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t-INTUITIONISTIC FUZZY SUBGROUPOIDS

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ABSTRACT. In this paper, we introduce the concepts of t-intuitionistic fuzzy products and t-intuitionistic fuzzy subgroupoids. And we study some properties of t-products and t-subgroupoids.

0. INTRODUCTION

In 1965, Zadeh [15] introduced the concept of fuzzy sets. After that time, several researchers Anthony & Sherwood [1], Liu [10], Rosenfeld [11], Sessa [13] have applied the notion of fuzzy sets to group theory. Moreover, Anthony & Sherwood [1] introduced the concept of t-fuzzy subgroups by using the t-norm introduced by Schweizer & Sklar [12].

In 1986, Atanassov [2] introduced the concept of intuitionistic fuzzy sets. Recently, Çoker [5], Çoker & Eş [6], Gürçay, Çoker & Eş [7], Lee & Lee [9] introduced the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets. In 1989, Biswas [4] introduced the concept of intuitionistic fuzzy subgroups and investigated some of it's properties. In 2003, Banerjee & Basnet [3] studied intuitionistic fuzzy subrings and ideals using intuitionistic fuzzy sets. Also Hur, Jang & Kang [8] applied the notion of intuitionistic fuzzy sets to groupoid theory.

In this paper, we introduce the concepts of t-intuitionistic fuzzy products and t-intuitionistic fuzzy subgroupoids by using the t-norm. And we investigate some properties of t-products and t-subgroupoids.

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1. PRELIMINARIES

We will list some concepts and results needed in the later sections.

Definition 1.1 (Atanassov [2]). Let X be a nonempty set. An *intuitionistic fuzzy set* (in short, *IFS*) on X is an object having the form

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$$

where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denoted the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (x, \mu_A, \nu_A)$ or $A = (\mu_A, \nu_A)$ for the IFS $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ (see, Atanassov [2]).

We will denote the set of all the IFSs in X as IFS(X).

Definition 1.2 (Atanassov [2]). Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ if and all if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ if and all if $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $\langle\,\rangle A = (1 - \nu_A, \nu_A)$.

Definition 1.3 (Çoker [5]). Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4 (Çoker [5]). $0_\sim = (0, 1)$ and $1_\sim = (1, 0)$.

Result 1.1 (Corollary 2.8 in Çoker [5]). Let A, B, C, D be IFSs in X. Then

(1) $A \subset B$ and $C \subset D \Rightarrow A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$.
(2) $A \subset B$ and $A \subset C \Rightarrow A \subset B \cap C$.
(3) $A \subset B$ and $B \subset C \Rightarrow A \cup B \subset C$.
(4) $A \subset B$ and $B \subset C \Rightarrow A \subset C$.
(5) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

(6) $A \subset B \Rightarrow B^c \subset A^c$.
(7) $(A^c)^c = A$.
(8) $1_\sim^c = 0$, $0_\sim^c = 1_\sim$.

Definition 1.5 (Hur, Jang & Kang [8]). Let X and Y be nonempty sets and let $f : X \to Y$ a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be an IFS on Y. Then

(a) The *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = \big(f^{-1}(\mu_B), f^{-1}(\nu_B)\big),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$.

(b) The *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = \big(f(\mu_A), f_-(\nu_A)\big),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{if } f^{-1}(y) = \varnothing \end{cases}$$

and

$$f_-(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{if } f^{-1}(y) = \varnothing. \end{cases}$$

Definition 1.6 (S. J. Lee & E. P. Lee [9]). Let $\lambda, \mu \in (0,1]$ and $\lambda + \mu \leq 1$. An *intuitionistic fuzzy point* (in short, *IFP*) $x_{(\lambda,\mu)}$ of X is the IFS in X defined by

$$x_{(\lambda,\mu)}(y) = \begin{cases} (\lambda, \mu) & \text{if } y = x, \\ (0,1) & \text{if } y \neq x \text{ for each } y \in Y. \end{cases}$$

In this case, x is called the *support* of $x_{(\lambda,\mu)}$ and λ and μ are called the *value* and *nonvalue* of $x_{(\lambda,\mu)}$, respectively.

An IFP $x_{(\lambda,\mu)}$ is said to *belong to* an IFS $A = (\mu_A, \nu_A)$ in X, denoted by $x_{(\lambda,\mu)} \in A$, if $\lambda \leq \mu_A(x)$ and $\mu \geq \nu_A(x)$.

We will denote the set of all IFPs of X as $\mathrm{IF_P}(X)$.

Result 1.2 (Theorem 2.4 in S. J. Lee & E. P. Lee [9]). Let $A = (\mu_A, \nu_A)$ be an IFS in X. Then

$$A = \bigcup \{x_{(\lambda,\mu)} \mid x_{(\lambda,\mu)} \in A\}.$$

Definition 1.7 (Hur, Jang & Kang [8]). Let $(X, \cdot)$ be a groupoid and let $A, B \in IFS(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$, is defined as follows: for any $x \in X$,

$$\mu_{A\circ B}(x) = \begin{cases} \bigvee_{yz=x} [\mu_A(y) \wedge \mu_B(z)] & \text{for each } (y,z) \in X \times X \text{ with } yz = x, \\ 0 & \text{for each } (y,z) \in X \times X \text{ with } yz \neq x \end{cases}$$

and

$$\nu_{A\circ B}(x) = \begin{cases} \bigwedge_{yz=x} [\nu_A(y) \vee \nu_B(z)] & \text{for each } (y,z) \in X \times X \text{ with } yz = x, \\ 1 & \text{for each } (y,z) \in X \times X \text{ with } yz \neq x. \end{cases}$$

Definition 1.8 (Hur, Jang & Kang [8]). Let $(G, \cdot)$ be a groupoid and let $0_\sim \neq A \in$ IFS(G). Then A is called an *intuitionistic fuzzy subgroupoid in* G(in short, *IFGP* in G) if $A \circ A \subset A$.

Definition 1.8′ (Hur, Jang & Kang [8]). Let $(G, \cdot)$ be a groupoid and let $A \in$ IFS(X). Then A is called an *intuitionistic fuzzy subgroupoid* (in short, *IFGP*) of G if for any $x, y \in G, \mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

It is clear that $0_\sim$ and $1_\sim$ are both IFGPs of G.

We will denote the set of all IFGPs of G as IFGP(G).

Definition 1.9 (Schweizer & Sklar [12]). A *t-norm* is a mapping $t : I \times I \longrightarrow I$ satisfying the following conditions: for any $x, y, z, u, v \in I$,

(i) $t(x, y) = t(y, x)$, *i. e.*, $xty = ytx$.

(ii) $xt(ytz) = (xty)tz$.

(iii) If $x \leq u$ and $y \leq v$, then $xty \leq utv$.

In particular, if $y \leq v$, then $xty \leq xtv$.

(iv) $xt1 = x$ and $xt0 = 0$.

t-norms which are frequently encountered are:

(a) $xt_0y = \min\{x, y\}$ for $x, y \in I$.

(a) $xt_1y = \text{Prod}\{x, y\} = xy$ for $x, y \in I$.

(a) $xt_2y = \max\{x + y - 1, 0\}$ for $x, y \in I$.

Definition 1.10 (Schweizer & Sklar [12]). A *t-conorm* or *s-norm* is a mapping $s_t : I \times I \longrightarrow I$ defined by: for any $u, v \in I, us_tv = 1 - (1 - u)t(1 - v)$.

It is clear that s_t satisfies the following conditions: for any $x, y, z, u, v \in I$,

(i) $xs_ty = ys_tx$.
(ii) $xs_t(ys_tz) = (xs_ty)s_tz$.
(iii) If $x \leq u$ and $y \leq v$, then $xs_ty \leq us_tv$.

In particular, if $y \leq v$, then $xs_ty \leq xs_tv$.

(iv) $xs_t0 = x$ and $xs_t1 = 1$.

t-conorms corresponding to the above t-conorms t_0, t_1, t_2 are as follows:

(a′) $xs_0y = \max\{x, y\}$ for any $x, y \in I$.
(b′) $xs_1y = x + y - xy$ for any $x, y \in I$.
(c′) $xs_2y = \min\{1, x + y\}$ for any $x, y \in I$.

2. t-INTUITIONISTIC FUZZY PRODUCTS

In this paper, X always denotes a nonempty set and s_t denotes the dual of t-norm t.

Definition 2.1. Let $(X, \cdot)$ be a groupoid and let $A, B \in \mathrm{IFS}(X)$. Then the *intuitionistic fuzzy product* of A and B under *t-norm* t (in short, *t-intuitionistic fuzzy product* of A and B), $A \circ_t B$, is defined as follows: for any $x \in X$,

$$\mu_{A\circ_t B}(x) = \begin{cases} \bigvee_{yz=x} [\mu_A(y) t \mu_B(z)] & \text{for each } (y,z) \in X \times X \text{ with } yz = x, \\ 0 & \text{for each } (y,z) \in X \times X \text{ with } yz \neq x \end{cases}$$

and

$$\nu_{A\circ_t B}(x) = \begin{cases} \bigwedge_{yz=x} [\nu_A(y) s_t \nu_B(z)] & \text{for each } (y,z) \in X \times X \text{ with } yz = x, \\ 1 & \text{for each } (y,z) \in X \times X \text{ with } yz \neq x. \end{cases}$$

It is clear that $A \circ_t B \in \mathrm{IFS}(X)$, *i. e.*, $(\mathrm{IFS}(X), \circ_t)$ is groupoid.

Proposition 2.2. *Let "$\circ_t$" be as above, let $x_{(\alpha,\beta)}, y_{(\alpha',\beta')} \in \mathrm{IF_P}(X)$ and let $A, B \in \mathrm{IFS}(X)$. Then:*

(i) $x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')} = (xy)_{(\alpha t \alpha', \beta s_t \beta')}$.
(ii) $A \circ_t B = \bigcup_{x_{(\alpha,\beta)} \in A,\, y_{(\alpha',\beta')} \in B} x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')}$.

Proof. (1) Let $z \in X$. Then:

$$\mu_{x_{(\alpha,\beta)}\circ_t y_{(\alpha',\beta')}}(z)$$

$$= \begin{cases} \bigvee_{x'y'=z} [\mu_{x_{(\alpha,\beta)}}(x') t \mu_{y_{(\alpha',\beta')}}(y')] & \text{for each } (x',y') \in X \times X \text{ with } x'y' = z, \\ 0 & \text{for each } (x',y') \in X \times X \text{ with } x'y' \neq z, \end{cases}$$

$$= \begin{cases} \alpha t \alpha' & \text{if } xy = z, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu_{x_{(\alpha,\beta)}\circ_t y_{(\alpha',\beta')}}(z)$$

$$= \begin{cases} \bigwedge_{x'y'=z} [\nu_{x_{(\alpha,\beta)}}(x') s_t \nu_{y_{(\alpha',\beta')}}(y')] & \text{for each } (x',y') \in X \times X \text{ with } x'y' = z, \\ 0 & \text{for each } (x',y') \in X \times X \text{ with } x'y' \neq z, \end{cases}$$

$$= \begin{cases} \beta s_t \beta' & \text{if } xy = z, \\ 1 & \text{otherwise.} \end{cases}$$

Hence $x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')} = (xy)_{(\alpha t \alpha', \beta s_t \beta')}$.

(2) Let $w \in X$ and we may assume that there exist $u, v \in X$ such that $uv = w$ and $\mu_A(u) > 0, \nu_A(u) \neq 1, \mu_B(v) > 0, \nu_B(v) \neq 1$ without loss of generality. Then:

$$\begin{aligned} \mu_{A \circ_t B}(w) &= \bigvee_{uv=w} [\mu_A(u) t \mu_B(v)] \\ &\geq \bigvee_{uv=w} \bigvee_{x_{(\alpha,\beta)} \in A, y_{(\alpha',\beta')} \in B} [\mu_{x_{(\alpha,\beta)}}(u) t \mu_{y_{(\alpha',\beta')}}(v)] \quad (t \text{ is increasing on } I) \\ &= \mu_C(w), \end{aligned} \tag{1}$$

where

$$C = \bigcup_{x_{(\alpha,\beta)} \in A, y_{(\alpha',\beta')} \in B} x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')}.$$

Since $u_{(\mu_A(u), \nu_A(u))} \in A$ and $v_{(\mu_B(v), \nu_B(v))} \in B$,

$$\begin{aligned} \mu_C(w) &= \bigvee_{x_{(\alpha,\beta)} \in A, y_{(\alpha',\beta')} \in B} \bigvee_{uv=w} [\mu_{x_{(\alpha,\beta)}}(u) t \mu_{y_{(\alpha',\beta')}}(v)] \\ &= \bigvee_{uv=w} \Big(\bigvee_{x_{(\alpha,\beta)} \in A, y_{(\alpha',\beta')} \in B} [\mu_{x_{(\alpha,\beta)}}(u) t \mu_{y_{(\alpha',\beta')}}(v)] \Big) \\ &\geq \bigvee_{uv=w} [\mu_{u_{(\mu_A(u), \nu_A(u))}}(u) t \mu_{v_{(\mu_B(v), \nu_B(v))}}(v)] \\ &= \bigvee_{uv=w} [\mu_A(u) t \mu_B(v)] \end{aligned}$$

$$= \mu_{A\circ_t B}(w).$$

Thus $\mu_{A\circ_t B} = \mu_C$. On the other hand:

$$\begin{aligned}
\nu_{A\circ_t B}(w) &= \bigwedge_{uv=w} [\nu_A(u) s_t \nu_B(v)] \\
&\leq \bigwedge_{uv=w, x_{(\alpha,\beta)}\in A, y_{(\alpha',\beta')}\in B} [\nu_{x_{(\alpha,\beta)}}(u) s_t \nu_{y_{(\alpha',\beta')}}(v)] \; (s_t \text{ is increasing on } I) \\
&= \nu_C(w)
\end{aligned}$$

and

$$\begin{aligned}
\nu_C(w) &= \bigwedge_{x_{(\alpha,\beta)}\in A, y_{(\alpha',\beta')}\in B} \bigwedge_{uv=w} [\nu_{x_{(\alpha,\beta)}}(u) s_t \nu_{y_{(\alpha',\beta')}}(v)] \\
&= \bigwedge_{uv=w} \Big(\bigwedge_{x_{(\alpha,\beta)}\in A, y_{(\alpha',\beta')}\in B} [\nu_{x_{(\alpha,\beta)}}(u) s_t \nu_{y_{(\alpha',\beta')}}(v)]\Big) \\
&\leq \bigwedge_{uv=w} [\nu_{u_{(\mu_A(u),\nu_A(u))}}(u) s_t \nu_{v_{(\mu_A(v),\nu_A(v))}}(v)] \\
&= \bigwedge_{uv=w} [\nu_A(u) s_t \nu_B(v)] \\
&= \nu_{A\circ_t B}(w).
\end{aligned}$$

Thus $\nu_{A\circ_t B} = \nu_C$. Hence

$$A \circ_t B = \bigcup_{x_{(\alpha,\beta)}\in A, y_{(\alpha',\beta)}\in B} x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')}.$$

□

Remark 2.1. Proposition 2.2 is the generalization of Proposition 2.2 in Hur, Jang & Kang [8]:

(1) $x_{(\alpha,\beta)} \circ y_{(\alpha',\beta')} = (xy)_{(\alpha\wedge\alpha',\beta\vee\beta')}$.

(2) $A \circ B = \bigcup_{x_{(\alpha,\beta)}\in A, y_{(\alpha',\beta)}\in B} x_{(\alpha,\beta)} \circ y_{(\alpha',\beta')}$.

The following is the immediate result of Definition 2.1.

Proposition 2.3. *Let $(X, \cdot)$ be a groupoid and let " $\circ_t$ " be as above.*

(1) *If " $\cdot$ " is associative [resp. commutative] in X, then so is " $\circ_t$ " in* IFS(X).

(2) *If " $\cdot$ " is commutative in X, then so is " $\circ_t$ " in* IFS(X).

(3) *If " $\cdot$ " has a unity $e \in X$, then $e_{(1,0)} \in \text{IF}_P(X)$ is a unity of " $\circ_t$ " in* IFS(X), *i.e., $A \circ_t e_{(1,0)} = A = e_{(1,0)} \circ_t A$ for each $A \in$* IFS(X).

Remark 2.2. Proposition 2.3 is generalization of Hur, Jang & Kang [8, Proposition 2.3]: Let $(X, \cdot)$ be a groupoid.

(1) If " $\cdot$ " is associative [*resp.* commutative] in X, then so is " $\circ$ " in IFS(X).

(2) If " $\cdot$ " has a unity $e \in X$, then $e_{(1,0)} \in \text{IF}_{\text{P}}(X)$ is a unity of " $\circ$ " in IFS(X), *i. e.*, $A \circ e_{(1,0)} = A = e_{(1,0)} \circ A$ for each $A \in \text{IFS}(X)$.

3. t-INTUITIONISTIC FUZZY SUBGROUPOIDS

Definition 3.1. Let $(X, \cdot)$ be a groupoid and let $0_\sim \neq A \in \text{IFS}(X)$. Then A is called an *intuitionistic fuzzy subgroupoid in* X *under a t-norm* t (in short, *t-IFGP* in X) if $A \circ_t A \subset A$.

It is clear that $0_\sim$ and $1_\sim$ are both t-IFGPs in X.

The followings are the immediate results of Definition 2.1 and Definition 3.1.

Proposition 3.2. *Let* $(X, \cdot)$ *be a groupoid and let* $0_\sim \neq A \in \text{IFS}(X)$*. Then the followings are equivalent:*

(1) *A is a t-IFGP in X.*

(2) *For any $x_{(\alpha,\beta)}, y_{(\alpha',\beta')} \in A$, $x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')} \in A$, i.e., $(A, \circ_t)$ is a groupoid.*

(3) *For any $x, y \in X$, $\mu_A(xy) \geq \mu_A(x) t \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) s_t \nu_A(y)$.*

Remark 3.1. Proposition 3.2 is a generalization of Hur, Jang & Kang [8, Proposition 3.2]:

(1) A is an IFGP in X.

(2) For any $x_{(\alpha,\beta)}, y_{(\alpha',\beta')} \in A$, $x_{(\alpha,\beta)} \circ y_{(\alpha',\beta')} \in A$, *i. e.*, $(A, \circ)$ is a groupoid.

(3) For any $x, y \in X$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

Proposition 3.3. *Let A be a t-IFGP in a groupoid $(X, \cdot)$.*

(1) *If "$\cdot$" is associative in X, then so is "$\circ_t$" in A, i.e., for any $x_{(\alpha,\beta)}, y_{(\alpha',\beta')}, z_{(\alpha'',\beta'')} \in A$, $(x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')}) \circ_t z_{(\alpha'',\beta'')} = x_{(\alpha,\beta)} \circ_t (y_{(\alpha',\beta')} \circ_t z_{(\alpha'',\beta'')})$.*

(2) *If " $\cdot$ " is commutative in X, then so is " $\circ_t$ " in A, i.e., for any $x_{(\alpha,\beta)}, y_{(\alpha',\beta')} \in A$, $x_{(\alpha,\beta)} \circ_t y_{(\alpha',\beta')} = y_{(\alpha',\beta')} \circ_t x_{(\alpha,\beta)}$.*

(3) *If " $\cdot$ " has a unity $e \in X$, then $e_{(1,0)} \circ_t x_{(\alpha,\beta)} = x_{(\alpha,\beta)} = x_{(\alpha,\beta)} \circ_t e_{(1,0)}$ for each $x_{(\alpha,\beta)} \in A$.*

Remark 3.2. Proposition 3.3 is the generalization of Hur, Jang & Kang [8, Proposition 3.3]:

(1) If "$\cdot$" is associative in X, then so is "$\circ$" in A, *i. e.*, for any $x_{(\alpha,\beta)}, y_{(\alpha',\beta')}, z_{(\alpha'',\beta'')}$ in A, $(x_{(\alpha,\beta)} \circ y_{(\alpha',\beta')}) \circ z_{(\alpha'',\beta'')} = x_{(\alpha,\beta)} \circ (y_{(\alpha',\beta')} \circ z_{(\alpha'',\beta'')})$.
(2) If "$\cdot$" is commutative in X, then so is "$\circ$" in A, *i. e.*, for any $x_{(\alpha,\beta)}, y_{(\alpha',\beta')}, z_{(\alpha'',\beta'')}$ in A, $x_{(\alpha,\beta)} \circ y_{(\alpha',\beta')} = y_{(\alpha',\beta')} \circ x_{(\alpha,\beta)}$.
(3) If "$\cdot$" has a unity $e \in X$, then $e_{(1,0)} \circ x_{(\alpha,\beta)} = x_{(\alpha,\beta)} \circ e_{(1,0)}$ for each $x_{(\alpha,\beta)} \in A$.

From Proposition 3.2, we can define a t-intuitionistic fuzzy subgroupoid of a groupoid X as follows:

Definition 3.1′. Let $(X, \cdot)$ be a groupoid and let $A \in \text{IFS}(X)$. Then A is called *t-intuitionistic fuzzy groupoid* (in short, *t-IFGP*) of X, if for any $x, y \in X$, $\mu_A(xy) \geq \mu_A(x)t\mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x)s_t\nu_A(y)$.

Proposition 3.4. *Let $(X, \cdot)$ be a groupoid and let $A \subset X$. Then $A = (\chi_A, \chi_A^c)$ is a t-IFGP in X if and only if A is a subgroupoid of X.*

Proof. ($\Longrightarrow$): Suppose $A = (\chi_A, \chi_A^c)$ is a t-IFGP in X and let $x, y \in X$. Then, by Proposition 3.3,

$$\mu_A(xy) \geq \chi_A(x)t\chi_A(y) = \begin{cases} 1 & \text{if } x, y \in A, \\ 0 & \text{if } x \notin A \text{ or } y \notin A \end{cases}$$

and, by Proposition 3.3,

$$\nu_A(xy) \leq \chi_{A^c}(x)s_t\chi_{A^c}(y) = \begin{cases} 0 & \text{if } x, y \in A, \\ 1 & \text{if } x \notin A \text{ or } y \notin A. \end{cases}$$

Thus $A(xy) = (1, 0)$. So $xy \in A$. Hence A is a subgroupoid of X.

($\Longleftarrow$): Suppose A is a subgroupoid of X and let $x, y \in X$. If $x, y \in A$, then $\chi_A(x) = \chi_A(y) = 1$. By the hypothesis, $xy \in A$. Thus $A(xy) = (1, 0)$, *i. e.*, $\mu_A(xy) = 1$ and $\nu_A(xy) = 0$. So $\mu_A(xy) \geq \chi_A(x)t\chi_A(y)$ and $\nu_A(xy) \leq \chi_{A^c}(x)s_t\chi_{A^c}(y)$. If $x \notin A$ or $y \notin A$, then $\chi_A(x) = 0$ or $\chi_A(y) = 0$. Thus $\chi_A(x)t\chi_A(y) = 0$ and $\chi_{A^c}(x)s_t\chi_{A^c}(y) = 1$. So $\mu_A(xy) \geq \chi_A(x)t\chi_A(y)$ and $\nu_A(xy) \leq \chi_{A^c}(x)s_t\chi_{A^c}(y)$. Hence A is a t-IFGP in X. □

Remark 3.3. Proposition 3.4 is the generalization of Hur, Jang & Kang [8, Proposition 3.8]:

Let $(X, \cdot)$ be a groupoid and let $A \subset X$. Then $A = (\chi_A, \chi_{A^c})$ is an IFGP in X if and only if A is a subgroupoid of X.

Definition 3.5. A t-norm t is said to be *continuous* if $t : I \times I \longrightarrow I$ is continuous with respect to the usual topologies.

It is clear that t_0, t_1 and t_2 are all continuous t-norms.

Proposition 3.6. *Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be any family of t-IFGPs in a groupoid $(X,\cdot)$. If t is continuous, then $\bigcap_{\alpha\in\Gamma} A_\alpha$ is a t-IFGP in X.*

Proof. Let $x, y \in X$. Since A_α is a t-IFGP in X for each $\alpha \in \beta$, $\mu_{A_\alpha}(xy) \geq \mu_{A_\alpha}(x)t\mu_{A_\alpha}(y)$ and $\nu_{A_\alpha}(xy) \leq \nu_{A_\alpha}(x)s_t\nu_{A_\alpha}(y)$ for each $\alpha \in \beta$. Then

$$\mu_{\cap_{\alpha\in\beta}A_\alpha}(xy) = \bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}(xy) \geq \bigwedge_{\alpha\in\Gamma} [\mu_{A_\alpha}(x)t\mu_{A_\alpha}(y)]$$

and

$$\nu_{\cap_{\alpha\in\beta}A_\alpha}(xy) = \bigvee_{\alpha\in\Gamma} \nu_{A_\alpha}(xy) \leq \bigvee_{\alpha\in\Gamma} [\nu_{A_\alpha}(x)s_t\nu_{A_\alpha}(y)].$$

Since t is continous, t is continuous at $\big(\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(x), \bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(y)\big)$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $r_1 \geq \bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(x)+\delta$ and $r_2 \geq \bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(y)+\delta$, then $r_1tr_2 \geq \big(\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(x)\big)t\big(\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(y)\big) + \epsilon$. Let us choose $\alpha_0 \in \Gamma$ such that $\mu_{A_{\alpha_0}}(x) \geq \bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(x) + \delta$ and $\mu_{A_{\alpha_0}}(y) \geq \bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(y) + \delta$. Then

$$\mu_{A_{\alpha_0}}(x)t\mu_{A_{\alpha_0}}(y) \geq \big(\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(x)\big)t\big(\mu_{A_{\alpha_0}}(y)\big) + \epsilon.$$

Thus

$$\bigwedge_{\alpha\in\Gamma}[\mu_{A_{\alpha_0}}(x)t\mu_{A_{\alpha_0}}(y)] \geq \big(\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(x)\big)t\big(\mu_{A_{\alpha_0}}(y)\big).$$

So

$$\mu_{\cap_{\alpha\in\Gamma}A_\alpha}(xy) \geq \mu_{\cap_{\alpha\in\Gamma}A_\alpha}(x)t\mu_{\cap_{\alpha\in\Gamma}A_\alpha}(y).$$

Similarly, we have $\nu_{\cap_{\alpha\in\Gamma}A_\alpha}(xy) \leq \nu_{\cap_{\alpha\in\Gamma}A_\alpha}(x)s_t\nu_{\cap_{\alpha\in\Gamma}A_\alpha}(y)$. Hence $\bigcap_{\alpha\in\Gamma} A_\alpha$ is a t-IFGP in X. □

Remark 3.4. Since $t_0 =$ " $\wedge$ " is continuous, Proposition 3.6 is a generalization of Hur, Jang & Kang [8, Proposition 3.9]:

If $\{A_\alpha\}_{\alpha\in\Gamma}$ is a family of IFGPs in a groupoid $(X,\cdot)$, then $\bigcap_{\alpha\in\Gamma} A_\alpha$ is an IFGP in X.

Proposition 3.7. *Let X and Y be groupoids and let $f : X \to Y$ an epimorphism. If A is a t-*IFGP *of X and t is continuous, then $f(A)$ is a t-*IFGP *of Y.*

Proof. By the proof of Anthony & Sherwood [1, Proposition 4],

$$\mu_{f(A)}(y_1y_2) \geq \mu_{f(A)}(y_1)t\mu_{f(A)}(y_2), \text{ for any } y_1, y_2 \in Y.$$

Thus it is enough to show that

$$\nu_{f(A)}(y_1y_2) \leq \nu_{f(A)}(y_1)s_t\nu_{f(A)}(y_2), \text{ for any } y_1, y_2 \in Y.$$

Let $y_1, y_2 \in Y$, let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$, $A_{12} = f^{-1}(y_1y_2)$ and let $A_1A_2 = \{x \in X : x = a_1a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2\}$. Then, by the proof of Anthony & Sherwood [1, Proposition 4], we have $A_1A_2 \subset A_{12}$. Since A is a t-IFGP of X,

$$\begin{aligned}\nu_{f(A)}(y_1y_2) &= f_-(\nu_A)(y_1y_2) \leq \bigwedge_{x\in A_{12}} \nu_A(x)\\ &\leq \bigwedge_{x\in A_1A_2} \nu_A(x) \leq \bigwedge_{x_1\in A_1, x_2\in A_2} \nu_A(x_1x_2)\\ &\leq \bigwedge_{x_1\in A_1, x_2\in A_2} [\nu_A(x_1)s_t\nu_A(x_2)].\end{aligned}$$

Let $\epsilon > 0$. Since s_t is continuous, there exists a $\delta > 0$ such that if $r_1 \leq \bigwedge_{x_1\in A_1} \nu_A(x_1) + \delta$ and $r_2 \leq \bigwedge_{x_2\in A_2} \nu_A(x_2) + \delta$, then $r_1s_tr_2 \leq (\bigwedge_{x_1\in A_1} \nu_A(x_1))s_t\ (\bigwedge_{x_2\in A_2} \nu_A(x_2)) + \epsilon$. Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\nu_A(a_1) \leq \bigwedge_{x_1\in A_1} \nu_A(x_1) + \delta$ and $\nu_A(a_2) \leq \bigwedge_{x_1\in A_1} \nu_A(x_1) + \delta$. Then:

$$\nu_A(a_1)s_t\nu_A(a_2) \leq (\bigwedge_{x_1\in A_1} \nu_A(x_1))s_t(\bigwedge_{x_2\in A_2} \nu_A(x_2)) + \epsilon.$$

So

$$\begin{aligned}\nu_{f(A)}(y_1y_2) &\leq \bigwedge_{x_1\in A_1, x_2\in A_2} [\nu_A(x_1)s_t\nu_A(x_2)]\\ &\leq (\bigwedge_{x_1\in A_1} \nu_A(x_1))s_t(\bigwedge_{x_2\in A_2} \nu_A(x_2))\\ &= \nu_{f(A)}(y_1)s_t\nu_{f(A)}(y_2).\end{aligned}$$

Hence $f(A)$ is a t-IFGP, of Y. □

Remark 3.5. Since $t_0 =$ " $\wedge$ " and $s_0 =$ " $\vee$ " are continuous, without the condition of "having the sup property", Hur, Jang & Kang [8, Proposition 4.4 (1)] holds.

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