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FOURIER MULTIPLIERS ON CERTAIN HARDY SPACES

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ABSTRACT. We prove de Leeuw's restriction theorem result Jodeit Jr. [4] for multipliers on H^p spaces, $p < 1$.

1. INTRODUCTION

Let $0 < p < 1$. We denote the quasi norm

$$\Big(\sup_{\alpha>0} \alpha^p |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|\Big)^{1/p}$$

of f in $L^{p,\infty}$ by $||f||_{L^{p,\infty}}$, and the inverse Fourier transform of f by $f^\vee$. Let $T_m f = m^\vee * f$. We define the class of Fourier multipliers $\mathcal{M}(H^p, L^{p,\infty})(\mathbb{R}^n)$ to be the set of all bounded measurable functions m so that for all $f \in C_0^\infty(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$,

$$||T_m f||_{L^{p,\infty}} \leq C||f||_{H^p}.$$

The best constant C is the quasi norm of the operator T_m, and we write $||m||_{\mathcal{M}}$ for this quantity.

Let $1 \leq p < 2$. We define the class of Fourier multipliers $\mathcal{M}(L^p_{rad}, L^{p,\infty})$ to be the set of all bounded measurable functions m so that for all $f \in C_0^\infty(\mathbb{R}^k) \cap L^p_{rad}(\mathbb{R}^k)$,

$$||T_m f||_{L^{p,\infty}} \leq C||f||_{L^p_{rad}}.$$

The best constant C is the norm of the operator T_m, and we write $||m||_{\mathcal{M}}$ for this quantity.

We now split $\mathbb{R}^{k+l} = \mathbb{R}^k \oplus \mathbb{R}^l$, and denote by $L^p_{rad}(L^p)$ the space of all measurable functions f of the form $f(x', x'') = g(|x'|, x'')$ where g is defined on $(0, \infty) \times \mathbb{R}^l$, for

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which

$$||f||_{L^p_{rad}(L^p)} = \left(\int_{\mathbb{R}^l} \int_0^\infty |g(s,z)|^p s^{k-1} \, ds \, dz \right)^{1/p}$$

is finite.

We define the class of Fourier multipliers $\mathcal{N}(L^p_{rad}(L^p), L^{p,\infty})$ as the set of all bounded measurable functions m so that for all $f(x', x'') = g(|x'|, x'') \in C_0^\infty(\mathbb{R}^{k+l})$,

$$||T_m f||_{L^{p,\infty}} \leq D \, ||f||_{L^p_{rad}(L^p)}.$$

The best constant D is the quasi norm of the operator T_m, and we write $||m||_{\mathcal{N}}$ for this quantity.

In Hong [2], the convolution operator generated by Fourier multipliers supported on the cone is of weak type (p,p) on $H^p(\mathbb{R}^{n+1})$, $0 < p < 1$ for the critical value $\delta_p = n(1/p - 1/2) - 1/2$. By de Leeuw type restriction theorem (see appendix in Hong [2]), this estimate implies the known result that the Bochner-Riesz means of the critical index $\delta_p = n(1/p - 1/2) - 1/2$ is of weak type (p,p) for functions in $H^p(\mathbb{R}^n)$ (see Stein, Taibleson & Weiss [7]).

Similarly, in Hong [3] the convolution operator associated with a localized height of cone multipliers is of weak type (p,p) for the functions of the form $f(x,t) = g(|x|,t)$ if $p = 2n/(n+1+2\delta)$ and $0 < \delta \leq (n-1)/2$. Then by de Leeuw type restriction theorem (see appendix in Hong [3]), this result implies the weak type endpoint estimate for the Bochner-Riesz means on radial functions in $L^p(\mathbb{R}^n)$ where $p = 2n/(n+1+2\delta)$ and $0 < \delta \leq (n-1)/2$, which is proved by Chanillo & Muckenhoupt [1]. For the related result on real Hardy spaces, see also Liu [5].

In the above examples, Fourier multipliers are continuous. Here, the purpose of this article is to prove de Leeuw's restriction theorem Jodeit Jr. [4] when Fourier multipliers are in the class L^∞, and belong to the class $\mathcal{M}(H^p, L^{p,\infty})(\mathbb{R}^{k+l})$, $p < 1$. Here H^p is the standard real Hardy space as defined in Stein [6]. Further, we also consider the restriction theorem when Fourier multipliers are in the class L^∞, and contained in the class $\mathcal{N}\big(L^p_{rad}(L^p), L^{p,\infty}\big)$, $1 \leq p < 2$.

Theorem 1. *Let $0 < p < 1$. Let $m(\xi', \xi'')$ be contained in the class*

$$\mathcal{M}(H^p, L^{p,\infty})(\mathbb{R}^{k+l}).$$

Then for almost every ξ'',

$$m_{\xi''}(\xi') \equiv m(\xi', \xi'')$$

is contained in the class $\mathcal{M}(H^p, L^{p,\infty})(\mathbb{R}^k)$ *and the multiplier norm of* $m_{\xi''}$ *does not exceed that of* m.

Remark 1. It is not known whether m can be contained in the class

$$\mathcal{M}(H^1, L^{1,\infty})(\mathbb{R}^{k+l}).$$

Corollary 1. *Let* $1 \le p < 2$. *Let* $m(\xi', \xi'')$ *be contained in the class*

$$\mathcal{N}\big(L^p_{rad}(L^p), L^{p,\infty}\big).$$

Then for almost every ξ'',

$$m_{\xi''}(\xi') \equiv m(\xi', \xi'')$$

is contained in the class $\mathcal{M}(L^p_{rad}, L^{p,\infty})$ *and the multiplier norm of* $m_{\xi''}$ *does not exceed that of* m.

Remark 2. When $p = 2$, there is an example that Fourier multiplier m is not contained in the class $\mathcal{N}\big(L^p_{rad}(L^p), L^{p,\infty}\big)$ (see Hong [3]).

2. Restriction theorem on H^p spaces, $p < 1$

An atom is defined as follows: Let $0 < p \le 1$ and s be an integer that satisfies $s \ge n(1/p - 1)$. A (p, s)-atom is a function $\mathfrak{a}$ which is supported on a cube Q_j with center x_j, and which satisfies

(i) $|\mathfrak{a}(x)| \le |Q_j|^{-1/p}$;

(ii) $\displaystyle\int_{\mathbb{R}^n} \mathfrak{a}(x)\, x^\gamma\, dx = 0$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a n-tuple of non-negative integers satisfying $|\gamma| \le \gamma_1 + \gamma_2 + \cdots + \gamma_n \le s$, and $x^\gamma = x^{\gamma_1} x^{\gamma_2} \cdots x^{\gamma_n}$.

If $\{\mathfrak{a}_j\}$ is a collection of (p, s)-atoms and $\{c_j\}$ is a sequence of complex numbers with $\sum_{j=1}^\infty |c_j|^p < \infty$, then the series $f = \sum_{j=1}^\infty c_j \mathfrak{a}_j$ converges in the sense of distributions, and its sum belongs to H^p with the quasi norm

$$||f||_{H^p} = \inf_{\sum_{j=1}^\infty c_j \mathfrak{a}_j = f} \Big(\sum_{j=1}^\infty |c_j|^p\Big)^{1/p}$$

(see Stein [6]).

Lemma 1. *Suppose* f_ϵ *and* f *are measurable functions on* $\mathbb{R}^n$ *and* $f_\epsilon \to f$ *almost everywhere. Assume that* $||f_\epsilon||_{L^{p,\infty}} \le C^{1/p}$ *for some* $C > 0$ *and for all* $\epsilon > 0$. *Let* $\alpha > 0$ *be fixed. Then* $\alpha^p |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \le C$.

Proof. See Hong [2, 3] □

Proposition 1. *Let $f(x',x'') = (f_1 \otimes f_2)(x',x'')$ for $f_1 \in C_0^\infty(\mathbb{R}^k) \cap H^p(\mathbb{R}^k)$ and $f_2 \in C_0^\infty(\mathbb{R}^l)$. Then*

$$||f||_{H^p(\mathbb{R}^{k+l})} \leq A\,||f_1||_{H^p(\mathbb{R}^k)}, \tag{1}$$

where A depends on the support of f_2.

Proof. We first consider the atomic decomposition of f_2. Suppose supp $f_2 \subset K$, where K is a compact set. Fix a positive smooth function ζ that equals 1 in the cube of side length 1 centered at the origin and vanishes out the concentric cube of side length 1. We set

$$\zeta_j(x) = \zeta\Big(\frac{|x - x_j|}{2^{R_j}}\Big),$$

where x_j is the center of the cube Q_j and 2^{R_j} is the side length. Write

$$\eta_j = \frac{\zeta_j}{\sum_k \zeta_k}, \quad \text{and} \quad Q_j^*$$

is the cube concentric with Q_j and with sides twice the length. The ζ_j form a partition of unity for the set K subordinate to the finite cover $\{Q_j^*\}_{j=1,\dots,n}$ of K, that is $\chi_K = \sum_{j=1}^n \eta_j$ with each η_j supported in the cube Q_j.

Set

$$f_j = f\eta_j \cdot c^{-1}\,|Q_j|^{-1/p}, \quad \text{and} \quad \lambda_j = |Q_j|^{1/p}.$$

It is clear from this that f_j is supported in Q_j and that $|f_j| \leq |Q_j|^{-1/p}$ there. Also it is easily seen that $\sum_{j=1}^n |\lambda_j|^p = \sum_{j=1}^n |Q_j|$. So $f_2 = \sum_{j=1}^n \lambda_j f_j$ is an atomic decomposition, except that f_j does not satisfy the cancellation property.

Fix a smooth function Φ on $B(0,1)$ (the unit ball about the origin) with

$$\int_{\mathbb{R}^l}\int_{\mathbb{R}^k} \Phi(x',x'')\,dx'\,dx'' \neq 0.$$

For $t > 0$, set $\Phi_t(x) = t^{-n}\,\Phi(x/t)$, so that

$$\int_{\mathbb{R}^l}\int_{\mathbb{R}^k} \Phi_t(x',x'')\,dx'\,dx'' \neq 0$$

for all t. Now let

$$M_\Phi f(x',x'') = \sup_{t>0} |(f * \Phi_t)(x',x'')|.$$

Suppose that f_1 is (p,N)-atom $\big(N \geq k(1/p - 1)\big)$ on $\mathbb{R}^k$ which is supported in a cube Q_1 of side length 2^{R_1} centered at $\overline{x'}$, and f_2 is an atom on $\mathbb{R}^l$ which is supported in the cube Q_2 of side length 2^{R_2} centered at $\overline{x''}$. Then $f_1 \otimes f_2$ is a $(p,\widetilde{N})$-atom

$\big(\widetilde{N} \geq (k+l)(1/p-1)\big)$ on $\mathbb{R}^k \times \mathbb{R}^l$, which is supported in a cube $Q_1 \otimes Q_2$ of diameter $(2^{R_1} + 2^{R_2})^{1/2}$ centered at $(\overline{x'}, \overline{x''})$.

Consider the case $(x', x'') \in (Q_1 \otimes Q_2)^*$. Since $M_\Phi f(x', x'') \leq C\,|Q_1 \otimes Q_2|^{-1/p}$, then

$$\int_{x'}\int_{x''} [M_\Phi f(x', x'')]^p \, dx' \, dx'' \leq C. \tag{2.1}$$

Next, consider the case $(x', x'') \in \{(Q_1 \otimes Q_2)^*\}^c$. Using the momentum conditions on $f_1 \otimes f_2$, that is

$$\int_{\mathbb{R}^l}\int_{\mathbb{R}^k} (x' \otimes x'')^\beta \, (f_1 \otimes f_2)(x', x'') \, dx' \, dx'' = 0$$

for all β with $|\beta| \leq \widetilde{N}$, we have

$$\begin{aligned}(f * \Phi_t)&(x', x'') \\ &= \int_{\mathbb{R}^l}\int_{\mathbb{R}^k} (f_1 \otimes f_2)(y', y'') \left[\Phi_t(x' - y', x'' - y'') - P_{x', x'', t}(y', y'')\right] dy' \, dy'',\end{aligned}$$

where $P_{x',x'',t}(y', y'')$ is the $\widetilde{N}$-th order Taylor polynomial of the function $(y', y'') \to \Phi_t(x' - y', x'' - y'')$ expanded about $(\overline{x'}, \overline{x''})$; here $\widetilde{N}$ is the smallest integer with $\widetilde{N} \geq (k + l)(1/p - 1)$.

Now, by the usual estimate of the remainder term in a Taylor expansion,

$$|\Phi_t(x' - y', x'' - y'') - P_{x',x'',t}(y', y'')| \leq C \, \frac{|(y' - \overline{x'}, y'' - \overline{x''})|^{\widetilde{N}+1}}{t^{k+l+\widetilde{N}+1}},$$

and since $(y', y'') \in Q_1 \otimes Q_2$, $(x', x'') \in \{(Q_1 \otimes Q_2)^*\}^c$ and Φ is supported in $B(0, 1)$, we have that $t > c|(x' - \overline{x'}, x'' - \overline{x''})|$. Hence

$$M_\Phi(f_1 \otimes f_2)(x', x'') \leq C \left(\frac{(2^{R_1} + 2^{R_2})^{1/2}}{|(x' - \overline{x'}, x'' - \overline{x''})|} \right)^{k+l+\widetilde{N}+1} |Q_1 \otimes Q_2|^{-1/p},$$

and since $(k + l + \widetilde{N} + 1)p > k + l$, we get (2.1) for $(x', x'') \in \{(Q_1 \otimes Q_2)^*\}^c$.

Finally, let $\{f_i\}$ be a collection of H^p atoms and $\{\lambda_i\}$ be a sequence of complex numbers with $\sum_i \lambda_i < \infty$. Set $f_1(x') = \sum_i \lambda_i f_i(x')$ and $f_2(x') = \sum_j \lambda_j f_j(x'')$. Since

$p < 1$, we have

$$\begin{aligned}\int_{x'}\int_{x''}&[M_\Phi f(x',x'')]^p\,dx'\,dx''\\ &\le \sum_i |\lambda_i|^p \sum_j |\lambda_j|^p \int_{x'}\int_{x''}[M_\Phi(f_i\otimes f_j)(x',x'')]^p\,dx'\,dx''\\ &\le C\sum_i |\lambda_i|^p \sum_j |\lambda_j|^p \;\le\; A\,||f_1||^p_{H^p(\mathbb{R}^k)},\end{aligned}$$

where $A = C\sum_j |\lambda_j|^p = C\sum_j |Q_j|$. □

Now we proceed to the proof of the restriction theorem. The method of proof is an adaptation of the argument in Hong [2].

Proof of **Theorem 1.** Let $f_1 \in C_0^\infty(\mathbb{R}^k)\cap H^p(\mathbb{R}^k)$ and $f_{2,\epsilon} \in C_0^\infty(\mathbb{R}^l)$ with $\widehat{f_{2,\epsilon}}(\xi'') = \epsilon^{l(1/p-1)}\phi(\frac{\xi''-a}{\epsilon})$ and supp $\phi \subset B(0,1)$. Set $f_\epsilon(x',x'') = (f_1\otimes f_{2,\epsilon})(x',x'')$.

Suppose that m is continuous. A straight forward calculation gives

$$T_m f_\epsilon(x',x'') = T_{m^\epsilon}(f_1\otimes \epsilon^{l/p}\check{\phi})(x',\epsilon x'')e^{i<x'',a>}$$

where $m^\epsilon(\xi',\xi'') = m(\xi',\epsilon\xi''+a)$. By (1) in Proposition 1 we have

$$||T_{m^\epsilon}(f_1\otimes\check{\phi})||_{L^{p,\infty}} \le C_{\check{\phi}}\,||m||_{\mathcal{M}}||f_1||_{H^p}. \tag{2.2}$$

Applying the Lebesgue dominated convergence theorem, we see that $T_{m^\epsilon}(f_1\otimes\check{\phi})$ converges to $(T_{m_a}f_1)\otimes\check{\phi}$ as $\epsilon\to 0$ where $m_a(\xi') = m(\xi',a)$. Thus, from Lemma 1 we have

$$||(T_{m_a}f_1)\otimes\check{\phi}||_{L^{p,\infty}} \le C_{\check{\phi}}\,||m||_{\mathcal{M}}||f_1||_{H^p}. \tag{2.3}$$

Suppose now that m is in the class L^∞. Let S denote the unit cube and set $\psi_\delta(\xi',\xi'') = \frac{1}{\delta^{k+l}}\chi_S(\frac{\xi'}{\delta},\frac{\xi''}{\delta})$. Now $\psi_\delta \in L^1$ and $m \in L^\infty$ yield that $\psi_\delta * m$ is continuous. Clearly,

$$T_{\psi_\delta * m}f_\epsilon(x',x'') = T_{\psi_\delta * m^\epsilon}(f_1\otimes\,\epsilon^{l/p}\,\check{\phi})(x',\epsilon x'')\,e^{i<x'',a>},$$

and from $||\psi_\delta||_{L^1} = 1$ and (2.2) we have

$$\begin{aligned}||T_{\psi_\delta * m^\epsilon}(f_1\otimes\check{\phi})||_{L^{p,\infty}} &\le ||\psi_\delta||_{L^1}\,||T_{m^\epsilon}(f_1\otimes\check{\phi})||_{L^{p,\infty}}\\ &\le C_{\check{\phi}}\,||m||_{\mathcal{M}}||f_1||_{H^p}.\end{aligned}$$

Likewise, $T_{\psi_\delta * m^\epsilon}(f_1\otimes\check{\phi})$ converges to $(T_{\psi_\delta * m_a}f_1)\otimes\check{\phi}$ as $\epsilon\to 0$ by the Lebesgue dominated convergence theorem. Moreover, for fixed ξ'' the definition of Lebesgue point ξ' is that

$$\begin{aligned}(2.4)\qquad \lim_{\delta\to 0}\psi_\delta * m(\xi',\xi'') &= \lim_{\delta\to 0}\frac{1}{\delta^{k+l}}\iint_{S_\delta} m(\xi'-\eta',\xi''-\eta'')\,d\eta'\,d\eta'' \\ &= m(\xi',\xi''),\end{aligned}$$

where S_δ is the cube of side length δ centered at the origin. By applying the Lebesgue dominated convergence theorem again and (2.4)

$$\lim_{\delta\to 0} T_{\psi_\delta * m_a}(f_1\otimes\check{\phi}) = T_{m_a}(f_1\otimes\check{\phi}),$$

and by (2.3)

$$\|\lim_{\delta\to 0} T_{\psi_\delta * m_a}(f_1\otimes\check{\phi})\|_{L^{p,\infty}} \le C_{\check{\phi}}\,\|m\|_{\mathcal{M}}\|f_1\|_{H^p}.$$

Therefore, m_a is contained in the class $\mathcal{M}(H^p, L^{p,\infty})(\mathbb{R}^k)$ and thus

$$\|m_a\|_{\mathcal{M}} \le \|m\|_{\mathcal{M}}.$$

This completes the proof. □

Remark 3. In order to prove Corollary 1, We take $f_1 \in C_0^\infty(\mathbb{R}^k)\cap L^p_{rad}(\mathbb{R}^k)$ and $f_{2,\epsilon}\in C_0^\infty(\mathbb{R}^l)$ with

$$\widehat{f_{2,\epsilon}}(\xi'') = \epsilon^{l(1/p-1)}\frac{\phi(\frac{\xi''-a}{\epsilon})}{\|\check{\phi}\|_p}$$

and supp $\phi\subset B(0,1)$. The remaining part follows the similar argument as above.

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