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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS WITHOUT CONTINUITY IN MENGER SPACES

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ABSTRACT. The aim of this paper is to prove some common fixed point theorems for the class of compatible maps to larger class of weakly compatible maps without appeal to continuity in Menger spaces and we also give a set of alternative conditions in place of completeness of the space. We improve and extend the results of Dedeic & Sarapa [A common fixed point theorem for three mappings on Menger spaces. *Math. Japon.* **34** (1989), no. 6, 919–923] and Rashwan & Hedar [On common fixed point theorems of compatible mappings in Menger spaces. *Demonstratio Math.* **31** (1998), no. 3, 537–546].

1. INTRODUCTION

Jungck [7] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. Banach fixed point theorem has many applications but suffers from one draw back, the definition requires continuity of the function. There then follows a flood of papers involving contractive definition that do not require the continuity of the function. This result was further generalized and extended in various ways by many authors.

Sessa [19] defined weak commutativity and proved common fixed point theorem for weakly commuting mappings. Further, Jungck [8] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. Since then various fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings in the compatible pair, have been obtained by many authors in different spaces.

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It has been known from the paper of Kannan [10] that there exists maps that have a discontinuity in the domain but which has a fixed point. Moreover the maps involved in every case were continuous at the fixed point.

In 1998, Jungck & Rhoades [9] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true.

Recently, Singh & Mishra [21] and Chugh & Kumar [3] proved some interesting results in metric spaces for weakly compatible maps without assuming any mapping continuous.

Menger [11] introduced the notion of probabilistic metric spaces, which is generalization of metric space, and the study of these spaces was expanded rapidly with the pioneering work of Schweizer & Sklar [17, 18]. The theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis.

The existence of fixed points for compatible mappings on probabilistic metric spaces is shown by Mishra [12]

Recently, fixed point theorems in Menger spaces have been proved by many authors including Bylka [1], Pathak, Kang & Baek [13], Stojakovic [22, 23, 24], Hadzic [5, 6], Dedeic & Sarapa [4], Rashwan & Hedar [16], Mishra [12], Radu [14, 15], Sehgal & Bhaucha-Reid [20] and Cho, Murthy & Stojakovic [2].

In this paper, we prove some common fixed point theorems for weakly compatible mappings in Menger spaces without using the condition of continuity. We also give a set of alternative conditions in place of completeness of the space. We improve results of Dedeic & Sarapa [4] and Rashwan & Hedar [16].

2. PRELIMINARIES

Let $\mathbb{R}$ denote the set of reals and $\mathbb{R}^+$ the non-negative reals. A mapping

$$F : \mathbb{R} \to \mathbb{R}^+$$

is called a *distribution function* if it is non-decreasing and left continuous with inf $F = 0$ and sup $F = 1$. We will denote by L the set of all distribution functions. A *probabilistic metric space* is a pair (X, F), where X is a non empty set and F is a mapping from $X \times X$ to L.

For $(u, v) \in X \times X$, the distribution function $F(u, v)$ is denoted by $F_{u,v}$. The functions $F_{u,v}$ are assumed to satisfy the following conditions:

(P_1) $F_{u,v}(x) = 1$ for every $x > 0$ if and only if $u = v$.
(P_2) $F_{u,v}(0) = 0$ for all $u, v \in X$.
(P_3) $F_{u,v}(x) = F_{v,u}(x)$ for every $u, v \in X$.
(P_4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$, then $F_{u,w}(x + y) = 1$ for all $u, v, w \in X$ and $x, y > 0$.

In a metric space (X, d), the metric d induces a mapping $F : X \times X \to L$ such that

$$F(u, v)(x) = F_{u,v}(x) = H(x - d(u, v)).$$

For every $u, v \in X$ and $x \in \mathbb{R}$, where H is a distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Definition 2.1. A function $t : [0, 1] \times [0, 1] \to [0, 1]$ is called T-*norm* if it satisfies the following conditions:

(t_1) $t(a, 1) = a$ for every $a \in [0, 1]$ and $t(0, 0) = 0$.
(t_2) $t(a, b) = t(b, a)$ for all $a, b \in [0, 1]$.
(t_3) If $c \geq a$ and $d \geq b$, then $t(c, d) \geq t(a, b)$.
(t_4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c \in [0, 1]$.

The concept of neighbourhood in PM-spaces was introduced by Schweizer & Sklar [17].

Definition 2.2. A Menger space is a triple (X, F, t), where (X, F) is a PM-space and t is a T-norm with the following condition:

(P5) $F_{u,v}(x + y) \geq t(F_{u,w}(x), F_{w,v}(y))$ for all $u, v, w \in X$ and $x, y \in \mathbb{R}^+$.

If $u \in X, \varepsilon > 0$ and $\lambda \in (0, 1)$, then an (ε, λ)-neighbourhood of u, denoted by $U_u(\varepsilon, \lambda)$, is defined by

$$U_u(\varepsilon, \lambda) = \{v \in X : F_{u,v}(\varepsilon) > 1 - \lambda\}$$

If (X, F, t) is a Menger space with the continuous T-norm t, then the family

$$\{U_u(\varepsilon, \lambda) : u \in X,\ \varepsilon > 0,\ \lambda \in (0, 1)\}$$

of neighbourhoods induces a Hausdorff topology on X and if $\sup_{a<1} t(a, a) = 1$, it is metrizable.

An important T-norm is the T-norm defined by

$$t(a, b) = \min\{a, b\}, \quad \text{for all} \quad a, b \in [0, 1]$$

and this is the unique T-norm such that

$$t(a,a) \geq a, \quad \text{for every } a \in [0,1].$$

Indeed if it satisfies this condition, we have

$$\min\{a,b\} \leq t(\min\{a,b\},\min\{a,b\}) \leq t(a,b) \leq t(\min\{a,b\},1) = \min\{a,b\}.$$

Therefore, $t(a,b) = \min\{a,b\}$.

In the sequel, we need the following definitions due to Radu [14].

Definition 2.3. Let (X,F,t) be a Menger space with continuous T-norm t. A sequence $\{x_n\}$ of points in X is said to be *convergent to a point* $x \in X$ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} F_{x_n,x}(\varepsilon) = 1.$$

Definition 2.4. Let (X,F,t) be a Menger space with continuous T-norm t. A sequence $\{x_n\}$ of points in X is said to be *Cauchy sequence* if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon,\lambda) > 0$ such that

$$F_{x_n,x_m}(\varepsilon) > 1-\lambda, \quad \text{for all } m,n \geq N.$$

Definition 2.5. A Menger space (X,F,t) with the continuous T-norm t is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Theorem 2.1 (Schweizer & Sklar [17]). *Let t be a T-norm defined by*

$$t(a,b) = \min\{a,b\}.$$

Then the induced Menger space (X,F,t) is complete if a metric space (X,d) is complete.

Definition 2.6 (Jungck & Rhoades [9]). Two maps A and B are said to be *weakly compatible* if they commute at a coincidence point.

Example 2.1. Define $A,S : [0,3] \to [0,3]$ by

$$A(x) = \begin{cases} x & \text{if } x \in [0,1), \\ 3 & \text{if } x \in [1,3]; \end{cases} \qquad S(x) = \begin{cases} 3-x & \text{if } x \in [0,1), \\ 3 & \text{if } x \in [1,3]. \end{cases}$$

Then for any $x \in [1,3]$, $ASx = SAx$, showing that A and S are weakly compatible maps on $[0,3]$.

Example 2.2. Let $X = \mathbb{R}$ and define $A, S : \mathbb{R} \to \mathbb{R}$ by $Ax = x/3$, $x \in \mathbb{R}$ and $Sx = x^2$, $x \in \mathbb{R}$. Hence 0 and 1/3 are two coincidence points for the maps A and S. Note that A and S commute at 0, *i. e.*, $AS(0) = SA(0) = 0$, but $AS(1/3) = A(1/9) = 1/27$ and $SA(1/3) = S(1/9) = 1/81$ and so A and S are not weakly compatible maps on $\mathbb{R}$.

Remark 2.1. Weakly compatible maps need not be compatible. Let $X = [2, 20]$ and d be the usual metric on X. Define mappings $A, S : X \to X$ by

$$Ax = \begin{cases} x & \text{if } x = 2 \text{ or } x > 5, \\ 6 & \text{if } 2 < x \leq 5; \end{cases} \qquad Sx = \begin{cases} x & \text{if } x = 2, \\ 12 & \text{if } 2 < x \leq 5, \\ x - 3 & \text{if } x > 5. \end{cases}$$

The mappings A and S are non-compatible consider the sequence $\{x_n\}$ defined by $x_n = 5 + (1/n), n \geq 1$. Then

$$\lim_{n\to\infty} Sx_n = 2, \lim_{n\to\infty} Ax_n = 2, \lim_{n\to\infty} SAx_n = 2 \text{ and } \lim_{n\to\infty} ASx_n = 6.$$

But they are weakly compatible since they commute at a coincidence point at $x = 2$.

3. Main results

Theorem 3.1. *Let A, B, S and T be self mappings on a Menger space (X, F, t) where t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$, satisfying the conditions:*

(3.1) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

(3.2) *There exists $k \in (0, 1)$ such that*

$$F_{Au,Bv}(kx) \geq t\Big(F_{Au,Su}(x), t\big(F_{Bv,Tv}(x), t(F_{Au,Tv}(\alpha x), F_{Bv,Su}(2x - \alpha x))\big)\Big)$$

for all $u, v \in X, x > 0$ and $\alpha \in (0, 2)$.

If

(3.3) *one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of X,*

then

(i) *A and S have a coincidence point, and*

(ii) *B and T have a coincidence point.*

Further if

(3.4) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible,*

then

(iii) *A, B, S and T have a unique fixed point in X.*

We need the following lemma proved by Mishra [12] for our first result.

Lemma 3.1. *Let A, B, S and T be self mappings of the Menger space (X, F, t), where t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$, satisfying the conditions (3.1) and (3.2). Then the sequence $\{y_n\}$ defined by condition (3.4) is a Cauchy sequence in X.*

Proof of **Theorem 3.1**. Since $A(X) \subset T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

(3.5) $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 1, 2, \ldots$

Let $\{y_n\}$ be the sequence in X defined above. By using Lemma 3.1, $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that $S(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $S(X)$ and has a limit z in $S(X)$. Let $p \in S^{-1}z$. Then $Sp = z$.

We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z. By (3.2), we have

$$F_{Ap,Bx_{2n+1}}(kx)$$
$$\geq t\Big(F_{Ap,Sp}(x), t(F_{Bx_{2n+1},Tx_{2n+1}}(x), t(F_{Ap,Tx_{2n+1}}(\alpha x), F_{Bx_{2n+1},Sp}(2x - \alpha x)))\Big)$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$\begin{aligned} F_{Ap,z}(kx) &\geq t\Big(F_{Ap,z}(x), t(F_{z,z}(x), t(F_{Ap,z}(x), F_{z,z}(x)))\Big) \\ &\geq F_{Ap,z}(x), \end{aligned}$$

which means that $Ap = z$. Hence $Ap = Sp = z$, *i. e.*, p is a coincidence point of A and S. This proves (i).

Since $A(X) \subset T(X)$, $Ap = z$ implies that $z \in T(X)$. Let $q \in T^{-1}z$. Then $Tq = z$.

It can easily verified by using similar arguments of the previous part of the proof that $Bq = z$. This proves (ii).

If we assume that $T(X)$ is complete, then argument analogous to the previous completeness argument establishes (i) and (ii).

The remaining two cases pertain essentially to the previous cases. Indeed, if $B(X)$ is complete, then by condition (3.1), $z \in B(X) \subset S(X)$. Similarly if $A(X)$ is complete the $z \in A(X) \subset T(X)$. Thus (i) and (ii) are completely established.

Now, we assume that condition (3.4) holds. Since the pair $\{A, S\}$ is weakly compatible therefore A and S commute at their coincidence point. *i. e.*, $ASp = SAp$ or $Az = Sz$. Similarly $BTq = TBq$ or $Bz = Tz$.

Now, we prove that $Az = z$. By (3.2), we have

$$F_{Ap,Bx_{2n+1}}(kx) \\ \geq t\big(F_{Az,Sz}(x), t\big(F_{Bx_{2n+1},Tx_{2n+1}}(x), t(F_{Az,Tx_{2n+1}}(\alpha x), F_{Bx_{2n+1},Sz}(2x - \alpha x))\big)\big)$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$F_{Az,z}(kx) \geq t\big(F_{Az,z}(x), t\big(F_{z,z}(x), t(F_{Az,z}(x), F_{z,Az}(x))\big)\big) \geq F_{Az,z}(x).$$

Therefore $Az = z$. Hence $Az = z = Sz$.

Similarly, we have $Bz = z = Tz$. This means that z is a common fixed point of mappings A, B, S and T.

For uniqueness of common fixed point let $w \neq z$ be another fixed point of mappings A, B, S and T.

Then by condition (3.2) and taking $\alpha \to 1$, we have

$$F_{z,w}(kx) \geq t\big(F_{z,z}(x), t\big(F_{w,w}(x), t(F_{z,w}(x), F_{w,z}(x))\big)\big) \geq F_{z,w}(x),$$

which means that $z = w$. This completes the proof. □

Remark 3.1. We note that Theorem 3.1 is still true if we replace the condition (3.2) by the following condition:

(3.6) there exists $k \in (0, 1)$ such that

$$F_{Au,Bv}(kx) \geq \min\big\{F_{Au,Su}(x), F_{Bv,Tv}(x), F_{Au,Tv}(\alpha x), F_{Bv,Su}(2x - \alpha x)\big\}$$

for all $u, v \in X$, $x > 0$ and $\alpha \in (0, 2)$.

Theorem 3.2. *Let A, B, S and T be self mappings on a Menger space (X, F, t), where t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$, satisfying the conditions* (3.1), (3.3), (3.4) *and*

(3.7) *there exists $k \in (0, 1)$ such that*

$$F_{Au,Bv}(kx) \geq \min\{F_{Au,Su}(x), F_{Bv,Tv}(x), F_{Su,Tv}(x)\},$$

for all $u, v \in X, x > 0$.

Then all the conclusions of Theorem 3.1 are true.

Proof. If the condition (3.7) is satisfied, then for any $\alpha \in (0,2)$, we have on the lines of Dedeic & Sarapa [4]

$$F_{Au,Bv}(kx) \geq \min\{F_{Au,Su}(x), F_{Bv,Tv}(x), F_{Su,Tv}(x)\},$$
$$\geq \min\{F_{Au,Su}(x), F_{Bv,Tv}(x), F_{Au,Tv}(x), F_{Bv,Su}(2x-\alpha x)\}.$$

Then using the Remark 3.1, the Theorem 3.2 is still true. □

The metric version of Theorem 3.1 is as follows:

Theorem 3.3. *Let A, B, S and T be self mappings on a metric space (X,d) satisfying the following conditions:*

(3.8) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(3.9) $d(Ax,By) \leq \max\left\{d(Ax,Sx), d(By,Ty), d(Sx,Ty), \frac{1}{2}\left[d(Ax,Ty)+d(Sy,By)\right]\right\}$ *for all* $x, y \in X$.

If

(3.10) *One of* $A(X)$, $B(X)$, $S(X)$ *or* $T(X)$ *is a complete subspace of* X,

then

(i) A *and* S *have a coincidence point, and*

(ii) B *and* T *have a coincidence point.*

Further if

(3.11) *the pairs* $\{A,S\}$ *and* $\{B,T\}$ *are weakly compatible,*

then

(iii) A, B, S *and* T *have a unique fixed point in* X.

Remark 3.2.

(i) Theorem 3.1 improves result of Rashwan & Hedar [16].

(ii) Theorem 3.2 improves and extends the main result of Dedeic & Sarapa [4].

Following Bylka [1], we consider the family G of functions $g : [0,\infty) \to [0,\infty)$ such that

(3.12) g is non-decreasing in $[0,\infty)$, and

(3.13) $\lim_{n\to\infty} g^n(x) = \infty$, for every $x > 0$, where g^n denotes the n-th iteration of g.

Theorem 3.4. *Let A, B, S and T be self mappings on a Menger space (X,F,t) where t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, satisfying the following conditions:*

(3.14) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$.*

(3.15) *There exists a function $g \in G$ such that*

$$F_{Au,Bv}(x) \geq F_{Su,Tv}(g(x)),$$

for all $u, v \in X$, $x > 0$ and $\alpha \in (0, 2)$.

(3.16) *One of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspace of X.*

Then

(i) *A and S have a coincidence point, and*

(ii) *B and T have a coincidence point.*

Further if

(3.17) *the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible,*

then

(iii) *A, B, S and T have a unique fixed point in X.*

In order to prove the theorem we need the following lemma due to Rashwan & Hedar [16].

Lemma 3.2. *Let $g \in G$, such that*

(i) *$g(x) \geq x$ for all $x \geq 0$, and*

(ii) *$F_{u,v}(x) \geq F_{u,v}\big(g(x)\big)$ for some $x > 0$,*

then $u = v$.

Proof of **Theorem 3.4**. Let $\{y_n\}$ be the sequence in X defined by (3.5). Then for all $x > 0$, $n = 1, 2, \ldots$, we have

$$\begin{aligned} F_{y_{2n+1},y_{2n+2}}(x) &= F_{Ax_{2n+2},Bx_{2n+1}}(x) \\ &\geq F_{Sx_{2n+2},Tx_{2n+1}}\big(g(x)\big) \\ &= F_{y_{2n},y_{2n+2}}\big(g(x)\big). \end{aligned}$$

Similarly, we have

$$\begin{aligned} F_{y_{2n},y_{2n+1}}(x) &= F_{Ax_{2n},Bx_{2n+1}}(x) \\ &\geq F_{Sx_{2n},Tx_{2n+1}}\big(g(x)\big) \\ &= F_{y_{2n-1},y_{2n}}\big(g(x)\big). \end{aligned}$$

Therefore

$$F_{y_n,y_{n+1}}(x) \geq F_{y_{n-1},y_n}\big(g(x)\big) \geq \cdots \geq F_{y_0,y_1}\big(g^n(x)\big). \tag{3.18}$$

Now, we show that the sequence $\{y_n\}$ is a Cauchy sequence in X. Let ε, λ be positive reals. Then for $m > n$ and $\ell = m - n$ and by using (3.18), we have

$$
\begin{aligned}
(3.19)\quad F_{y_n,y_m}(\varepsilon) &\geq t(F_{y_n,y_{n+1}}(\varepsilon/\ell),\ F_{y_{n+1},y_m}(\varepsilon(\ell-1)/\ell)) \\
&\geq t\big(F_{y_0,y_1}(g^n(\varepsilon/\ell),\ F_{y_{n+1},y_m}(\varepsilon(\ell-1)/\ell))\big) \\
&\geq t\big(F_{y_0,y_1}(g^n(\varepsilon/\ell),\ t(F_{y_{n+1},y_{n+2}}(\varepsilon/\ell)),\ F_{y_{n+2},y_m}(\varepsilon(\ell-2)/\ell))\big) \\
&\geq t\Big(F_{y_0,y_1}(g^n(\varepsilon/\ell),\ t(F_{y_0,y_1}(g^{n+1}(\varepsilon/\ell)),\ F_{y_{n+2},y_m}(\varepsilon(\ell-2)/\ell)))\Big) \\
&\geq t\Big(t(F_{y_0,y_1}(g^n(\varepsilon/\ell),\ (F_{y_0,y_1}(g^{n+1}(\varepsilon/\ell))),\ F_{y_{n+2},y_m}(\varepsilon(\ell-2)/\ell))\Big).
\end{aligned}
$$

Since $\lim_{n\to\infty} g^n(x) = \infty$, we have $g^n(\varepsilon/\ell) \leq g^{n+1}(\varepsilon/\ell)$ and by the hypothesis $t(a,a) \geq a$. Then from the last in equality of (3.19) we obtain

$$
(3.20)\qquad F_{y_n,y_m}(\varepsilon) \geq t\Big(F_{y_0,y_1}(g^n(\varepsilon/\ell)), F_{y_{n+2},y_m}(\varepsilon(\ell-2)/\ell)\Big).
$$

Using the induction argument we obtain from (3.20) that

$$
\begin{aligned}
F_{y_n,y_m}(\varepsilon) &\geq t\Big(F_{y_0,y_1}(g^n(\varepsilon/\ell), t(F_{y_{n+k-2},y_{n+k-1}}(\varepsilon/\ell), F_{y_{m-1},y_m}(\varepsilon/\ell)))\Big) \\
&\geq t\Big(F_{y_0,y_1}(g^n(\varepsilon/\ell), t(F_{y_0,y_1}(g^{n+k-2}(\varepsilon/\ell), F_{y_0,y_1}(g^{m-1}(\varepsilon/\ell)))))\Big) \\
&\geq F_{y_0,y_1}(g^n(\varepsilon/\ell)).
\end{aligned}
$$

Hence, we can choose $N \leq n$ such that

$$
F_{y_0,y_1}(g^n(\varepsilon/\ell)) > 1 - \lambda,
$$

and then $F_{y_n,y_m}(\varepsilon) > 1 - \lambda$ for all $m > n \geq N$.

This means that $\{y_n\}$ is a Cauchy sequence in X. Now suppose that $S(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $S(X)$ and has a limit z in $S(X)$. Let $p \in S^{-1}z$. Then $Sp = z$. We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z. By (3.15), we have

$$
F_{Ap,Bx_{2n+1}}(x) \geq F_{Sp,Tx_{2n+1}}(gx).
$$

Taking $n \to \infty$, we have

$$
F_{Ap,z}(x) \geq F_{z,z}(gx) = 1,
$$

which implies that $Ap = z$. Hence $Ap = Sp$, *i. e.*, p is a coincidence point of A and S. This proves (i). Since $A(X) \subset T(X)$, $Ap = z$ implies that $z \in T(X)$.

Let $q \in T^{-1}z$. Then $Tq = z$. It can easily verified by using similar arguments of the previous part of the proof that $Bq = z$.

If we assume that $T(X)$ is complete, then argument analogous to the previous completeness argument establishes (i) and (ii). The remaining two cases pertain essentially to the previous cases. Indeed, if $B(X)$ is complete, then by (3.14), $z \in B(X) \subset S(X)$.

Similarly if $A(X)$ is complete then $z \in A(X) \subset T(X)$. Thus (i) and (ii) are completely established.

Since the pair $\{A, S\}$ is weakly compatible therefore A and S commute at their coincidence point, *i. e.*, $ASp = SAp$ or $Az = Sz$. Similarly $BTq = TBq$ or $Bz = Tz$.

By (3.15), we have

$$F_{Az,y_{2n+1}}(x) = F_{Az,Bx_{2n+1}}(g(x)) \geq F_{Sz,Tx_{2n+1}}(g(x)).$$

Taking $n \to \infty$, we have

$$F_{Az,z}(x) \geq F_{Az,z}(g(x)).$$

By Lemma 3.2, we have $Az = z$. Therefore $Az = z = Sz$. Similarly, we have $Bz = z = Tz$. This means that z is a common fixed point of A, B, S and T. It follows easily from (3.15) that z is a unique common fixed point of A, B, S and T. This completes the proof. □

References

1. C. Bylka: Fixed point theorems of Matkowski on probabilistic metric spaces. *Demonstratio Math.* **29** (1996), no. 1, 158–164. CMP 1 398 738
2. Y. J. Cho, P. P. Murthy & M. Stojakovic: Compatible mappings of type (A) and common fixed points in Menger spaces. *Com. Korean Math. J.* **7** (1992), 325–339.
3. R. Chugh & S. Kumar: Common fixed points for weakly compatible maps. *Proc. Indian Acad. Sci.* **111** (2001), no. 2, 241–247. CMP 1 836 370
4. R. Dedeic & N. Sarapa: A common fixed point theorem for three mappings on Menger spaces. *Math. Japon.* **34** (1989), no. 6, 919–923. MR **90j**:54045
5. O. Hadzic: A fixed point theorem in Menger spaces. *Publ. Inst. Math. (Beograd)* **20** (1979), 107–112. MR **81j**:54071
6. ______: Some theorems on the fixed points in probabilistic metric and Random normed spaces. *Boll. Un. Mat. Ital. B(6)* **1** (1982), no. 1, 381–391. MR **84g**:54057
7. G. Jungck: Commuting mappings and fixed points. *Amer. Math. Monthly* **83** (1976), 261–263. MR **53#**4031
8. ______: Compatible mappings and common fixed points. *Internat. J. Math. Math. Sci.* **9** (1986), 771–779. MR **87m**:54122

9. G. Jungck & B. E. Rhoades: Fixed points for set valued functions without continuity. *Indian J. Pure Appl. Math.* **29** (1998), no. 3, 227–238. CMP 1 617 919
10. R. Kannan: Some results on fixed points. *Bull. Calcutta Math. Soc.* **60** (1968), 71–76. MR **41**#2486
11. K. Menger: Statistical metric. *Proc. Nat. Acad. Sci. U. S. A.* **28** (1942), 535–537. MR **4**,163e
12. S. N. Mishra: Common fixed points of compatible mappings in PM-spaces, *Math. Japon.* **36** (1991), no. 2, 283–289. CMP 095 742
13. H. K. Pathak, S. M. Kang & J. H. Baek: Weak compatible mappings of type (A) and common fixed points in Menger spaces. *Commun. Korean Math. Soc.* **10** (1995), no. 1, 67–83. CMP 1 430 174
14. V. Radu: On the contraction principle in Menger spaces. *An Univ. Timisoara Ser. Stiint. Mat.* **22** (1984), 83–88. MR **86e**:47062
15. ______: On some contraction-type mappings in Menger spaces. *An Univ. Timisoara Ser. Stiint. Mat.* **23** (1985), 61–65. MR **86k**:54068
16. R. A. Rashwan & A. Hedar: On common fixed point theorems of compatible mappings in Menger spaces. *Demonstratio Math.* **31** (1998), no. 3, 537-546. CMP 1 658 442
17. B. Schweizer & A. Sklar: Statistical metric spaces. *Pacific J. Math.* **10** (1960), 313–334. MR **22**#5955
18. ______: *Statistical metric spaces.* North-Holland Series in Probability and Applied Mathematics 5, North Holland, Amsterdam 1983.
19. S. Sessa: On a weak commutativity condition of mappings in fixed point considerations. *Publ. Inst. Math.* **32** (1982), no. 46, 149–153. MR **85f**:54107
20. V. M. Sehgal & A. T. Bharucha-Reid: Fixed points of contraction mappings on probabilistic metric spaces. *Math. Systems Theory* **6** (1972), 97–102. MR **46**#9956
21. S. L. Singh & S. N. Mishra: Remarks on Jachymski's fixed point theorems for compatible maps. *Indian J. Pure Appl. Math.* **28** (1997), no. 5, 611–615. MR **98c**:54036
22. M. Stojakovic: Fixed point theorems in probabilistic metric spaces. *Kobe J. Math.* **2** (1985), no. 1, 1–9. MR **87a**:54071
23. ______: Common fixed point theorems in complete metric and probabilistic metric spaces. *Bull. Austral. Math. Soc.* **36** (1987), 73–88. MR **88j**:54066
24. ______: A common fixed point theorems in probabilistic metric spaces and its applications. *Glasnik Mat.* **23** (1988) no. 1, 203–211. MR **90d**:54097

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