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GENERALIZED CR-SUBMANIFOLDS OF A T-MANIFOLD

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Abstract. The purpose of the present paper is to study the generalized CR-submanifold of a T-manifold. After preliminaries we have studied the integrability of the distributions and obtained the conditions for integrability. Then geometry of leaves are being studied. Finally it is proved that every totally umbilical generalized CR-submanifold of a T-manifold is totally geodesic.

1. Introduction

Bejancu [1, 2] defined and studied CR-submanifolds of Kaehlerian manifolds. CR-submanifolds of Sasakian manifold were studied by Kobayashi [7] and Shahid, Sharfuddin & Husain [11]. Chen [6] introduced the notion of a generic submanifold of a Kaehler manifold. Generic submanifolds of Sasakian manifolds were studied by many authors including Verheyen [12]. Blair [3] studied the metric the f-structure with complemented frames. If the stucture is normal and has closed fundamental 2-form (Φ), then f-structure is called a *K-structure*, as an analogue of Kaehler. If in addition each of the 1-forms $\eta_1, \eta_2, \ldots, \eta_s$ is closed, Blair [3] called it a *C-structure*, as an analogue of cosymplectic. If the K-structure satisfies $\Phi = d\eta_1 = d\eta_2 = \cdots = d\eta_s$, he called the K-structure a *T-structure* (*cf.* Blair [3]), as an analogue of Sasakian structure. It is possible to have a Darboux theorem in a T-structure. Mihai [8] have defined a new class of submanifolds called a *generalized CR-submanifold* of a Kaehler manifold. This class contains both CR-submanifolds and slant submanifolds. The Riemannian product of a slant submanifold and a totally real submanifold of a Hermitian manifold is a generalized CR-submanifold. Also the Riemannian product of a complex submanifold, a slant submanifold and a totally real submanifold of a Hermitian manifold is a generalized CR-submanifold. Mihai [9] also studied gereralised

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CR-submanifold of a Sasakian manifold. Sengupta & De [10] studied a generalized CR-submanifolds of a trans-Sasakian manifold. Calin [5] studied CR-submanifolds of a T-manifold.

The purpose of the present paper is to study the gereralised CR-submanifold of a T-manifold. In section 4 we have studied the integrability of the distributions and obtained the conditions for integrability. Geometry of leaves are being studied in Section 5. In Section 6 it is proved that every totally umbilical generalized CR-submanifold of a T-manifold is totally geodesic.

2. PRELIMINARIES

Let $\bar{M}$ be a $(2n+s)$-dimensional differentiable manifold of class C^∞ endowed with an f-structure of rank $2n$ (*cf.* Yano [13]). The f-structure ϕ is said to be *complemented frame* if there exist the structure vector fields ξ_α, $\alpha = 1, 2, \ldots, s$ and its dual 1-form η_α such that

$$\begin{cases} \phi^2 & = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha, \quad \phi(\xi_\alpha) = 0, \\ \eta_\alpha(\xi_\beta) & = \delta_{\alpha\beta} \ (\delta_{\alpha\beta} : \text{Kroneker delta}), \quad \eta_\alpha \circ \phi = 0, \end{cases} \tag{2.1}$$

$\alpha, \beta = 1, 2, \ldots, s$, where I is the indentity tensor of the tangent bundle $T(\bar{M})$ (*cf.* Blair [3]). The manifold $\bar{M}$ is said to have a *metric f-structure* if there exists a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y), \tag{2.2}$$

for $X, Y \in T(\bar{M})$.

The fundanmental 2-form Φ on $\bar{M}$ is given by

$$\Phi(X, Y) = g(X, \phi Y), \text{ for } X,\ Y \in T(\bar{M}).$$

The Nijenhuis torsion $N_\phi(X, Y)$ of ϕ is defined by

$$N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \text{ for } X,\ Y \in T(\bar{M}).$$

The manifold $\bar{M}$ is said to be a *K-manifold* if the fundamental 2-form is closed and the metric f-structure is normal (*cf.* Blair [3]), that is

$$N_\phi(X, Y) + 2\sum_{\alpha=1}^{s} d\eta_\alpha(X, Y)\xi_\alpha = 0, \text{ for all } X,\ Y \in T(\bar{M}).$$

A K-manifold with $d\eta_\alpha = 0, \alpha = 1, 2, \ldots, s$ is called a *T-manifold* (*cf.* Blair [3]).

In a T-manifold, we have (*cf.* Blair [3])

$$(\bar{\nabla}_X\phi)Y = 0 \tag{2.3}$$

and

$$\bar{\nabla}_X\xi_\alpha = 0, \quad \alpha = 1, 2, \ldots, s, \tag{2.4}$$

for all $X, Y \in T(\bar{M})$, where $\bar{\nabla}$ is the Levi-Civita connection with respect to metric g given by eq. (2.2).

Let M be an m-dimensional submanifold isometrically immersed in a T-manifold $\bar{M}$ such that the structure vector fields $\xi_\alpha, \alpha = 1, 2, \ldots, s$ are tangent to the submanifold M. We denote $\{\xi\} = \text{span}\{\xi_1, \ldots, \xi_s\}$ and by $\{\xi\}^\perp$ the complementary orthogonal distribution to $\{\xi\}$ in $T(M)$.
For any $X \in T(M)$ we have $g(\phi X, \xi_\alpha) = 0$, $\alpha = 1, 2, \ldots, s$.

Then we put

$$\phi X = bX + cX, \tag{2.5}$$

where $bX \in \{\xi\}^\perp$ and $cX \in T^\perp(M)$. Thus $X \to bX$ is an endomorphism of the tangent bundle $T(M)$ and $X \to cX$ is a normal bundle valued 1-form on M.

Definition 2.1. A submanifold M of a T-manifold $\bar{M}$ is said to be a *generalized CR-submanifold* if

$$D_x^\perp = T_x(M) \bigcap \phi T_x^\perp(M); \quad x \in M$$

defines a differentiable subbundle of $T_x(M)$.

Thus for $X \in D^\perp$ one has $bX = 0$. We denote by D the complementary orthogonal subbundle to $D^\perp \oplus \{\xi\}$ in $T(M)$. For any $X \in D$, $bX \neq 0$. Also we have $bD = D$.

Thus for a generalized CR-submanifold M we have

$$T(M) = D \oplus D^\perp \oplus \{\xi\}. \tag{2.6}$$

3. Basic Lemmas

Let M be a generalized CR-submanifold of the T-manifold $\bar{M}$. We denote by g both the Riemannian metrics on $\bar{M}$ and M. For each $X \in T(M)$ we put

$$X = PX + QX + \sum_{\alpha=1}^{s} \eta_\alpha(X)\xi_\alpha, \tag{3.1}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. For any $N \in T^{\perp}(M)$ we put

$$\phi N = tN + fN, \tag{3.2}$$

where tN is the tangential part of ϕN and fN is the normal part of ϕN.

By using (2.2) we have

$$g(\phi X, cY) = g(\phi X, \phi Y) = g(X, Y) = 0, \text{ for } X \in D^{\perp} \text{ and } Y \in D.$$

Therefore

$$g(\phi D^{\perp}, cD) = 0. \tag{3.3}$$

We denote by ν the orthogonal complementary vector bundle to $\phi D^{\perp} \oplus cD$ in $T^{\perp}(M)$. Thus we have

$$T^{\perp}(M) = \phi D^{\perp} \oplus cD \oplus \nu. \tag{3.4}$$

Lemma 3.1. *The morphisms t and f satisfy*

$$t(\phi D^{\perp}) = D^{\perp} \tag{3.5}$$

and

$$t(cD) \subset D. \tag{3.6}$$

Proof. For $X \in D^{\perp}$ and $Y \in D$,

$$g(t\phi X, Y) = g(t\phi X + f\phi X, Y) = g(\phi^2 X, Y) = -g(\phi X, \phi Y) = -g(X, Y) = 0.$$

Also

$$g(t\phi X, \xi_\alpha) = g(\phi^2 X, \xi_\alpha) = -g(\phi X, \phi \xi_\alpha) = 0, \text{ for } \alpha = 1, 2, \dots, s.$$

Therefore $t(\phi D^{\perp}) \subset D^{\perp}$.

For $X \in D^{\perp}$, we have $-X = \phi^2 X = t\phi X + f\phi X$ which implies $-X = t\phi X$. Consequently, $D^{\perp} \subset t(\phi D^{\perp})$. Hence the relation (3.5) follows. The relation (3.6) is trivial.

Now we denote by $\bar{\nabla}$(*resp.* ∇) the Riemannian connection on $\bar{M}$ (*resp.* M) with respect to the Riemannian metric g. The linear connection induced by $\bar{\nabla}$ on the normal bundle $T^{\perp}(M)$ is denoted by $\nabla^{\perp}$. Then the equations of Gauss and Weingarten are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3.7}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{3.8}$$

for $X, Y \in T(M)$ and $N \in T^{\perp}(M)$, where h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. These tensor fields are related by

$$g(h(X,Y),N) = g(A_N X, Y) \tag{3.9}$$

for $X, Y \in T(M)$ and $N \in T^{\perp}(M)$.

We denote

$$u(X,Y) = \nabla_X bPY - A_{cPY}X - A_{\phi QY}X.$$

□

Lemma 3.2. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then we have*

$$P\big(u(X,Y)\big) - bP\nabla_X Y - Pth(X,Y) = 0 \tag{3.10}$$

$$Q\big(u(X,Y)\big) - Qth(X,Y) = 0 \tag{3.11}$$

$$\eta_\alpha\big(u(X,Y)\big) = 0, \quad \text{for } \alpha = 1,2,\ldots,s \tag{3.12}$$

and

$$h(X,bPY) + \nabla^{\perp}_X cPY + \nabla^{\perp}_X \phi QY - cP\nabla_X Y - \phi Q\nabla_X Y - fh(X,Y) = 0, \tag{3.13}$$

for $X, Y \in T(M)$.

Proof. For $X, Y \in T(M)$ by using (2.5), (3.1), (3.2) and (3.7), (3.8) in (2.3), we have

$$\begin{aligned}\nabla_X bPY &+ h(X,bPY) - A_{cPY}X + \nabla^{\perp}_X cPY - A_{\phi QY}X + \nabla^{\perp}_X \phi QY \\ &- bP\nabla_X Y - cP\nabla_X Y - \phi Q\nabla_X Y - Pth(X,Y) - Qth(X,Y) - fh(X,Y) = 0.\end{aligned}$$

Then (3.10), (3.11), (3.12) and (3.13) follow by taking components on each of the vector bundles $D, D^{\perp}, \{\xi\}$ *and* $T^{\perp}(M)$ respectively. □

Lemma 3.3. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then we have*

$$P(t\nabla^{\perp}_X N + A_{fN}X - \nabla_X tN) = bPA_N X, \tag{3.14}$$

$$Q(t\nabla^{\perp}_X N + A_{fN}X - \nabla_X tN) = 0, \tag{3.15}$$

$$\eta_\alpha(A_{fN}X - \nabla_X tN) = 0, \quad \textit{and} \tag{3.16}$$

$$h(X,tN) + \phi QA_N X + \nabla^{\perp}_X fN + cPA_N X = f\nabla^{\perp}_X N \tag{3.17}$$

for $X \in T(M)$ and $N \in T^{\perp}(M)$.

Proof. For $X \in T(M)$ and $N \in T^{\perp}(M)$ by using (2.5), (3.1), (3.2) and (3.7), (3.8) in (2.3) we get

$$\begin{aligned} P\nabla_X tN + Q\nabla_X tN + \sum_{\alpha=1}^{s} \eta_\alpha(\nabla_X tN)\xi_\alpha + h(X, tN) - PA_{fN}X - QA_{fN}X \\ - \sum_{\alpha=1}^{s} \eta_\alpha(A_{fN}X)\xi_\alpha + \nabla_X^{\perp} fN + bPA_NX + cPA_NX + \phi QA_NX \\ - Pt\nabla_X^{\perp}N - Qt\nabla_X^{\perp}N - f\nabla_X^{\perp}N = 0. \end{aligned}$$

Then (3.14), (3.15), (3.16) and (3.17) follow by taking components on each of the vector bundles $D, D^{\perp}, \{\xi\}$ and $T^{\perp}(M)$ respectively. □

Lemma 3.4. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then we have*

$$\nabla_X \xi_\alpha = 0, \tag{3.18}$$

$$h(X, \xi_\alpha) = 0, \quad \text{and} \tag{3.19}$$

$$A_N \xi_\alpha = 0, \tag{3.20}$$

for $X \in T(M)$, $N \in T^{\perp}(M)$, $\alpha = 1, 2, \ldots, s$.

Proof. The lemma follows from (2.4) by (2.5), (3.1) and (3.7), (3.8). □

Lemma 3.5. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then we have*

$$A_{\phi X}Y = A_{\phi Y}X \tag{3.21}$$

for $X, Y \in D^{\perp}$.

Proof. For $X, Y \in D^{\perp}$ and $Z \in T(M)$ by using (2.3), (3.7) and (3.9) we obtain

$$\begin{aligned} g(A_{\phi X}Y, Z) = g(h(Z, Y), \phi X) = g(\bar{\nabla}_Z Y, \phi X) = -g(\phi\bar{\nabla}_Z Y, X) \\ = -g(\bar{\nabla}_Z \phi Y, X) = g(A_{\phi Y}Z, X) = g(A_{\phi Y}X, Z), \end{aligned}$$

because A_N is a symmetric tensor with respect to the metric tensor g. Hence the lemma follows. □

Lemma 3.6. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then we have*

$$\nabla_{\xi_\alpha} V \in D^{\perp}, \quad \text{for} \quad V \in D^{\perp}, \quad \text{and} \tag{3.22}$$

$$\nabla_{\xi_\alpha} W \in D, \quad \text{for} \quad W \in D, \tag{3.23}$$

for $\alpha = 1, 2, \dots, s$.

Proof. Let us take $X = \xi_\alpha$, $\alpha = 1, 2, \dots, s$ and $V = \phi N$ in (3.14) where $N \in \phi D^\perp$. Taking into account that $tN = \phi N, fN = 0$ we get

$$P\nabla_{\xi_\alpha} V = Pt\nabla^\perp_{\xi_\alpha} N - bPA_N\xi_\alpha, \quad \alpha = 1, 2, \dots, s \tag{3.24}$$

Using (3.20), we have

$$g(PA_N\xi_\alpha, W) = g(A_N\xi_\alpha, W) = 0, \quad \alpha = 1, 2, \dots, s$$

for $W \in D$.

Hence (3.24) becomes

$$P\nabla_{\xi_\alpha} V = Pt\nabla^\perp_{\xi_\alpha} N. \tag{3.25}$$

On the other hand, using (3.19) and (3.20) in (3.17) we have

$$f\nabla^\perp_{\xi_\alpha} N = 0$$

and hence from (3.25) it follows that

$$P\nabla_{\xi_\alpha} V = 0, \;\; for \;\; V \in D^\perp, \tag{3.26}$$

$\alpha = 1, 2, \dots, s$

Next from (3.16) we have

$$\eta(\nabla_{\xi_\alpha} V) = 0, \quad \alpha = 1, 2, \dots, s, \tag{3.27}$$

for all $V = \phi N \in D^\perp$, where $N \in \phi D^\perp$.

Hence (3.22) follows from (3.26) and (3.27).

Finally by using (3.1), (3.18) and (3.22), we have

$$g(\nabla_{\xi_\alpha} W, X) = g(\nabla_{\xi_\alpha} W, PX), \quad \text{for} \quad X \in T(M) \quad \text{and} \quad W \in D.$$

Thus we have $\nabla_{\xi_\alpha} W \in D$, for $W \in D$, $\alpha = 1, 2, \dots, s$ and this completes the proof. □

Corollary 3.1. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then we have*

$$[Y, \xi_\alpha] \in D^\perp, \quad \text{for} \quad Y \in D^\perp, \quad \text{and} \tag{3.28}$$

$$[X, \xi_\alpha] \in D, \quad \text{for} \quad X \in D, \tag{3.29}$$

$\alpha = 1, 2, \dots, s$.

The above corollary follows immediately from the Lemma 3.4 and 3.6.

4. Integrability of Distributions

Theorem 4.1. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then the distribution $D^{\perp}$ is always involutive.*

Proof. For $X, Y \in D^{\perp}$ by using (3.18) we get

$$g([X,Y],\xi_\alpha) = g(\nabla_X Y,\xi_\alpha) - g(\nabla_Y X,\xi_\alpha) = g(X,\nabla_Y\xi_\alpha) - g(Y,\nabla_X\xi_\alpha) = 0, \quad (4.1)$$

for $\alpha = 1, 2, \ldots, s$

On the otherhand from (3.10) we have

$$bP\nabla_X Y = -PA_{\phi Y}X - Pth(X,Y), \quad \text{for} \quad X,\, Y \in D^{\perp}$$

and then by using Lemma 3.5 we get

$$bP[X,Y] = 0, \quad \text{for} \quad X,\, Y \in D^{\perp}. \quad (4.2)$$

As b is automorphism of D, the theorem follows from (4.1) and (4.2). By virtue of (3.28) and (4.2) the following theorem follows immediately. □

Theorem 4.2. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then the distribution $D^{\perp}\oplus\{\xi\}$ is integrable.*

Theorem 4.3. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then the distribution D is involutive if and only if*

$$h(bX,Y) - h(X,bY) + \nabla^{\perp}_Y cX - \nabla^{\perp}_X cY \in cD\oplus\nu. \quad (4.3)$$

Proof. For $X, Y \in D$ by using (3.18) we have

$$g\big([X,Y],\xi_\alpha\big) = 0, \quad \alpha = 1, 2, \ldots, s \quad (4.4)$$

Also applying ϕ to (3.13) and then taking component in $D^{\perp}$ we have

$$Q\nabla_X Y = -Qt\big(h(X,bY) + \nabla^{\perp}_X cPY - fh(X,Y)\big), \quad \text{for} \quad X, Y \in D$$

and thus

$$Q[X,Y] = Qt\big(h(Y,bX) - h(X,bY) + \nabla^{\perp}_Y cX - \nabla^{\perp}_X cY\big), \quad (4.5)$$

for $X, Y \in D$.

Hence the theorem follows from (4.4) and (4.5).

Next by virtue of (3.29) and (4.5) we have. □

Theorem 4.4. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then the distribution $D\oplus\{\xi\}$ is involutive if and only if* (4.3) *holds good.*

Remark. CR-submanifold is a particular case of generalized CR-submanifold. For a generalized CR-submanifold M of a T-manifold $\bar{M}$, if for $X \in D$, we have $\phi X = bX$ *i. e.*, $cX = 0$ then M becomes a CR-submanifold of $\bar{M}$.

Corollary 4.1 (Calin [5]). *Let M be a CR-submanifold of a T-manifold $\bar{M}$. Then the invariant distribution D and $D \oplus \{\xi\}$ are involutive if and only if*

$$h(bX, Y) = h(X, bY), \quad \text{for all} \quad X, Y \in D.$$

Proof. For $X, Y \in D$, by using (2.3), (2.5) (3.7), (3.8), (3.2) and $cX = CY = 0$, we have

$$\nabla_X bY + h(X, bY) = b\nabla_X Y + c\nabla_X Y + th(X, Y) + fh(X, Y). \tag{4.6}$$

Taking normal components of (4.6), we get

$$c\nabla_X Y = h(X, bY) - fh(X, Y),$$

which implies

$$c[X, Y] = h(X, bY) - h(Y, bX). \tag{4.7}$$

For a CR-submanifold M, from (3.4) we have

$$T^{\perp}(M) = \phi D^{\perp} \oplus \nu.$$

From (4.7) it follows that

$$h(X, bY) - h(Y, bX) \in \phi D^{\perp}. \tag{4.8}$$

On the otherhand, by virtue of Theorem 4.3 and Theorem 4.4 it follows that the invariant distribution D and $D \oplus \{\xi\}$ of a CR-submanifold M are involutive if and only if

$$h(X, bY) - h(Y, bX) \in \nu. \tag{4.9}$$

Hence the corollary follows from (4.8) and (4.9). □

5. Geometry of Leaves

Definition 5.1. An integrable ditribution on a manifold is a distribution with the property that through each point of the manifold passes an integral submanifold of the dimension equal with the dimension of the fibre of the distribution. It is proved that there exists a maximal integral manifold (which is connected) passing through each point. Such maximal integral manifold is called a *leaf* of the foliation determined by the integrable distribution.

Theorem 5.1. *Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. Then the leaves of distribution $D^{\perp}$(or $D^{\perp}\oplus\{\xi\}$) are totally geodesic in M if and only if*

$$h(X, bZ) \in cD\oplus\nu, \tag{5.1}$$

for $X \in D^{\perp}$ and $Z \in D\oplus\{\xi\}$

Proof. For $X, Y \in D^{\perp}$ and $Z \in D$, by using (2.2), (2.3), (3.7)and (3.8) we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X Y, Z) \\ &= -g(Y, \bar{\nabla}_X Z) \\ &= -g(\phi\bar{\nabla}_X Z, \phi Y) - \sum_{\alpha=1}^{s} \eta_\alpha(\bar{\nabla}_X Z)\eta_\alpha(Y) \\ &= -g(\phi\bar{\nabla}_X Z, \phi Y) \\ &= -g(\bar{\nabla}_X \phi Z, \phi Y) \\ &= -g\big(\nabla_X bZ + h(X, bZ) - A_{cZ}X + \nabla^{\perp}_X cZ, \phi Y\big) \\ &= -g\big(h(X, bZ) + \nabla^{\perp}_X cZ, \phi Y\big). \end{aligned} \tag{5.2}$$

Also for $X, Y \in D^{\perp}$, using (3.18) we have

$$g(\nabla_X Y, \xi_\alpha) = -g(Y, \nabla_X \xi_\alpha) = 0, \tag{5.3}$$

for $\alpha = 1, 2, \ldots, s$.

Hence the theorem follows from (5.2) and (5.3). □

Corollary 5.1 (Calin [5]). *Let M be a CR-submanifold of a T-manifold $\bar{M}$. Then the leaf of distribution $D^{\perp}$(or, $D^{\perp}\oplus\{\xi\}$) is totally geodesic in M if and only if*

$$h(X, Z) \in \nu, \quad \text{for} \quad X \in D^{\perp} \quad \text{and} \quad Z \in D.$$

Proof. As a particular case of theorem 5.1 with $cD = \{0\}$, it follows that the leaf of the distribution $D^{\perp}$(*or* $D^{\perp}\oplus\{\xi\}$) of a CR-submanifold M is totally geodesic if and only if

$$h(X, \phi Z) \in \nu, \tag{5.4}$$

for $X \in D^{\perp}$ and $Z \in D$ Taking $X \in D^{\perp}$ and $\phi Z \in D$ in (5.4), we get

$$h(X, Z) \in \nu, \quad \text{for} \quad X \in D^{\perp} \quad \text{and} \quad Z \in D.$$

Hence the corollary follows. □

Theorem 5.2. *Let M be generalized CR-submanifold of a T-manifold $\bar{M}$. Then the leaves of distribution D (or $D\oplus\{\xi\}$) are totally geodesic in M if and only if*

$$h(X,bY)+\nabla_X^{\perp}cY\in cD\oplus\nu, \tag{5.5}$$

for $X,Y\in D$.

Proof. For $X,Y\in D$ and $Z\in D^{\perp}$ by using (2.2), (2.3), (2.5), (3.7) and (3.8), we have

$$\begin{aligned} g(\nabla_XY,Z)&=g(\bar{\nabla}_XY,Z)\\ &=g(\phi\bar{\nabla}_XY,\phi Z)+\sum_{\alpha=1}^{s}\eta_\alpha(\bar{\nabla}_XY)\eta_\alpha(Z)\\ &=g(\phi\bar{\nabla}_XY,\phi Z)\\ &=g(\bar{\nabla}_X\phi Y,\phi Z)\\ &=g(\nabla_XbY+h(X,bY)-A_{cY}X+\nabla_X^{\perp}cY,\phi Z)\\ &=g(h(X,bY)+\nabla_X^{\perp}cY,\phi Z). \end{aligned}\tag{5.6}$$

Also by using (3.18), we have

$$g(\nabla_XY,\xi_\alpha)=-g(Y,\nabla_X\xi_\alpha)=0, \tag{5.7}$$

for $X,Y\in D$ and $\alpha=1,2,\ldots,s$.

Hence the theorem follows from (5.6) and (5.7). □

Corollary 5.2 (Calin [5]). *Let M be a CR-submanifold of a T-manifold $\bar{M}$. Then the leaf of distribution D (or $D\oplus\{\xi\}$) is totally geodesic in M if and only if*

$$h(X,Y)\in\nu,\quad \text{for}\quad X,\,Y\in D.$$

Proof. Taking $cD=\{0\}$ in theorem 5.2, it follows that the leaf of the distribution D(or $D\oplus\{\xi\}$) of a CR-submanifold M is totally geodesic if and only if

$$h(X,\phi Y)\in\nu, \tag{5.8}$$

for X, $Y\in D$. Taking $X,\phi Y\in D$ in (5.8), we get

$$h(X,Y)\in\nu,\quad \text{for}\quad X,\,Y\in D.$$

Hence the corollary follows. □

6. Totally geodesic generalized CR-submanifolds of a T-manifold

A submanifold M of a T-manifold $\bar{M}$ is said to to be totally umbilical if there exists a vector field $H \in T^{\perp}(M)$ such that $h(X,Y) = g(X,Y)H$, X, $Y \in T(M)$. H is called the *mean curvature vector field* of M. M is called a *totally geodesic submanifold* if $h(X,Y) = 0$, X, $Y \in T(M)$.

Let M be a generalized CR-submanifold of a T-manifold $\bar{M}$. It is obvious that the structure vector fields ξ_α, $\alpha = 1, 2, \ldots, s$ are tangents to Mihai [9]. Otherwise if ξ_α, $\alpha = 1, 2, \ldots s$ are normals to M then $\phi(T_x(M)) \subset T_x^{\perp}(M)$, for all $x \in M$.

Theorem 6.1. *Every totally umbilical generalized CR-submanifold M of a T-manifold $\bar{M}$ is totally geodesic.*

Proof. Using (3.19) and the fact that M is totally umbilical, we deduce for any $\alpha \in \{1, 2, \ldots, s\}$ and $X \in T(M)$, $0 = h(X, \xi_\alpha) = g(X, \xi_\alpha)H$, which gives $H = 0$ and consequently M becomes totally geodesic. □

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