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THE CENTRAL LIMIT THEOREMS FOR THE MULTIVARIATE LINEAR PROCESSES GENERATED BY NEGATIVELY ASSOCIATED RANDOM VECTORS

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ABSTRACT. Let $\{\mathbb{X}_t\}$ be an m-dimensional linear process of the form

$$\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \mathbb{Z}_{t-j},$$

where $\{\mathbb{Z}_t\}$ is a sequence of stationary m-dimensional negatively associated random vectors with $E\mathbb{Z}_t = \mathbb{O}$ and $E||\mathbb{Z}_t||^2 < \infty$. In this paper we prove the central limit theorems for multivariate linear processes generated by negatively associated random vectors.

1. INTRODUCTION

Notions of negative dependence for collections of random variables have been much studied in recent years. The most prevalent negatively dependent notion is that of negative association. A finite collection $\{Y_i, 1 \le i \le m\}$ of random variables is said to be negatively associated (NA) if for any disjoint subsets A, B of $\{1, 2, \ldots, m\}$ and for all coordinatewise nondecreasing functions

$$f : \mathbb{R}^A \to \mathbb{R},\ g : \mathbb{R}^B \to \mathbb{R}\, \mathrm{Cov}\big(f(Y_i : i \in A), g(Y_j : j \in B)\big) \le 0,$$

where the covariance is defined. An infinite collection of random variables is negatively associated if every finite subcollection is negatively associated. This negatively dependent notion was first defined by Joag-Dev & Proschan [6]. Negatively associated sequences are widely encountered in multivariate statistical analysis and reliability theory, and the notions of negative association have more attention recently.

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We refer to Joag-Dev & Proschan [6] for fundamental properties of negatively associated sequences, Newman [8], Birkel [2] and Zhang [10] for the central limit theorem, Matula [7] for the three series theorem, Shao [9] for the Rosenthal-type maximal inequality and the Kolmogorov exponential inequality.

Let $\mathbb{X}_t, t = 0, \pm 1, \ldots$, be an m-dimensional linear process of the form

$$\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \mathbb{Z}_{t-j} \tag{1}$$

defined on a probability space (Ω, A, P), where $\mathbb{Z}_t, t = 0, \pm 1, \ldots$, is a sequence of strictly stationary m-dimensional random vectors with mean $\mathbb{0} : m \times 1$ and positive definite covariance matrix $\Gamma : m \times m$. The class of linear processes defined in (1) contains stationary multivariate autoregressive moving average processes (MARMA) that satisfy certain condition (See Brockwell & Davis [3]). Fakhre-Zakeri & Farshidi [4] established a central limit theorem for linear process generated by iid random variables, and Fakhre-Zakeri & Lee [5] derived a central limit theorem for multivariate linear process generated by martingale difference random vectors.

In this paper we introduce the notion of negatively associated random vectors and prove the central limit theorems for stationary multivariate linear processes generated by negatively associated random vectors.

2. RESULTS

Definition 2.1. A finite sequence $\{\mathbb{Z}_t, 1 \leq t \leq n\}$ of m-dimensional random vectors is said to be negatively associated if for any disjoint subsets A,B of $\{1, \ldots, n\}$ and for all coordinatewise nondecreasing functions f and g we have

$$\mathrm{Cov}\big(f(\mathbb{Z}_i : i \in A),\ g(\mathbb{Z}_j : j \in B)\big) \leq 0, \tag{2}$$

whenever this covariance is defined. Infinite collection of m-dimensional random vectors is negatively associated if every finite subcollection is negatively associated.

Lemma 2.2. *Let $\{Y_1, \cdots, Y_n\}$ be a strictly sationary sequence of negatively associated random variables with $EY_1 = 0, EY_1^2 < \infty$. Then*

$$E(\max_{1 \leq k \leq n} |Y_1 + \cdots + Y_k|^2) \leq AnEY_1^2$$

where A is a positive constant.

Proof. See the proof of Lemma 4 of Matula [7]. □

Lemma 2.3. *Let $\{\mathbb{Z}_t : 1 \le t \le n\}$ be a strictly stationary sequence of negatively associated m-dimensional random vectors with $E\mathbb{Z}_1 = 0$ and $E \parallel \mathbb{Z}_1 \parallel^2 < \infty$, where for a vector $x \in \mathbb{R}^m$, denote its Euclidean norm by $\|x\|$. Then, there is a positive constant A such that*

$$E \max_{1\le k\le n} \parallel \sum_{t=1}^{k} \mathbb{Z}_t \parallel^2 \le Am^2 nE \parallel \mathbb{Z}_1 \parallel^2 . \tag{3}$$

Proof. Note that

$$\max_{1\le k\le n} \parallel \sum_{t=1}^{k} \mathbb{Z}_t \parallel^2 \le m \sum_{j=1}^{m} \max_{1\le k\le n} \mid \sum_{t=1}^{k} Z_t^{(j)} \mid^2$$

and by Lemma 2.2,

$$\begin{aligned} E \max_{1\le k\le n} \mid \sum_{t=1}^{k} Z_t^{(j)} \mid^2 &\le A \sum_{t=1}^{n} E \mid Z_t^{(j)} \mid^2 \\ &\le A \sum_{t=1}^{n} E \parallel \mathbb{Z}_t \parallel^2 \\ &= AnE \parallel \mathbb{Z}_1 \parallel^2, \end{aligned}$$

where $Z_t^{(j)}$ is the j-th component of $\mathbb{Z}_t$. Thus (3) follows. □

Lemma 2.4. *Let $\{\mathbb{Z}_t, t \ge 1\}$ be a strictly stationary sequence of negatively associated m-dimensional random vectors with $E(\mathbb{Z}_1) = \mathbb{O}, E||\mathbb{Z}_1||^2 < \infty$. Let*

$$\mathbb{X}_t = \sum_{j=1}^{\infty} A_j \mathbb{Z}_{t-j}, \qquad \mathbb{S}_k = \sum_{t=1}^{k} \mathbb{X}_t,$$

$$\tilde{\mathbb{X}}_t = \Big(\sum_{j=1}^{\infty} A_j\Big)\mathbb{Z}_t \quad \textit{and} \quad \tilde{\mathbb{S}}_k = \sum_{t=1}^{k} \tilde{\mathbb{X}}_t.$$

Assume

$$\sum_{j=1}^{\infty} ||A_j|| < \infty \text{ and } \sum_{j=1}^{\infty} A_j \ne \mathbb{O}_{m\times m}, \tag{4}$$

where for any $m \times m, m \ge 1$, matrix $A = (a_{ij}), ||A|| = \sum_{i=1}^{m} \sum_{j1}^{m} |a_{ij}|$ and $\mathbb{O}_{m\times m}$ denotes the $m \times m$ zero matrix. Then

$$n^{-\frac{1}{2}} \max_{1\le k\le n} ||\tilde{\mathbb{S}}_k - \mathbb{S}_k|| = o_p(1).$$

Proof. First observe that

$$\begin{aligned}\tilde{\mathbb{S}}_k &= \sum_{t=1}^{k}\left(\sum_{j=0}^{k-t} A_j\right)\mathbb{Z}_t + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\ &= \sum_{t=1}^{k}\left(\sum_{j=0}^{t-1} A_j\mathbb{Z}_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t\end{aligned}$$

and thus,

$$\begin{aligned}\tilde{\mathbb{S}}_k - \mathbb{S}_k &= -\sum_{t=1}^{k}\left(\sum_{j=t}^{\infty} A_j\mathbb{Z}_{t-j}\right) + \sum_{t=1}^{k}\left(\sum_{j=k-t+1}^{\infty} A_j\right)\mathbb{Z}_t \\ &= I_1 + I_2 \text{ (say)}.\end{aligned}$$

To prove

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_1\| = o_p(1), \tag{5}$$

note that

$$\begin{aligned}&n^{-1}E \max_{1\le k\le n}\left\|\sum_{t=1}^{k}\sum_{j=t}^{\infty} A_j\,\mathbb{Z}_{t-j}\right\|^2 \\ &= n^{-1}\,E \max_{1\le k\le n}\left\|\sum_{j=1}^{\infty}\sum_{t=1}^{j\wedge k} A_j\,\mathbb{Z}_{t-j}\right\|^2 \\ &\le n^{-1}\left(\sum_{j=1}^{\infty}\|A_j\|\left\{E \max_{1\le k\le n}\left\|\sum_{t=1}^{j\wedge k}\mathbb{Z}_{t-j}\right\|^2\right\}^{\frac{1}{2}}\right)^2 \text{ by Minkowski inequality} \\ &\le Am^2E\,\|\,Z_1\,\|^2\left[\sum_{j=1}^{\infty}\|A_j\|\left(\frac{j\wedge n}{n}\right)^{\frac{1}{2}}\right]^2\end{aligned}$$

by (3) and (4) and E $\|\,Z_1\,\|^2 < \infty$. By the dominated convergence theorem the last term above tends to zero as $n \longrightarrow \infty$. Thus (5) is proved by the Markov inequality. Next, we show that

$$n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_2\| = o_p(1). \tag{6}$$

Write

$$I_2 = II_1 + II_2,$$

where

$$II_1 = A_1\mathbb{Z}_k + A_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + A_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1)$$

and

$$II_2 = (A_{k+1} + A_{k+2} + \cdots)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_1).$$

Let p_n be a sequence of positive integers such that

$$p_n \to \infty \text{ and } p_n/n \to 0 \text{ as } n \to \infty. \tag{7}$$

Then

$$\begin{aligned} n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_2\| &\le \left(\sum_{i=0}^{\infty} \|A_i\|\right) n^{-\frac{1}{2}} \max_{1\le k\le p_n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ &\quad + \left(\sum_{i>p_n} \|A_i\|\right) n^{-\frac{1}{2}} \max_{1\le k\le n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ &= o_p\left(1\right) + o_p\left(\sum_{i>p_n} \|A_i\|\right) \\ &= o_p(1) \end{aligned}$$

by (4), (7) and $E \parallel \mathbb{Z}_1 \parallel^2 < \infty$. It remains to prove that

$$Y_n := n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_1\| = o_p(1).$$

To this end, define for each $l \ge 1$

$$II_{1,l} = B_1\mathbb{Z}_k + B_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + B_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1),$$

where

$$B_k = \begin{cases} A_k, & k \le l \\ \mathbb{O}_{m\times m}, & k > l. \end{cases}$$

Let $Y_{n,l} = n^{-\frac{1}{2}} \max_{1\le k\le n} \|II_{1,l}\|$. Clearly, for each $l \ge 1$,

$$Y_{n,l} = o_p(1). \tag{8}$$

On the other hand,

$$\begin{aligned} n(Y_{n,l} - Y_n)^2 &\le \max_{1\le k\le n} \left\| \sum_{i=1}^{k} (A_i - B_i)(\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}) \right\|^2 \\ &\le \max_{l<k\le n} \left(\sum_{i=l+1}^{k} \|A_i\| \cdot \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\| \right)^2 \\ &\le \left(\sum_{i>l} \|A_i\| \right)^2 \max_{l<k\le n} \max_{l<i\le k} \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\|^2 \end{aligned}$$

$$\leq 4\left(\sum_{i>l}\|A_i\|\right)^2 \max_{l\leq j\leq n}\|\mathbb{Z}_1+\cdots+\mathbb{Z}_j\|^2.$$

From this result, (4) and $E\parallel Z_1\parallel^2<\infty$, for any $\delta>0$,

$$\lim_{l\to\infty}\lim_{n\to\infty}\sup P(|Y_{n,l}-Y_n|^2>\delta)$$

$$\leq \lim_{l\to\infty}\lim_{n\to\infty}\sup 4\delta^{-1}\left(\sum_{i>l}\|A_i\|\right)^2 n^{-1}E\max_{1\leq j\leq n}\|\mathbb{Z}_1+\cdots+\mathbb{Z}_j\|^2$$

$$\leq 4Am^2\delta^{-1}E\parallel Z_1\parallel^2 \lim_{l\to\infty}\left(\sum_{i>l}\|A_i\|\right)^2=0. \quad (9)$$

In view of (8) and (9), it follows from Theorem 4.2 of Billingsley [1, p. 25] that $Y_n=o_p(1)$. This completes the proof of Lemma 2.4. □

Theorem 2.5. *Let $\{\mathbb{Z}_t : t\geq 1\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_1)=\mathbb{O}$ and $E\parallel\mathbb{Z}_1\parallel^2<\infty$.*

Let $S_n=\sum_{t=1}^n\mathbb{Z}_t$. If

$$E\parallel\mathbb{Z}_1\parallel^2+2\sum_{t=2}^{\infty}\sum_{j=1}^{m}EZ_1^{(j)}Z_t^{(j)}=\sigma^2<\infty \quad (10)$$

holds, then, as $n\to\infty$,

$$n^{-\frac{1}{2}}S_n\xrightarrow{D}N(\mathbb{O},\Gamma)$$

with covariance matrix $\Gamma=[\sigma_{kj}]$, $k=1,\ldots,m$; $j=1,\ldots,m$,

$$\sigma_{kj}=E(Z_1^{(k)}Z_1^{(j)})+\sum_{t=2}^{\infty}\left[E(Z_1^{(k)}Z_t^{(j)})+E(Z_1^{(j)}Z_t^{(k)})\right]. \quad (11)$$

Proof. By Theorem 12 in Newman [8] it follows from (10) that, for each $j(1\leq j\leq m)$,

$$\lim_{n\to\infty}n^{-\frac{1}{2}}\sum_{t=1}^{n}Z_t^{(j)}=\sigma_j'Z_j'$$

where Z_j' is standard normal and

$$\sigma_j'^2=E(Z_1^{(j)})^2+2\sum_{t=2}^{\infty}E(Z_1^{(j)}Z_t^{(j)})<\infty.$$

Hence by the Cramer-Wold device (See Billingsley [1, pp. 48–49]) the desired result follows. □

Theorem 2.6. *Let $\{\mathbb{Z}_t, t \geq 1\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_1) = \mathbb{O}, E||\mathbb{X}_t||^2 < \infty$ and $\{\mathbb{X}_t\}$ an m-dimensional linear process defined in* (1). *Set*

$$\mathbb{S}_n = \sum_{t=1}^{n} \mathbb{X}_t (\mathbb{S}_0 = \mathbb{O}),\ \tilde{\mathbb{S}}_n = \sum_{t=1}^{n} \tilde{\mathbb{X}}_t$$

as in Lemma 2.4. If (4) and (10) hold then

$$n^{-\frac{1}{2}} \mathbb{S}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T) \quad as \quad n \to \infty, \tag{12}$$

where $T = (\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$ and Γ is defined as in (11).

Proof. First note that

$$n^{-\frac{1}{2}} \tilde{\mathbb{S}}_n = n^{-\frac{1}{2}} \Big(\sum_{j=1}^{\infty} A_j \Big) \sum_{t=1}^{n} \mathbb{Z}_t$$

and that $n^{-\frac{1}{2}} \tilde{\mathbb{S}}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T)$ according to Theorem 2.5. Hence, $n^{-\frac{1}{2}} \mathbb{S}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T)$ follows by applying Lemma 2.4 and Theorem 4.1 of Billingsley [1]. □

We now introduce another central limit theorem.

Theorem 2.7. *Let $\{\mathbb{Z}_t, t \geq 1\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors with $E(\mathbb{Z}_1) = \mathbb{O}, E||\mathbb{Z}_1||^2 < \infty$ and let $\{\mathbb{X}_t\}$ be an m-dimensional linear process defined in* (1). *If*

$$\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} ||A_j|| < \infty \tag{13}$$

hold. Then

$$n^{-\frac{1}{2}} \tilde{\mathbb{S}}_n \xrightarrow{\mathcal{D}} N(\mathbb{O}, T) \quad as \quad n \to \infty,$$

where $T = (\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$ and Γ is defined as in (11).

Proof. Letting $\tilde{A}_i = \sum_{j=i+1}^{\infty} A_j$ and $\mathbb{Y}_t = \sum_{i=0}^{\infty} \tilde{A}_i \mathbb{Z}_{t-i}$, which is well defined since $\sum_{i=0}^{\infty} ||\tilde{A}_i|| < \infty$ by (13), we have

$$\begin{aligned} \mathbb{X}_t &= \Big(\sum_{i=0}^{\infty} A_i \Big) \mathbb{Z}_t - \tilde{A}_0 \mathbb{Z}_t + \sum_{i=1}^{\infty} (\tilde{A}_i - \tilde{A}_{i-1}) \mathbb{Z}_{t-i} \\ &= \Big(\sum_{i=0}^{\infty} A_i \Big) \mathbb{Z}_t + \mathbb{Y}_{t-1} - \mathbb{Y}_t \end{aligned}$$

which implies that

$$\mathbb{S}_n = \left(\sum_{i=0}^{\infty} A_i \right) \sum_{t=1}^{n} \mathbb{Z}_t + \mathbb{Y}_0 - \mathbb{Y}_n.$$

According to Theorem 2.5 we have $n^{-\frac{1}{2}} \sum_{t=1}^{n} \mathbb{Z}_t \to N(\mathbb{O}, \Gamma)$ as $n \to \infty$ and thus using this result on $(\sum_{i=0}^{\infty} A_i) \sum_{t=1}^{n} \mathbb{Z}_t$, we have

$$(\sum_{i=0}^{\infty} A_i) \sum_{t=1}^{n} \mathbb{Z}_t \to N(\mathbb{O}, T) \text{ as } n \to \infty.$$

Hence, this theorem is proved if

$$\frac{\mathbb{Y}_n}{\sqrt{n}} \xrightarrow{p} \mathbb{O} \text{ as } n \to \infty. \tag{14}$$

To prove (14) it is sufficient to show that

$$\frac{\mathbb{Y}_n}{\sqrt{n}} \to \mathbb{O} \ a.s. \text{ as } n \to \infty. \tag{15}$$

But (15) follows from the fact that for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\left(\frac{|Y_{n,j}|}{\sqrt{n}} > \epsilon\right) = \sum_{n=1}^{\infty} P(|Y_{0,j}| > \sqrt{n}\epsilon) < \infty,$$

for all j, where $Y_{n,j}$ denotes the j-th component of $\mathbb{Y}_n$. □

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