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A NOTE ON STRONG REDUCEDNESS IN NEAR-RINGS

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ABSTRACT. Let N be a right near-ring. N is said to be *strongly reduced* if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$, or equivalently, for $a \in N$ and any positive integer n, $a^n \in N_c$ implies $a \in N_c$, where N_c denotes the constant part of N.

We will show that strong reducedness is equivalent to condition (ii) of Reddy and Murty's property (*) (*cf.* [Reddy & Murty: On strongly regular near-rings. *Proc. Edinburgh Math. Soc. (2)* **27** (1984), no. 1, 61–64]), and that condition (i) of Reddy and Murty's property (*) follows from strong reducedness. Also, we will investigate some characterizations of strongly reduced near-rings and their properties. Using strong reducedness, we characterize left strongly regular near-rings and (P_0)-near-rings.

1. INTRODUCTION

Throughout this paper we will work with right near-rings. For notations and basic concepts, we shall refer to Pilz [7].

Let N be a right near-ring. N is said to be *left strongly regular* if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$. Right strong regularity is defined in a symmetric way. Mason [4] introduced these notions and characterized left strongly regular zero-symmetric unital near-rings. Several authors (*cf.* Hongan [2], Mason [5], Murty [6] and Reddy & Murty [8]) have studied them. In particular, Reddy & Murty [8] extended some results in Mason [4] to the non-zero symmetric case. They observed that every left strongly regular near-ring has an interesting property. In this paper, we consider the property (it is called *Reddy and Murty's property* (*)) in Reddy & Murty [8]:

(i) For any $a, b \in N$, $ab = 0$ implies $ba = b0$.
(ii) For $a \in N$, $a^3 = a^2$ implies $a^2 = a$.

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Let N_c denote the constant part of N, that is, $N_c = \{a \mid a = a0, a \in N\}$.

Now we define a new concept for near-rings, that is, a near-ring N is said to be *strongly reduced* if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$.

Recall that a near-ring N is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$. As we shall show later, a strongly reduced near-ring N is reduced. We will show that strong reducedness is equivalent to condition (ii) of Reddy and Murty's property (*) and condition (i) of Reddy and Murty's property (*) follows from strong reducedness. Consequently, we see that condition (i) of Reddy and Murty's property (*) is not needed.

Left or right strongly regular near-rings form one of the important classes of strongly reduced near-rings. We will investigate some properties of strongly reduced near-rings. Using strong reducedness, we characterize left strongly regular near-rings and (P_0)-near-rings.

2. RESULTS

A subnear-ring H of a near-ring N is said to be *left invariant* if $NH \subseteq H$, *right invariant* if $HN \subseteq H$ and *invariant* if it is both left and right invariant. For a subset S of N,

$$\langle S|, \quad |S\rangle \quad \text{and} \quad \langle S\rangle$$

(*resp.*) stand for the left invariant, right invariant and invariant (*resp.*) subnear-rings of N generated by S. For any element $a \in N$,

$$\langle a|, \quad |a\rangle \quad \text{and} \quad \langle a\rangle$$

(*resp.*) are called *the principal left invariant*, *principal right invariant* and *principal invariant* (*resp.*) subnear-rings of N generated by a.

There are slightly generalized new concepts of left strong regularity and right strong regularity. A near-ring N is said to be *quasi left strongly regular* if $a \in \langle a^2|$ for each $a \in N$, *quasi right strongly regular* if $a \in |a^2\rangle$ for each $a \in N$.

There are lots of quasi left (*resp.* right) strongly regular near-rings which are not left (*resp.* right) strongly regular.

First, we introduce the following lemma.

Lemma 1. *We have the following properties.*

(1) *The direct product of strongly reduced near-rings is strongly reduced.*

(2) *Every subnear-ring of a strongly reduced near-ring is strongly reduced.*

(3) *Every homomorphic image of a strongly reduced contant near-ring is strongly reduced.*

Proof. (3) A constant near-ring is strongly reduced, and the homomorphic image of a contant near-ring is constant. □

Now we give some sufficient conditions for quasi left strongly regular near-rings or quasi right strongly regular near-rings to be strongly reduced.

Proposition 1. *We have the following properties.*

(1) *All quasi left strongly regular near-rings and quasi right strongly regular near-rings are strongly reduced. In particular, right or left strongly regular near-rings are strongly reduced.*

(2) *Every integral near-ring N is strongly reduced. Hence a subdirect product of integral near-rings is strongly reduced.*

Proof. (1) Note that the constant part N_c is an invariant subnear-ring of N. Suppose N is a quasi left strongly regular near-ring. Then $a \in \langle a^2|$ for each $a \in N$. If $a^2 \in N_c$ then $a \in \langle a^2| \subseteq N_c$. Hence N is strongly reduced. Similarly, all quasi right strongly regular near-rings are strongly reduced.

(2) Let $a \in N$ with $a^2 \in N_c$. Then $(a - a^2)a = 0$, and hence $a = a^2 \in N_c$. □

Proposition 2. *If N is a unital quasi left strongly regular near-ring, then every completely prime ideal is maximal.*

Proof. Let P be a completely prime ideal which is not maximal, so suppose that $P \subsetneq M$ for some maximal M. Let $a \in M \setminus P$. Since N is quasi left strongly regular, we see that $a = a^2$ or $a = xa^2$ for some $x \in N$. Then $0 = (1-a)a$ or $0 = (1-xa)a$. Since P is completely prime, $1 - a \in P \subseteq M$ or $1 - xa \in P \subseteq M$. In any case, $1 \in M$, this is a contradiction. □

From now on, we consider on strongly reduced near-rings and left strongly regular near-rings. Now, we state some basic and useful properties of a strongly reduced near-ring.

Proposition 3. *Let N be a strongly reduced near-ring and let $a, b \in N$. Then we have the following properties.*

(1) *N is reduced.*

(2) *If* $ab^n \in N_c$ *for any positive integer* n, *then* $\{ab, ba\} \cup aNb \cup bNa \subseteq N_c$. *In particular,* $ab \in N_c$ *implies* $ba \in N_c$, $aNb \subseteq N_c$ *and* $bNa \subseteq N_c$.

(3) *If* $ab^n = 0$ *for any positive integer* n, *then* $ab = 0$ *and* $ba = b0$. *In particular,* $ab = 0$ *implies* $ba = b0$, *that is,* N *has condition (i) of Reddy and Murty's property (*).*

Proof. (1) Assume that $a^2 = 0$. Then $a^2 \in N_c$, hence $a \in N_c$. Then we see $a = a0 = a0a = aa = 0$.

(2) First, suppose $ab \in N_c$. Then $(ba)^2 = baba = bab0a = bab0 \in N_c$. Since N is strongly reduced, we have $ba \in N_c$. Then we obtain $xba \in N_c$ for each $x \in N$, whence $(axb)^2 \in N_c$. By the strong reducibility of N, we obtain $axb \in N_c$ for each $x \in N$. Since $ba \in N_c$, we also obtain $bNa \subseteq N_c$. Now suppose $ab^n \in N_c$. Then $(ab)^n \in N_c$ by the above argument. Since N is strongly reduced, this implies $ab \in N_c$. Hence by the first paragraph, the claim is proved.

(3) If $ab^n = 0$ for some $n \geq 1$, then $ab \in N_c$ by (2). Hence $ab = abb^{n-1} = ab^n = 0$. Then $(ba)^2 = baba = b0 \in N_c$. Hence $ba \in N_c$. Therefore $(ba)^2 - ba \in N_c$. Then $(ba)^2 - ba = \{(ba)^2 - ba\}b = babab - bab = b0 - b0 = 0$. Hence we obtain $ba = (ba)^2 = b0$. □

Clearly, if N is a zero-symmetric near-ring, then N is strongly reduced if and only if N is reduced. The following example shows that, in general, a reduced near-ring is not necessarily strongly reduced.

Example 1. Let $N = \{0, 1, 2, 3, 4, 5\}$ be an additive group of integers modulo 6 and multiplication as follows (see Pilz [7] for near-rings of low order; $\mathbb{Z}_6$ No. 32):

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	4	4	4	1	4	1
2	2	2	2	2	2	2
3	0	0	0	3	0	3
4	4	4	4	4	4	4
5	2	2	2	5	2	5

Clearly, this near-ring N is reduced. The constant part of N is $\{0, 2, 4\}$. We see that this near-ring N is not strongly reduced, because $1^2 = 4$ is a constant element but 1 is not a constant element. On the other hand, this near-ring N is an example of π-regular but not a regular near-ring.

Example 2. Let $V = \{0, a, b, c\}$ be Klein's four group under addition.

(1) We define multiplication as follows (see Pilz [7] near-rings of low order; V No. 20):

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

The constant part of this near-ring is $\{0, a\}$. Clearly, this near-ring is reduced and strongly reduced.

(2) We have multiplication table as follows (see Pilz [7] near-rings of low order; V No. 19):

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	a	a	a	c

The constant part of this near-ring is $\{0, a\}$. Obviously, this near-ring is not reduced, for $b^2 = 0$; and it is also not strongly reduced.

Now we consider polynomial near-rings over commutative unital rings and polynomial near-rings on groups (*cf.* Lausch & Nöbauer [3, §8.11 and §9.11], Pilz [7, §7.61]). Let R be a commutative ring with unity 1, G an additive group, x an indeterminate variable, $R[x]$ the set of all polynomials over R and

$$G[x] = \{a_0 + n_1x + a_1 + n_2x + a_2 + \cdots + a_{t-1} + n_tx + a_t \mid t \in \mathbb{N}_0, a_i \in G,\ n_i \in \mathbb{Z}^* \text{ and } a_1 \neq 0, a_2 \neq 0, \ldots, a_{t-1} \neq 0\}.$$

Then $(R[x], +, \circ)$ and $(G[x], +, \circ)$ are near-rings with unity x respectively, where $\circ$ is substitution. In this case, we say that $R[x]$ is a polynomial near-ring over R and $G[x]$ is a polynomial near-ring on G. We see that

$$(R[x])_c = R \text{ and } (R[x])_0 = \Big\{\sum_{i=1}^{n} a_ix^i \mid i \in \mathbb{Z}^+\Big\},$$

so that $R[x] = (R[x])_c + (R[x])_0$.

Next, for any $f(x) \in R[x]$, the map $f : R \longrightarrow R$ given by $a \rightsquigarrow f(x) \circ a = f(a)$ is called the *polynomial function induced by f(x)*. We let $P(R) = \{f \mid f(x) \in R[x]\}$ be the set of all polynomial functions on R. Similarly, one can define f for $f(x) \in G[x]$ and let $P(G)$ be the set of all polynomial functions on G. It is well known that

$P(R)$ and $P(G)$ are subnear-rings of $M(R)$ (*resp.* $M(G)$), and they are called the *near-rings of polynomial functions* on R (*resp.* on G) (*cf.* Pilz [7, §7.65 and §7.66]).

Example 3. Consider the group $(\mathbb{Z}_2, +)$ and the commutative ring $(\mathbb{Z}_2, +, \cdot)$. The two kinds of near-rings (see Pilz [7] for near-rings of low order; $\mathbb{Z}_2$ No. 2 and $\mathbb{Z}_2$ No. 3) on a group $(\mathbb{Z}_2, +)$ are strongly reduced, and $\mathbb{Z}_2[x]$ and $P(\mathbb{Z}_2) = \{0, 1, x, x+1\}$ are strongly reduced.

Example 4. The four kinds of near-rings (see Pilz [7] for near-rings of low order; $\mathbb{Z}_4$ No. 8, $\mathbb{Z}_4$ No. 9, $\mathbb{Z}_4$ No. 10 and $\mathbb{Z}_4$ No. 11) on a group $(\mathbb{Z}_4, +)$ are strongly reduced. However, $\mathbb{Z}_4[x]$ and $P(\mathbb{Z}_4) = \{0, 1, x, 2x, \cdots\}$ are not strongly reduced.

We give equivalent conditions for a near-ring N to be strongly reduced.

Theorem 1. *The following statements are equivalent for a near-ring N:*

(1) *N is strongly reduced.*
(2) *For $a \in N$, $a^3 = a^2$ implies $a^2 = a$, that is, N has condition (ii) of Reddy and Murty's property (*).*
(3) *If $a^{n+1} = xa^{n+1}$ for $a, x \in N$ and some nonnegative integer n, then $a = xa = ax$.*

Proof. (1) $\Rightarrow$ (3). Suppose $a^{n+1} = xa^{n+1}$ for some $n \geq 0$. We will show $a = xa = ax$. If $n = 0$, then immediately $a = xa$. Now $(a - ax)a = a^2 - axa = a^2 - a^2 = 0 \in N_c$. Hence $(a - ax)^2 = a(a - ax) - ax(a - ax) \in N_c$ by property (2) of Proposition 3, and so $a - ax \in N_c$. Therefore $a - ax = (a - ax)a = 0$. If $n \geq 1$, then $(a - xa)a^n = 0$. Hence $(a - xa)a = 0$ by property (3) of Proposition 3, and so $(a - xa)^2 \in N_c$ by property (2) of Proposition 3. Since N is strongly reduced, we have $a - xa \in N_c$. Then $a - xa = (a - xa)a = 0$, that is $a = xa$. Obviously as above $a = ax$.

(3) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (1). Assume $a^2 \in N_c$. Then $a^3 = a^2a = a^2$. By condition (2), this implies $a = a^2 \in N_c$. □

Left strongly regular near-rings has been studied by several authors (*cf.* Lausch & Nöbauer [3], Mason [4, 5], Murty [6], Reddy & Murty [8], etc.) Since all left strongly regular near-rings are strongly reduced, the following is a generalization of Reddy & Murty [8, Theorem 3].

Lemma 2. *Let N be a strongly reduced near-ring and let $a, x \in N$. If $a^n = xa^{n+1}$ for some positive integer n, then $a = xa^2 = axa$ and $ax = xa$.*

Proof. Assume that $a^n = xa^{n+1}$ for some $n \geq 1$. By condition (3) of Theorem 1, $a = xa^2 = axa$. Then $(ax - xa)a = 0$. Hence, by property (2) of Proposition 3, $(ax - xa)^2 = ax(ax - xa) - xa(ax - xa) \in N_c$. Since N is strongly reduced, $ax - xa \in N_c$. Hence $ax - xa = (ax - xa)a = 0$. □

A near-ring N is said to be *left strongly π-regular* if, for each $a \in N$, there exists a positive integer n and an element $x \in N$ such that $a^n = xa^{n+1}$. This equation is equivalent to $a^n = ya^{2n}$, for some $y \in N$. Here we give some characterizations of left strongly regular near-rings.

Theorem 2. *Let N be a near-ring. Then the following statements are equivalent:*

(1) *N is left strongly regular.*

(2) *N is strongly reduced and left strongly π-regular.*

(3) *For each $a \in N$, there exists $x, y \in N$ such that $a = xa^2ya$.*

(4) *For each $a \in N$, $a \in \langle a^2 \rangle \cap aNa$.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) follow easily from property (1) of Proposition 1 and Lemma 2.

(3) $\Rightarrow$ (1). The hypothesis implies N is strongly reduced. If $a = xa^2ya$, then $ya = yxa^2(ya)$. By Theorem 1, $ya = yayxa^2$. Thus $a = xa^2yayxa^2$. This implies that N is left strongly regular.

(4) $\Rightarrow$ (3). Since $a \in \langle a^2 \rangle$ for each $a \in N$, N is strongly reduced by an argument similar to that in the proof for property (1) of Proposition 1. Hence N satisfies (3) in Theorem 1. Since $a \in aNa$, there exists $x \in N$ such that $a = axa$. Hence $a = (ax)a = a(ax) = a^2x$. Then we have $a = axa = (a^2x)xa = a^2x^2a = a^2x^2a^2x^2a$. (3) holds. □

A near-ring is said to be *periodic* if, for each $a \in N$, there exist distinct positive integers m, n such that $a^m = a^n$. A near-ring N is called a (P_0)-near-ring if, for each $a \in N$, there exists an integer $n\rangle 1$ such that $a = a^n$ (see [7, §9.4, p. 289]). Obviously a (P_0)-near-ring is strongly reduced. Hence the proof of the following corollary follows directly from Lemma 2.

Corollary 1. *Let N be a near-ring. Then the following statements are equivalent:*

(1) *N is periodic and strongly reduced.*

(2) *N is a (P_0)-near-ring.*

As a special case of this corollary, we have

Corollary 2. *Let N be a finite near-ring. Then the following statements are equivalent:*

(1) *N is strongly reduced.*

(2) *N is left strongly regular.*

(3) *N is a (P_0)-near-ring.*

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