quotient. Since it is a general principle that group extensions with $\mathbb{Z}$ as quotients are always split, there is a splitting φ from $\mathbb{Z}$ to E. Set $\varphi(n) = (\psi(n), n) \in E$. Then

$$\begin{aligned}\varphi(m)\varphi(n) &= \varphi(m+n) = (\psi(m+n), m+n), \\ \varphi(m)\varphi(n) &= (\psi(m), m)(\psi(n), n) \\ &= (\psi(m)\alpha_m(\psi(n))u(m,n), m+n).\end{aligned}$$

Thus, it follows that $\psi(m)\alpha_m(\psi(n))u(m,n)\psi(m+n)^* = 1$. Now define $\beta_n(a) = \psi(n)\alpha_n(a)\psi(n)^*$. Then β is an action of $\mathbb{Z}$ on $\mathfrak{A}$, and it is exterior equivalent to (α, u). By Packer & Raeburn [PR1, Definition 3.1, Remarks 3.2 and Lemma 3.3], we deduce the conclusion. The case for $\mathbb{Z}_n$ follows from the same argument as above. Note also that a twisted action of $\mathbb{Z}_n$ is naturally extended to a twisted action of $\mathbb{Z}$. □

Remark. Note that crossed products $\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n$ can be regarded as quotients of $\mathfrak{A} \rtimes_{\alpha^\sim} \mathbb{Z}$ with $\alpha^\sim$ the canonically extended action of α (*cf.* Blackadar [Bl, Section 10.3]), so that $\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}_n) \le \mathrm{sr}(\mathfrak{A} \rtimes_{\alpha^\sim} \mathbb{Z})$ by Rieffel [Rf1, Theorem 4.3]. It might be possible to replace the tori in Theorem 1 with the direct products $\mathbb{T}^l \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ for some $l \ge 0$ and $n_1, \ldots, n_k \ge 2$ since they are compact Abelian. However, their twisted group C^*-algebras are noncommutative in general (*cf.* *Remark* of Corollary 8 and Raeburn & Rosenberg [RR, Example 4.19]).

Now we consider applications in the following.

Recall that a commutative Lie group Q generated by a compact neighborhood of the identity is isomorphic to the direct product $\mathbb{T}^l \times \mathbb{R}^s \times \mathbb{Z}^t \times F$ for F a finite Abelian group and $l, s, t \ge 0$ (*cf.* Pontrjagin [Pt]). We say that such a group Q is an elementary topological Abelian group. Note also that F is isomorphic to a direct product $\Pi_{j=1}^u \mathbb{Z}_{v_j}$ of finite cyclic groups $\mathbb{Z}_{v_j}$ of order $v_j \ge 2$. Set $\text{rd-grk}(Q) = 1 + s + t + u$ if $l \ge 1$, and $\text{rd-grk}(Q) = s + t + u$ if $l = 0$, where $\text{rd-grk}(\cdot)$ means (the sum of) the (reduced) dimension and (generalized) rank of Q. Note that if $Q = \mathbb{T}^l \times \mathbb{R}^s \times \mathbb{Z}^t$ $(l \ge 1)$, then

$$\text{rd-grk}(Q) = \dim(\hat{Q}) + 1,$$

where $\hat{Q} = \mathbb{Z}^l \times \mathbb{R}^s \times \mathbb{T}^t$ is the dual group of Q. We consider solvable Lie groups G which have (finite) normal series $\{H_j\}_{j=0}^n$ with $H_0 = \{1\}$ and $H_n = G$ such that H_{j-1} is normal and closed in H_j, and each subquotient H_j/H_{j-1} is an elementary topological Abelian group $(1 \le j \le n)$ (*cf.* Ragunathan [Rg] for G discrete). Define the (reduced) dimension and (generalized) rank of G to be

$$\text{rd-grk}(G) = \sum_{j=1}^{n} \text{rd-grk}(H_j/H_{j-1}).$$

Then we obtain the following theorem.

Theorem 7. *Let $\mathfrak{A}$ be a unital C^*-algebra and G a solvable Lie group with a normal series with subquotients elementary topological Abelian. Then*

$$\begin{cases} \text{sr}(\mathfrak{A} \rtimes_{\alpha,u} G) \le \text{sr}(\mathfrak{A}) + \text{rd-grk}(G), \\ \text{csr}(\mathfrak{A} \rtimes_{\alpha,u} G) \le \text{sr}(\mathfrak{A}) + \text{rd-grk}(G). \end{cases}$$

Proof. Let $\{H_j\}_{j=0}^n$ be a normal series of G with $H_0 = \{1\}$ and $H_n = \Gamma$ such that H_j/H_{j-1} is elementary topological Abelian and closed in $G/H_{j-1}(1 \le j \le n)$. By using the decomposition theorem of twisted crossed products Packer & Raeburn [PR1, Theorem 4.1] repeatedly,

$$\begin{aligned} \mathfrak{A} \rtimes_{\alpha,u} G &\cong (\mathfrak{A} \rtimes_{\alpha,u} H_1) \rtimes_{\alpha_2,u_2} (G/H_1) \\ &\cong ((\mathfrak{A} \rtimes_{\alpha,u} H_1) \rtimes_{\alpha_2,u_2} (H_2/H_1)) \rtimes_{\alpha_3,u_3} (G/H_2) \\ &\cong (\cdots(((\mathfrak{A} \rtimes_{\alpha,u} H_1) \rtimes_{\alpha_2,u_2} (H_2/H_1)) \rtimes_{\alpha_3,u_3} (H_3/H_2))\cdots) \rtimes_{\alpha_n,u_n} H_n/H_{n-1}. \end{aligned}$$

Put

$$\mathfrak{B}_j = \mathfrak{A}_j \rtimes_{\alpha_j,u_j} (H_j/H_{j-1}) \ (1 \le j \le n)$$

with $\alpha_1 = \alpha$ and $u_1 = u$, where $\mathfrak{A}_2 = \mathfrak{A} \rtimes_{\alpha,u} H_1$, $\mathfrak{A}_3 = (\mathfrak{A} \rtimes_{\alpha,u} H_1) \rtimes_{\alpha_2,u_2} (H_2/H_1)$ and $\mathfrak{A}_j$ is defined inductively the same way as in the decomposition. Since H_j/H_{j-1} is elementary topological Abelian, $H_j/H_{j-1} \cong \mathbb{T}^{l_j} \times \mathbb{R}^{s_j} \times \mathbb{Z}^{t_j} \times F_j$ for a finite Abelian group F_j and some $l_j, s_j, t_j \ge 0$. Let $F_j = \Pi_{k=1}^{u_j} \mathbb{Z}_{v_k}$ for some $v_k \ge 2$ and $u_j \ge 0$. By using the decomposition theorem again repeatedly,

$$\begin{aligned} \mathfrak{B}_j \rtimes_{\alpha_j,u_j} (H_j/H_{j-1}) &\cong \mathfrak{B}_j \rtimes_{\alpha_j,u_j} (\mathbb{T}^{l_j} \times \mathbb{R}^{s_j} \times \mathbb{Z}^{t_j} \times F_j) \\ &\cong (\mathfrak{B}_j \rtimes_{\alpha_j,u_j} \mathbb{T}^{l_j}) \rtimes (\mathbb{R}^{s_j} \times \mathbb{Z}^{t_j} \times F_j) \\ &\cong (\cdots(((\cdots((\mathfrak{B}_j \rtimes_{\alpha_j,u_j} \mathbb{T}^{l_j}) \rtimes \mathbb{R})\cdots) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}_{v_1})\cdots) \rtimes \mathbb{Z}_{v_{u_j}}. \end{aligned}$$

By using Theorems 1 and 3, Proposition 6 and the formula (F) repeatedly,

$$\begin{aligned}\mathrm{sr}(\mathfrak{A} \rtimes_{\alpha,u} G) &= \mathrm{sr}(\mathfrak{A}_n \rtimes_{\alpha_n,u_n} (H_n/H_{n-1})) \\ &\leq \mathrm{sr}(\mathfrak{A}_n) + \text{rd-grk}(H_n/H_{n-1}) \\ &\leq \mathrm{sr}(\mathfrak{A}_{n-1}) + \text{rd-grk}(H_{n-1}/H_{n-2}) + \text{rd-grk}(H_n/H_{n-1}) \\ &\leq \mathrm{sr}(\mathfrak{A}) + \sum_{j=1}^{n} \text{rd-grk}(H_j/H_{j-1})\end{aligned}$$

and $\text{rd-grk}(G) = \sum_{j=1}^{n} \text{rd-grk}(H_j/H_{j-1})$. We use the same argument and Corollary 2 for the connected stable rank estimate in the statement. In particular,

$$\begin{aligned}\mathrm{csr}(\mathfrak{A} \rtimes_{\alpha,u} G) &= \mathrm{csr}(\mathfrak{B}_n \rtimes_{\alpha_n,u_n} (\mathbb{T}^{l_n} \times \mathbb{R}^{s_n} \times \mathbb{Z}^{t_n} \times F_n)) \\ &\leq \mathrm{sr}(\mathfrak{B}_n \rtimes_{\alpha_n,u_n} (\mathbb{R}^{s_n} \times \mathbb{Z}^{t_n} \times F_n)) + 1. \qquad \square\end{aligned}$$

Remark. If G is nilpotent, each subquotient H_j/H_{j-1} is central in G/H_{j-1}. In particular, we take G as either the real $2n+1$-dimensional Heisenberg groups or the discrete Heisenberg groups of rank $2n+1$, which are central extensions of $\mathbb{R}^{2n}$ by $\mathbb{R}$ and those of $\mathbb{Z}^{2n}$ by the center $\mathbb{Z}$, and isomorphic to the semi-direct products $\mathbb{R}^{n+1} \rtimes_\alpha \mathbb{R}^n$ and $\mathbb{Z}^{n+1} \rtimes_\alpha \mathbb{Z}^n$ respectively, where $\alpha_n(l,m) = (l + \sum_{j=1}^{n} n_j m_j, m)$ for $l \in \mathbb{R}$ (or $\mathbb{Z}$), $n = (n_j), m = (m_j) \in \mathbb{R}^n$ (or $\mathbb{Z}^n$) (*cf.* Lee & Packer [LP]; Sudo [Sd5]).

Also, it is well known that a finite group is solvable if and only if it has a composition series (a refinement of normal series) with its subquotients cyclic groups of prime orders (*cf.* Ragunathan [Rg]).

Corollary 8. *Let G be as in Theorem 7 and $C^*(G,\sigma)$ the twisted group C^*-algebra of G with a cocycle σ and $[\sigma] \in H^2(G,\mathbb{T})$. Then*

$$\begin{cases}\mathrm{sr}(C^*(G,\sigma)) \leq \text{rd-grk}(G), \\ \mathrm{csr}(C^*(G,\sigma)) \leq \max\{2, \text{rd-grk}(G)\}, \\ \mathrm{gsr}(C^*(G,\sigma)) \leq \text{rd-grk}(G).\end{cases}$$

In particular, for $C^*(G)$ the group C^*-algebra of G,

$$\begin{cases}\mathrm{sr}(C^*(G)) \leq \text{rd-grk}(G), \\ \mathrm{csr}(C^*(G)) \leq \max\{2, \text{rd-grk}(G)\}, \\ \mathrm{gsr}(C^*(G)) \leq \text{rd-grk}(G).\end{cases}$$

Proof. Note that $C^*(G,\sigma) \cong \mathbb{C} \rtimes_{\mathrm{id},\sigma} G$ a twisted crossed product of $\mathbb{C}$ by G, where id means the trivial action of G on $\mathbb{C}$. Also note that $\mathbb{C} \rtimes_{\mathrm{id},\sigma} \mathbb{Z} \cong \mathbb{C} \rtimes \mathbb{Z}$ since the cohomology group $H^2(\mathbb{Z},\mathbb{T})$ is trivial (*cf.* Packer [Pk, Example 1.2]), and $\mathbb{C} \rtimes \mathbb{Z} \cong$

$C^*(\mathbb{Z}) \cong C(\mathbb{T})$ by the Fourier transform. Similarly, $\mathbb{C} \rtimes_{\mathrm{id},\sigma} \mathbb{Z}_n \cong \mathbb{C} \rtimes \mathbb{Z}_n \cong \mathbb{C}^n$. Moreover, we have $\mathrm{sr}(C(\mathbb{T})) = 1$, and $\mathrm{csr}(C(\mathbb{T})) = 2$ (*cf.* Rieffel [Rf1, Proposition 1.7]; Sheu [Sh, p. 381]) while $\mathrm{gsr}(C(\mathbb{T})) = 1$ since $C(\mathbb{T})$ is commutative and $\mathrm{gsr}(C(\mathbb{T})) \leq 2$ (*cf.* Rieffel [Rf1, remark of Proposition 10.2]). □

Remark. When $G = \mathbb{Z}^n$, the twisted group C^*-algebras $C^*(\mathbb{Z}^n, \sigma)$ are called noncommutative tori. In particular, $C^*(\mathbb{Z}^2, \sigma_\theta) \cong C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z}$ the rotation algebra, where $\sigma_\theta((x_1, x_2), (y_1, y_2)) = e^{2\pi i \theta x_2 y_1}$ for $(x_1, x_2), (y_1, y_2) \in \mathbb{Z}^2$, $\theta \in \mathbb{R}$ and $\alpha_\theta(z) = e^{2\pi i \theta} z$ for $z \in \mathbb{T}$ (*cf.* Packer [Pk, p. 192]), so that

$$C^*((\mathbb{Z}_n)^2, \sigma_{n^{-1}}) \cong \mathbb{C}^n \rtimes_{\alpha_{n^{-1}}} \mathbb{Z}_n \cong M_n(\mathbb{C})$$

the $n \times n$ matrix algebra over $\mathbb{C}$. It is known by Blackadar, Kumjian & Rodam [BKR] that any simple noncommutative torus has the stable rank one. On the other hand, the connected stable rank of simple noncommutative tori is 2 by this fact, Rieffel [Rf1, Corollary 4.10] and the fact that their K_1-groups are nontrivial (*cf.* Hassan [Eh, Corollary 1.6 & Theorem 2.2]). Also, $C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$. Thus,

$$\mathrm{sr}(C^*(\mathbb{Z}^n)) = \mathrm{sr}(C(\mathbb{T}^n)) = [n/2] + 1 \leq n = \mathrm{rank}(\mathbb{Z}^n).$$

Hence, it would be desirable to replace rd-grk(G) with [rd-grk$(G)/2] + 1$ in the estimates obtained above.

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