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REPRESENTING NATURAL NUMBERS AS UNIQUE SUMS OF POSITIVE INTEGERS

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ABSTRACT. It is known that each natural number can be written uniquely as a sum of Fibonacci numbers with suitably increasing indices. In 1960, Daykin showed that the sequence of Fibonacci numbers is the only sequence with this property. Consider here the problem of representing each natural number uniquely as a sum of positive integers taken from certain sequence allowing a fixed number, $l \geq 2$, of repetitions. It is shown that the $(l+1)$-adic expansion is the only such representation possible.

1. INTRODUCTION

The problem of representing natural numbers by sums of Fibonacci numbers was first reported in 1952 by Lekkerkerker [2], who attributed it to E. Zeckendorf, with the result:

I. For each natural number N, there is one, and only one system of natural numbers $i_1, i_2, \ldots, i_d$ such that

$$N = u_{i_1} + u_{i_2} + \cdots + u_{i_d} \quad \text{and} \quad i_{\gamma+1} \geq i_\gamma + 2 \quad \text{for} \quad 1 \leq \gamma < d,$$

when (u_n) is the sequence of Fibonacci numbers.

Later in 1960, Daykin [1] established the following converse of I.

II. The Fibonacci numbers form the only sequence of natural numbers (u_n) for which I holds.

In so doing, Daykin introduced the concepts of Property P and the (h, k)-th Fibonacci sequence as follows: let (a_n), (k_n) be two sequences of natural numbers.

Property P. For each natural number N, there is one and only one system

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of natural numbers $i_1, i_2, \ldots, i_d$ such that

$$N = a_{i_1} + a_{i_2} + \cdots + a_{i_d}, \quad \text{with} \quad i_{\gamma+1} \geq i_\gamma + k_\gamma \quad \text{for} \quad 1 \leq \gamma < d. \tag{1}$$

If h, k are natural numbers such that $h \leq k \leq h+1$, then the (h,k)-th Fibonacci sequence (v_n) is defined by

$$\begin{aligned} v_n &= n && \text{for } 1 \leq n \leq k, \\ v_n &= v_{n-1} + v_{n-h} && \text{for } k < n < h+k, \\ v_n &= v_{n-1} + v_{n-k} + (k-h) && \text{for } n \geq h+k. \end{aligned}$$

With these concepts, Daykin also proved:

III. If (v_n) is the (h,k)-th Fibonacci sequence, then for each natural number N, there is one and only one system of natural numbers $i_1, i_2, \ldots, i_d$ such that

$$N = v_{i_1} + v_{i_2} + \cdots + v_{i_d}, \tag{2}$$

where $i_2 \geq i_1 + h$ and $i_{\gamma+1} \geq i_\gamma + k$ for $2 \leq \gamma < d$.
In this representation, we also have $v_{i_d} \leq N < v_{i_{d+1}}$.

IV. If $(a_n), (k_n)$ are two sequences of natural numbers with Property P, and (a_n) increasing, then $k_1 \leq k_2 \leq k_1 + 1$, $k_n = k_2$ for $n \geq 3$, and (a_n) is the (h,k)-th Fibonacci sequence with $h = k_1 and k = k_2$.

V. Let (v_n) be the (h,k)-th Fibonacci sequence. For each natural number n, let Ψ_n be the average number of summands required in the representation (2) for all those natural numbers N such that $v_n \leq N < v_{n+1}$. Then

$$\begin{aligned} &\text{for} \quad k \geq 1 : \frac{v_{n+1}}{v_n} \to \theta \quad \text{as} \quad n \to \infty, \\ &\text{for} \quad k = 1 : \Psi_n = \frac{1}{2}(n+1), \\ \text{and for} \quad &k \geq 2 : \frac{\Psi_n}{n} \to \frac{\theta - 1}{1 + k(\theta - 1)} \quad \text{as} \quad n \to \infty, \end{aligned}$$

where $\theta = \theta(k)$ is the positive real solution of the equation $z - 1 = z^{1-k}$.

In this paper, we complement the results in I, II and V by considering representations which allow repetition of each constituent up to a fixed number, l, of times. It turns out that such representation coincides with the $(l+1)$-adic representation. The difficult part of the proof is that of showing uniqueness.

2. Prelimilaries

Our results are based on two notions complementing Property P_l which we now describe.

Let (v_n) and (k_n) be two sequences of natural numbers. We say that the two sequences $(v_n), (k_n)$ have **Property $\mathbf{P}_l$**, $l \geq 2$, when:

Property $\mathbf{P}_l$. Each natural number N can be written uniquely in the form

$$N = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \cdots + \alpha_d v_{i_d}, \tag{M1}$$

where α_γ and $\alpha_d \in \{1, 2, \ldots, l\}, i_{\gamma+1} \geq i_\gamma + k_\gamma$ and $1 \leq \gamma < d$.

Lemma 1. *If $(v_n), (k_n)$ are two sequences of natural numbers with Property P_l and (v_n) is increasing, then v_n is the least natural number which cannot be written in the form (M1) using only $v_1, v_2, \ldots, v_{n-1}$.*

Proof. By Property P_l, the first l natural numbers are uniquely represented in the form (M1) as $v_1 = 1, 2v_1 = 2, 3v_1 = 3, \ldots, lv_1 = l$. Since $l + 1$ cannot be written in the form (M1) using only v_1, then $v_2 = l + 1$. Suppose $n > 2$. If v_n could be written in the form (M1) using only $v_1, v_2, \ldots, v_{n-1}$, then v_n could be written by two distinct forms as $v_n = v_n$ and as a combination of $v_1, v_2, \ldots, v_{n-1}$, which contradicts the uniqueness of the representation (M1). If we have a number between v_{n-1} and v_n which cannot be written in the form (M1) using only $v_1, v_2, \ldots, v_{n-1}$, then this number cannot be written in the form (M1) at all. □

Let (v_n) be a sequence of natural numbers and $(k_n^{(1)})$ be the sequence $(1, 1, 1, 1, \ldots)$ all of whose elements are 1. We say that the two sequences (v_n), $(k_n^{(1)})$, or simply the sequence (v_n), have **Property $\mathbf{P}_l^*$**, $l \geq 2$, when:

Property $\mathbf{P}_l^*$. Each natural number N can be written uniquely in the form

$$N = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \cdots + \alpha_d v_{i_d}, \tag{M2}$$

where $i_1 < i_2 < \cdots < i_d$ and $\alpha_i \in \{1, 2, \ldots, l\}, 1 \leq i \leq d$.

Note that $\mathbf{P}_l^*$ is just $\mathbf{P}_l$ when the sequence (k_n) is specialized as $(k_n^{(1)})$; we define it separately for emphasis.

3. Results

Theorem 1. *If (v_n) is an increasing sequence of natural numbers with Property P_l^*, then $v_n = (l + 1)^{n-1}$.*

Proof. By Property P_l^*, the first l natural numbers are uniquely represented in the form (M2) as $v_1 = 1, 2v_1 = 2, 3v_1 = 3, \ldots, lv_1 = l$. Since $l+1$ is the least natural number not representable in the form (M2) using only v_1, then by Lemma 1, $v_2 = l+1$. Assume the results hold for $v_1, v_2, \ldots, v_k$. By Lemma 1, v_{k+1} is the least natural number not representable in the form (M2) using only $v_1 = 1, v_2 = l+1, \ldots, v_k = (l+1)^{k-1}$. This is just the $(l+1)$-adic expansion, *i. e.*, expansion in base $(l+1)$, and so $v_{k+1} = (l+1)^k$. □

Theorem 2. *Let (v_n) and (k_n) be two sequences of natural numbers with (v_n) increasing. If $(v_n), (k_n)$ have Property P_l, $l \geq 2$, then $(k_n) = (k_n^{(1)}) = (1, 1, 1, 1, \ldots)$ and $(v_n) = \left((l+1)^{n-1}\right)$.*

Proof. The shape of v_n follows from Theorem 1 if we show that $(k_n) = (k_n^{(1)}) = (1, 1, 1, 1, \ldots)$. We proceed to derive contradiction in each possible case assuming $(k_n) \neq (1, 1, 1, 1, \ldots)$. There are two possibilities.

Case I. $l \geq 3$.

Subcase I.1. $(k_n) = (h \geq 2, k_2, k_3, \ldots)$.

Claim 1. For $h = 2$, we have $v_1 = 1, v_2 = l+1$, and $v_3 = l+2$.

Proof of **Claim 1.** Clearly $v_1 = 1$. By Lemma 1, $v_2 = l+1$. Since $i_2 \geq i_1 + 2$, $l+2$ cannot be written in the form (M1) using only v_1, v_2, then by Lemma 1, $v_3 = l+2$, and Claim 1 is proved.

The case $h = 2$ is now eliminated by noting that $2v_2 = 2l + 2 = lv_1 + v_3$ contradicting the uniqueness of (M1).

For $h = 3$, by the same reasoning as in Claim 1, we have $v_1 = 1$, $v_2 = l+1$, $v_3 = l+2$, $v_4 = l+3$.

The case $h = 3$ is now eliminated by noting that $2v_2 = 2l + 2 = (l-1)v_1 + v_4$ contradicting the uniqueness of (M1).

For $h = 4$, by the same reasoning as in Claim 1, we have $v_1 = 1$, $v_2 = l+1$, $v_3 = l+2$, $v_4 = l+3$, $v_5 = l+4$.

The case $h = 4$ is now eliminated by noting that $2v_2 = 2l + 2 = (l-2)v_1 + v_5$ contradicting the uniqueness of (M1).

Claim 2. For $h \geq 5$, we have

$$\begin{aligned} v_1 &= 1, & v_2 &= l+1, \\ v_3 &= l+2, \ldots, & v_{h-2} &= l+h-3, \\ v_{h-1} &= l+h-2, & v_h &= l+h-1, \end{aligned}$$

$$v_{h+1} = \begin{cases} l+h+1 & \text{for } l = h-2 \\ l+h & \text{for } l > h-2 \end{cases}.$$

(Note that the values of v_{h+1} for $3 \le l < h-2$ are not explicitly stated because they somewhat fluctuate and to obtain a desired contradiction later, their information is not needed.)

Proof of **Claim 2.** Clearly $v_1 = 1$. By Lemma 1, $v_2 = l+1$. Since $i_2 \ge i_1 + 5$, $l+2$ cannot be written in the form (M1) using only v_1, v_2, then $v_3 = l+2$. Similarly $v_4 = l+3$ and $v_5 = l+4$.

For $l = 3$, $l+5$ can be written in the form (M1) using only v_1, v_2, v_3, v_4, v_5, viz., $l+5 = 2(l+1) = 2v_2$. Thus $v_6 \ne l+5$ for $l = 3$. For $l > 3$, $l+5$ cannot be written in the form (M1) using only v_1, v_2, v_3, v_4, v_5, so $v_6 = l+5$. To find v_6 for $l = 3$, by direct checking, $l+6$ cannot be written in the form (M1) using only v_1, v_2, v_3, v_4, v_5 so $v_6 = l+6$.

To sum up for $l = 3$, we have

$$\begin{aligned} &v_1 = 1, && v_2 = l+1, && v_3 = l+2, \\ &v_4 = l+3, && v_5 = l+4, && v_6 = l+6 \end{aligned}$$

and for $l > 3$,

$$\begin{aligned} &v_1 = 1, && v_2 = l+1, && v_3 = l+2, \\ &v_4 = l+3, && v_5 = l+4, && v_6 = l+5; \end{aligned}$$

so Claim 2 holds for $h = 5$.

Assuming the results for h, *i. e.*,

$$\begin{aligned} &v_1 = 1, && v_2 = l+1, \\ &v_3 = l+2, \ldots, && v_{h-1} = l+h-2, \\ &v_h = l+h-1, && v_{h+1} = \begin{cases} l+h+1 & \text{for } l = h-2 \\ l+h & \text{for } l > h-2 \end{cases}, \end{aligned}$$

we proceed to prove the result for $h+1$. Observe that we must start with

$$v_1 = 1, v_2 = l+1, v_3 = l+2, \ldots, v_h = l+h-1, v_{h+1} = l+h.$$

Consider the next value $l+h+1$. If $l = h-1$, then $l+h+1 = 2(l+1) = 2v_2$, so $v_{h+2} \ne l+h+1$ in this case. By direct checking, $l+h+2$ cannot be written in the form (M1) using only $v_1, v_2, \ldots, v_{h+1}$ and so $v_{h+2} = l+h+2$. But for $l > h-1$ by direct checking, $l+h+1$ cannot be written in the form (M1) using only $v_1, v_2, \ldots, v_{h+1}$ and so $v_{h+2} = l+h+1$ in this case, which ends the proof of Claim 2.

The case $h \geq 5$ is now eliminated by noting that $2v_3 = 2l+4 = (l-(h-3))\,v_1 + v_{h+1}$ for $l = h-2$ and $2v_2 = 2l+2 = (l-(h-2))\,v_1 + v_{h+1}$ for $l > h-2$, both of which contradict the uniqueness of representation.

Subcase I.2. $(k_n) = \left(\underbrace{1, 1, \ldots, 1,}_{f-1 \text{ terms } (f \geq 2)} \; h \geq 2, \ldots \right)$, where $f \geq 2$.

The next Claim works for all $l \geq 2$ which will also be needed later.

Claim 3. Let $l \geq 2$. Then

$$v_1 = 1, v_2 = l+1, v_3 = (l+1)^2, \ldots, v_{f+1} = (l+1)^f, v_{f+2} = \frac{(l+1)^{f+1}-1}{l}.$$

Proof of **Claim 3.** Clearly $v_1 = 1$, $v_2 = l+1$ and the integers $l+2, \ldots, l+l(l+1) = l^2+2l$ can be written in the form (M1) using only v_1, v_2, but $(l+1)^2$ cannot, so $v_3 = (l+1)^2$. The values $v_4, v_5, v_6, \ldots, v_{f+1}$ are found analogously. The integers with values

$$v_{f+1} + 1 = (l+1)^f + 1, (l+1)^f + 2, \ldots, v_{f+1} + v_f + \cdots + v_2$$

can be written in the form (M1) using only $v_1, v_2, v_3, \ldots, v_{f+1}$, but the next integer

$$\begin{aligned} v_{f+1} + v_f + \cdots + v_2 + v_1 &= (l+1)^f + (l+1)^{f-1} + \cdots + (l+1) + 1 \\ &= \frac{(l+1)^{f+1}-1}{l} \end{aligned}$$

cannot because $i_2 \geq i_1 + 1, i_3 \geq i_2 + 1, \ldots, i_f \geq i_{f-1} + 1, i_{f+1} \geq i_f + h \geq i_f + 2$, and so

$$v_{f+2} = \frac{(l+1)^{f+1}-1}{l}.$$

This proves Claim 3.

The case $l \geq 2$ is now eliminated by nothing that

$$lv_1 + v_{f+2} = 2(l+1) + (l+1)^2 + \cdots + (l+1)^f = 2v_2 + v_3 + \cdots + v_{f+1},$$

contradicting the uniqueness of (M1).

Case II. $l = 2$.

Define (P_2, h) to be a set of all possible pairs of sequences $\{(v_n), (k_n)\}$ with Property P_2 and $(k_n) = (h, k_2, k_3, \ldots)$.

Subcase II.1. $(k_n) = (h \geq 2, k_2, k_3, \ldots)$.

Claim 4. For fixed $H \in \mathbb{N}$, if $h \geq H$, each pair of sequences in (P_2, h) has the same first $H+1$ values in (v_n), *i. e.*, the same value of $v_1, v_2, \ldots, v_{H+1}$.

Proof of **Claim 4.** For $H = 1$, for each pair in (P_2, h), we clearly have $v_1 = 1$, $v_2 = 3$ (because $2 = 2v_1$). For $H = 2$ in $(\mathrm{P}_2, h \geq 2)$, direct computation gives

$v_1 = 1$, $v_2 = 3$ and $v_3 = 4$ (because $4 \neq 2v_1, 2v_2$). For general H, we determine $v_1, v_2, \ldots, v_{H+1}$ in (P_2, h) by using Lemma 1. Observe that in (P_2, h), where $h \geq H$, again using Lemma 1, the first $H+1$ values must be the same as in (P_2, h) because $i_2 \geq i_1 + h \geq i_1 + H$, and Claim 4 is proved.

Claim 5. Let $\{(v_n), (k_n)\} \in (\mathrm{P}_2, h)$ with $h \geq 2$. Then $v_1, v_2, \ldots, v_{h+1}$ can be explicitly determined using 2-adic expansion as follows:

(i) Each positive integer written 2-adically ending with an odd number of 0's cannot represent any of $v_1, v_2, \ldots, v_{h+1}$.

In the 2-adic expansion, for $n = 1, 2, \ldots, h$,

(ii) if $v_n = \boxtimes 1\underbrace{00\ldots00}_{2m \text{ terms}}$, then $v_{n+1} = \boxtimes 1\underbrace{00\ldots01}_{2m \text{ terms}}$,

(iii) if $v_n = \boxtimes 0\underbrace{11\ldots11}_{2m \text{ terms}}$, then $v_{n+1} = \boxtimes 1\underbrace{00\ldots00}_{2m \text{ terms}}$, and

(iv) if $v_n = \boxtimes 0\,\underbrace{11\ldots11}_{2m-1 \text{ terms}}$, then $v_{n+1} = \boxtimes 1\,\underbrace{00\ldots01}_{2m-1 \text{ terms}}$.

Proof of **Claim 5.** From numerical values displayed in the following table, we see that

valid for $h \geq$		base 2
	$v_1 = 1$	01
1	$v_2 = 3$	11
2	$v_3 = 4$	100
3	$v_4 = 5$	101
4	$v_5 = 7$	111
5	$v_6 = 9$	1001
6	$v_7 = 11$	1011
7	$v_8 = 12$	1100
8	$v_9 = 13$	1101
9	$v_{10} = 15$	1111
10	$v_{11} = 16$	10000
11	$v_{12} = 17$	10001
12	$v_{13} = 19$	10011
13	$v_{14} = 20$	10100
14	$v_{15} = 21$	10101
15	$v_{16} = 23$	10111
16	$v_{17} = 25$	11001
17	$v_{18} = 27$	11011
18	$v_{19} = 28$	11100
19	$v_{20} = 29$	11101
20	$v_{21} = 31$	11111

the results (i)-(iv) hold for v_n written 2-adically with ≤ 5 digits. Note that (i) follows from (ii)-(iv), so we need only check (ii)-(iv). Assume the 2-adic shape of $v_1, v_2, \ldots, v_n$. Since $1 \leq n \leq h$, by Property P_2, then v_{n+1} is the first integer greater than v_n which is not a double of v_1, or $v_2, \ldots$, or v_n. The three assertions in (ii)-(iv) follows immediately by induction, which ends the proof of Claim 5.

The subcase II.1 is now eliminated by the following claim:

Claim 6.

(i) If $v_{h+1} = \boxtimes 1\underbrace{00\ldots00}_{2m \text{ terms}}$, then $2v_1 + v_{h+1} = 2v_n$ for some $1 < n \leq h+1$.

(ii) If $v_{h+1} = \boxtimes 0\, \underbrace{11\ldots11}_{2m-1 \text{ terms}}$, then $v_1 + v_{h+1} = 2v_n$ for some $1 < n \leq h+1$.

(iii) If $v_{h+1} = \boxtimes 0\underbrace{11\ldots11}_{2m \text{ terms}}$, then $v_2 + v_{h+2} = 2v_n$ for some $1 < n \leq h+1$.

Proof of **Claim 6.**

(i) Writting in base 2, we have

$$2v_1 + v_{h+1} = 2\,(01) + \boxtimes 1\underbrace{00\ldots00}_{2m \text{ terms}} = \boxtimes 1\underbrace{00\ldots010}_{2m \text{ terms}} = 2\left(\boxtimes 1\, \underbrace{00\ldots01}_{2m-1 \text{ terms}}\right) = 2v_n$$

for some $1 < n \leq h+1$.

(ii) Writting in base 2, we have

$$v_1 + v_{h+1} = 01 + \boxtimes 0\, \underbrace{11\ldots11}_{2m-1 \text{ terms}} = \boxtimes 1\, \underbrace{00\ldots00}_{2m-1 \text{ terms}} = 2\left(\boxtimes 1\, \underbrace{00\ldots00}_{2m-2 \text{ terms}}\right) = 2v_n$$

for some $1 < n \leq h+1$.

(iii) We first show that

$$v_{h+2} = \boxtimes 1\underbrace{00\ldots011}_{2m \text{ terms}}.$$

Recall from Lemma 1 that v_{h+2} is the least natural number which cannot be written in the form (M1) using only $v_1, v_2, \ldots, v_{h+1}$ with the difference in the first two indices being $\geq h$. Since

$$v_1 + v_{h+1} = 01 + \boxtimes 0\underbrace{11\ldots11}_{2m \text{ terms}} = \boxtimes 1\underbrace{00\ldots00}_{2m \text{ terms}},$$

and

$$2v_1 + v_{h+1} = 2\,(01) + \boxtimes 0\underbrace{11\ldots11}_{2m \text{ terms}} = \boxtimes 1\underbrace{00\ldots01}_{2m \text{ terms}},$$

then v_{h+2} is not equal in value to $v_1 + v_{h+1}$, $2v_1 + v_{h+1}$. Now

$$v_2 + v_{h+1} = 11 + \boxtimes 0\underbrace{11\ldots 11}_{2m \text{ terms}} = \boxtimes 1\underbrace{00\ldots 010}_{2m \text{ terms}} = 2\left(\boxtimes 1 \underbrace{00\ldots 01}_{2m-1 \text{ terms}}\right).$$

Thus $v_2 + v_{h+1}$ is equal to a double of some previous v_n, so that v_{h+2} is not equal in value to $v_2 + v_{h+1}$. Since

$$v_3 + v_{h+1} = 100 + \boxtimes 0\underbrace{11\ldots 11}_{2m \text{ terms}} = \boxtimes 1\underbrace{00\ldots 011}_{2m \text{ terms}},$$

then $v_3 + v_{h+1}$ is not equal to a double of some previous v_n. Thus the value of

$$v_{h+2}\,(= v_3 + v_{h+1}) = \boxtimes 1\underbrace{00\ldots 011}_{2m \text{ terms}},$$

and so

$$v_2 + v_{h+2} = 11 + \boxtimes 1\underbrace{00\ldots 011}_{2m \text{ terms}} = \boxtimes 1\underbrace{00\ldots 0110}_{2m \text{ terms}} = 2\left(\boxtimes 1\underbrace{00\ldots 011}_{2m-1 \text{ terms}}\right) = 2v_n$$

for some $1 < n \leq h+1$, which completes the proof of Claim 6.

Subcase II.2.

$$(k_n) = \left(\underbrace{1, 1, \ldots, 1}_{f-1 \text{ terms } (f \geq 2)},\ h \geq 2, \ldots\right),$$

where $f \geq 2$.

This subcase is eliminated by Claim 3 in subcase I.2.

To sum up, we must have $(k_n) = (k_n^{(1)}) = (1, 1, 1, 1, \ldots)$ and by Theorem 1, $(v_n) = \big((l+1)^{n-1}\big)$. □

Theorem 3. *Let the increasing sequence (v_n) have the Property P_l^*. For each natural number n, let Ψ_n be the average number of summands required in the representation (M2) for all natural numbers N such that $v_n \leq N < v_{n+1}$. Then*

$$\Psi_n = \frac{nl+1}{l+1} \quad \text{and} \quad \lim_{n\to\infty} \frac{\Psi_n}{n} = \frac{l}{l+1}.$$

Proof. For each natural number N, let $f(N)$ be the unique number of summands d used in representing N in the form (M2). Since Ψ_n is defined as the average value of $f(N)$ for those numbers N such that $v_n \leq N < v_{n+1}$, we have

$$\Psi_n = \frac{\sum\limits_{v_n \leq N < v_{n+1}} f(N)}{v_{n+1} - v_n}.$$

Let us first compute a few values of n.

$$\Psi_1 = \frac{\sum_{1 \le N < l+1} f(N)}{l+1-1} = \frac{l}{l} = 1.$$

For

$$\Psi_2 = \frac{\sum_{l+1 \le N < (l+1)^2} f(N)}{(l+1)^2 - (l+1)},$$

N is of the form $av_2 + bv_1 = a(l+1) + b$; $a \in \{1, 2, \ldots, l\}, b \in \{0, 1, \ldots, l\}$, so

$$\Psi_2 = \frac{l + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} l^2}{l^2 + l}.$$

For arbitrary Ψ_n, N is of the form $a_n v_n + \cdots + a_1 v_1$; $a_n \in \{1, 2, \ldots, l\}$; $a_{n-1}, \ldots, a_1 \in \{0, 1, \ldots, l\}$. Each summand of k terms, $1 \le k \le n$, contributes to Ψ_n with

$$k \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} l^k.$$

Hence

$$\Psi_n = \frac{\sum_{k=1}^{n} k \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} l^k}{(l+1)^{n-1} l} = \frac{nl+1}{l+1},$$

and $\frac{\Psi_n}{n} \to \frac{l}{l+1} (n \to \infty)$. □

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