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GENERALIZED VECTOR VARIATIONAL-TYPE INEQUALITIES FOR SET-VALUED MAPPINGS

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ABSTRACT. In this paper, we consider the existence of the solutions to the generalized vector variational-type inequalities for set-valued mappings on Hausdorff topological vector spaces using Fan's geometrical lemma.

1. INTRODUCTION AND PRELIMINARIES

A vector variational inequality in a finite-dimensional Euclidean space was first introduced by Giannessi [5], which is the vector-valued version of the variational inequality of Hartman & Stampacchia [6]. Over the past two decades, various vector variational inequalities and their applications have been intensively studied by Chen [3], Konnov & Yao [7], B. S. Lee & G. M. Lee [9], B. S. Lee & G. M. Lee & Kim [10, 11], B. S. Lee & S. J. Lee [12, 13], G. M. Lee, Kim & B. S. Lee [14, 15], G. M. Lee, Kim, B. S. Lee & Cho [16], G. M. Lee, Kim, B. S. Lee & Yen [17], G. M. Lee, B. S. Lee, Kim & Chen [18], Siddiqi, Ahmad & Khan [19], Siddiqi, Ansari & Ahmad [20], Siddiqi, Ansari & Khaliq [22], Yu & Yao [23] and others.

Ansari [1] introduced and considered vector variational-like inequalities. Since then, B. S. Lee & G. M. Lee [9], B. S. Lee, G. M. Lee & Kim [11] and Siddiqi, Ansari & Ahmad [20] have been studied various vector variational-like inequalities.

B. S. Lee & S. J. Lee [12, 13] introduced and considered vector variational-type inequalities, which was generalized form vector variational-like inequality.

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Recently, Siddiqi, Ansari & Khan [21] considered scalar generalized variational-type inequalities for set-valued mappings with monotonicity assumption on Banach spaces.

Our motivation for this paper is to consider generalized vector variational-type inequalities for set-valued mappings without the monotonicity assumption on Hausdorff topological vector spaces. In the proof of our main theorem, we use Fan's geometrical lemma Fan [4], which has been applied to variational problems, complementarity problems, game theory, and so on.

Let X, Y be topological vector spaces, K a nonempty subset of X and N a nonempty subset of $L(X,Y)$, where $L(X,Y)$ is the space of all linear continuous operators from X to Y. Let $M : K \times N \to L(X,Y), \theta : K \times K \to X$ and $\eta : K \times K \to Y$ be mappings, and $\{C(x) : x \in K\}$ a family of closed convex cones in Y. A partial order $\leq_{C(x)}$ in Y with the closed convex cone $C(x)$ is defined as for $y_1, y_2 \in Y$,

$$y_1 \leq_{C(x)} y_2 \text{ if and only if } y_2 - y_1 \in C(x).$$

Definition 1.1 (Kuroiwa [8]). Let K be a convex subset of X. A mapping $f : K \to Y$ is convex if for every $x_1, x_2 \in K$ and $t \in (0,1)$,

$$f(tx_1 + (1-t)x_2) \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

i. e., $tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x)$.

We consider the following Generalized Vector Variational-Type Inequality Problem (GVVTIP):

(GVVTIP). Find $x_0 \in K$ such that for all $y \in K$ there exists $u_0 \in T(x_0)$ satisfying

$$\big\langle M(x_0, u_0), \theta(y, x_0)\big\rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin -\operatorname{int} C(x_0),$$

where $\big\langle M(x_0, u_0), \theta(y, x_0)\big\rangle$ denotes the evaluation of $M(x_0, u_0)$ at $\theta(y, x_0)$.

Now, we introduce the following famous Fan's geometrical lemma.

Lemma 1.1 (Fan [4]). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let A be a subset of $K \times K$ satisfying the following conditions;*

(1) *for each $x \in K$, $(x, x) \in A$,*

(2) *for each fixed $y \in K$, the set $A_y := \{x \in K : (x,y) \in A\}$ is closed in K,*

(3) *for each fixed $x \in K$, the set $A_x := \{y \in K : (x,y) \notin A\}$ is convex in K.*

Then there exists an $x_0 \in K$ *such that* $\{x_0\} \times K \subset A$.

Definition 1.2 (B. S. Lee, G. M. Lee & Kim [11]). Let X, Y be topological vector spaces and $T : X \to 2^Y$ a set-valued mapping.

(1) T is said to be upper semicontinuous (briefly, u.s.c.) at $x_0 \in X$ if for any open neighborhood N containing $T(x_0)$ there exists a neighborhood M of x_0 such that $T(M) \subset N$. T is said to be u.s.c. if T is u.s.c. at every point $x \in X$.

(2) T is said to be closed at $x \in X$ if for each nets $\{x_\alpha\}$ converging to x and $\{y_\alpha\}$ converging to y such that $y_\alpha \in T(x_\alpha)$ for all α, we have $y \in T(x)$. T is said to be closed if it is closed at every point $x \in X$.

Lemma 1.2 (Aubin & Cellina [2]). *Let* X, Y *be topological vector spaces and* $T : X \to 2^Y$ *be a set-valued mapping.*

(1) *If* K *is a compact subset of* X, *and* T *is u.s.c. and compact-valued, then* $T(K)$ *is compact.*

(2) *If* T *is u.s.c. and compact-valued, then* T *is closed.*

2. Main results

Now we consider the existence theorem of solution to **(GVVTIP)**.

Theorem 2.1. *Let* X *be a Hausdorff topological vector space,* Y *a topological vector space. Let* K *be a nonempty compact convex subset of* X, N *a nonempty subset of* $L(X,Y)$ *and* $\{C(x) : x \in K\}$ *a family of closed convex cones in* Y. *Let a set-valued mapping* $W : K \to 2^Y$ *defined by* $W(x) = Y \setminus \{-\operatorname{int} C(x)\}$ *has a closed graph. Assume that* $M : K \times N \to L(X,Y)$ *is a continuous mapping,* $\theta : K \times K \to X$ *is a mapping such that* $x \mapsto \theta(x, \cdot)$ *is convex,* $x \mapsto \theta(\cdot, x)$ *is continuous and* $\theta(x,x) = 0$, *and* $\eta : K \times K \to Y$ *is a continuous mapping such that* $x \mapsto \eta(\cdot, x)$ *is convex for all* $x \in K$. *Let* $T : K \to 2^N$ *be an u.s.c. mapping with compact values. Then* **(GVVTIP)** *is solvable.*

Proof. Let

$$A := \{(x,y) \in K \times K : \text{ there exists } u \in T(x) \text{ such that } \langle M(x,u), \theta(y,x) \rangle + \eta(x,y) - \eta(x,x) \notin -\operatorname{int} C(x)\}$$

then A is nonempty.

For each fixed $y \in K$,

$$\begin{aligned} A_y &:= \{x \in K : (x,y) \in A\} \\ &= \{x \in K : \text{there exists } u \in T(x) \text{ such that} \\ &\qquad \langle M(x,u), \theta(y,x)\rangle + \eta(x,y) - \eta(x,x) \notin -\operatorname{int} C(x)\} \end{aligned}$$

is closed in K. Indeed, let $\{x_\lambda\}$ be a net in A_y such that $x_\lambda \to x_0$. Since $x_\lambda \in A_y$, we have there exists $u_\lambda \in T(x_\lambda)$ such that

$$\langle M(x_\lambda, u_\lambda), \theta(y, x_\lambda)\rangle + \eta(x_\lambda, y) - \eta(x_\lambda, x_\lambda) \in W(x_\lambda).$$

Since $T(K)$ is compact, we can assume that there exists $u_0 \in T(x_0)$ such that $u_\lambda \to u_0$. By Lemma 1.2 (2), T is closed and hence $u_0 \in T(x_0)$. By assumption of M, θ and η, and W has a closed graph. Thus we have there exists $u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \theta(y, x_0)\rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin -\operatorname{int} C(x_0).$$

Hence $x_0 \in A_y$, A_y is closed in K.

On the other hand, for each fixed $x \in K$,

$$\begin{aligned} A_x &:= \{y \in K : (x,y) \notin A\} \\ &= \{y \in K : \text{for all } u \in T(x) \langle M(x,u), \theta(y,x)\rangle + \eta(x,y) - \eta(x,x) \in -\operatorname{int} C(x)\} \end{aligned}$$

is convex in K. In fact, let $y_1, y_2 \in A_x$ and $t \in (0,1)$, we have for all $x \in K$ and $u \in T(x)$,

$$\begin{aligned} &\left[\langle M(x,u), \theta(ty_1 + (1-t)y_2, x)\rangle + \eta(x, ty_1 + (1-t)y_2) - \eta(x,x)\right] \\ &\qquad \leq_{C(x)} t\left[\langle M(x,u), \theta(y_1,x)\rangle + \eta(x,y_1) - \eta(x,x)\right] \\ &\qquad\qquad + (1-t)\left[\langle M(x,u), \theta(y_2,x)\rangle + \eta(x,y_2) - \eta(x,x)\right]. \end{aligned}$$

That is,

$$\begin{aligned} &t\left[\langle M(x,u), \theta(y_1,x)\rangle + \eta(x,y_1) - \eta(x,x)\right] \\ &\quad + (1-t)\left[\langle M(x,u), \theta(y_2,x)\rangle + \eta(x,y_2) - \eta(x,x)\right] \\ &\qquad - \left[\langle M(x,u), \theta(ty_1 + (1-t)y_2, x)\rangle + \eta(x, ty_1 + (1-t)y_2) - \eta(x,x)\right] \in C(x). \end{aligned}$$

Since $\langle M(x,u), \theta(y_1,x)\rangle + \eta(x,y_1) - \eta(x,x) \in -\operatorname{int} C(x)$ and $\langle M(x,u), \theta(y_2,x)\rangle + \eta(x,y_2) - \eta(x,x) \in -\operatorname{int} C(x)$, we have

$$\langle M(x,u), \theta(ty_1 + (1-t)y_2, x)\rangle + \eta(x, ty_1 + (1-t)y_2) - \eta(x,x) \in -\operatorname{int} C(x).$$

Hence $ty_1 + (1-t)y_2 \in A_x$, A_x is convex in K. By Lemma 1.1, there exists $x_0 \in K$ such that $\{x_0\} \times K \subset A$. That is, there exists $x_0 \in K$ such that for all $y \in K$ there exists $u_0 \in T(x_0)$ satisfying

$$\langle M(x_0, u_0), \theta(y, x_0)\rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin \operatorname{int} C(x_0).$$

□

If we take $M(x,u) = u$ and $\eta(x,x) = 0$, then we obtain B. S. Lee & G. M. Lee [12, Theorem 2.1] as a corollary.

When X is a reflexive Banach space, $Y = \mathbb{R}$, $L(X,Y) = X^*$ and $C(x) = \mathbb{R}^+$, we obtain Siddiqi, Ansari & Khan [21, Theorem 2.1] as a corollary.

In Theorem 2.1, we considered K to be a nonempty compact convex subset of a Hausdorff topological vector space X. But in the following theorem, we do not assume that K is compact.

Theorem 2.2. *Let X be a Hausdorff topological vector space, Y a topological vector space. Let K be a nonempty convex subset of X, N a nonempty subset of $L(X,Y)$ and $\{C(x) : x \in K\}$ a family of closed convex cones in Y. Let a set-valued mapping $W : K \to 2^Y$ defined by $W(x) = Y \setminus \{-\operatorname{int} C(x)\}$ has a closed graph. Assume that $M : K \times N \to L(X,Y)$ is a continuous mapping, $\theta : K \times K \to X$ is a mapping such that $x \mapsto \theta(x,\cdot)$ is convex, $x \mapsto \theta(\cdot, x)$ is continuous and $\theta(x,x) = 0$, and $\eta : K \times K \to Y$ is a continuous mapping such that $x \mapsto \eta(\cdot, x)$ is convex for all $x \in K$. Let $T : K \to 2^N$ be an u.s.c. mapping with compact values. And the following coercive condition is satisfied;*
there exists a nonempty compact convex subset D of K and $z \in D$ such that for all $x \in K \setminus D$ there exists $u \in T(x)$ satisfying

$$\langle M(x,u), \theta(z,x)\rangle + \eta(x,z) - \eta(x,x) \in -\operatorname{int} C(x).$$

Then **(GVVTIP)** *is solvable in D.*

Proof. For each $y \in K$,

$$B_y := \{x \in D : \text{there exists } u \in T(x) \text{such that } \langle M(x,u), \theta(y,x)\rangle + \eta(x,y) - \eta(x,x) \notin -\operatorname{int} C(x)\}$$

is nonempty. And for each $y \in K$,

$$C_y := \{x \in K : \text{there existsu} \in T(x) \text{ such that } \langle M(x,u), \theta(y,x)\rangle + \eta(x,y) - \eta(x,x) \notin -\operatorname{int} C(x)\}$$

then C_y is closed in K by the same method in the proof of Theorem 2.1. Since D is closed in X, $B_y = D \cap C_y$ is closed subset of D. It is clear that **(GVVTIP)** has a solution in D if $\bigcap_{y \in K} B_y \neq \varnothing$. For this, it is sufficient to prove the family $\{B_y : y \in K\}$ has the finite intersection property. Let $y_1, y_2, \ldots, y_n$ be arbitrary finite elements of K and let $D_h = \mathrm{co}(D \cup \{y_1, y_2, \ldots, y_n\})$, where *co* denote convex hull. Then D_h is a compact convex subset of K. By Theorem 2.1, there exists $x_0 \in D_h$ such that for all $y \in D_h$ there exists $u_0 \in T(x_0)$ satisfying

$$\langle M(x_0, u_0), \theta(y, x_0) \rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin -\operatorname{int} C(x_0). \tag{2.1}$$

It can be shown that $x_0 \in D$. In fact, if $x_0 \notin D$ then by the coercive condition, there exists $z \in D$ such that for such $x_0 \in K \setminus D$, there exists $u_0 \in T(x_0)$ satisfying

$$\langle M(x_0, u_0), \theta(z, x_0) \rangle + \eta(x_0, z) - \eta(x_0, x_0) \in -\operatorname{int} C(x_0),$$

which contradicts (2.1), when $z = y$. In particular,$x_0 \in C_{y_i}$ for all y_i. In fact, if $x_0 \notin C_{y_i}$ for some y_i then for all $u_0 \in T(x_0)$,

$$\langle M(x_0, u_0), \theta(y_i, x_0) \rangle + \eta(x_0, y_i) - \eta(x_0, x_0) \in -\operatorname{int} C(x_0). \tag{2.2}$$

But since $y_i \in D_h$, we can choose $u_0 \in T(x_0)$ such that

$$\langle M(x_0, u_0), \theta(y_i, x_0) \rangle + \eta(x_0, y_i) - \eta(x_0, x_0) \notin -\operatorname{int} C(x_0),$$

which contradicts (2.2). Hence $x_0 \in B_{y_i}$ for $i = 1, 2, \ldots, n$. Therefore

$$\bigcap_{i=1}^{n} B_{y_i} \neq \varnothing.$$

Hence, the family $\{B_y : y \in K\}$ has the finite intersection property, so there exists $x_0 \in D$ such that for all $y \in K$ there exists $u_0 \in T(x_0)$ satisfying

$$\langle M(x_0, u_0), \theta(y, x_0) \rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin -\operatorname{int} C(x_0).$$

□

If we take $M(x, u) = u$ and $\eta(x, x) = 0$, then we obtain B. S. Lee & G. M. Lee [12, Theorem 2.3] as a corollary.

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