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RS-COMPACTNESS IN A REDEFINED FUZZY TOPOLOGICAL SPACE

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ABSTRACT. In this paper, we introduce the concepts of interior of a fuzzy set and several types of fuzzy compactness and fuzzy RS-compactness in a redefined fuzzy topological space and investigate their properties.

1. INTRODUCTION

Hazra, Samanta & Chattopadhyay [5] introduced a fuzzy topology on X as a mapping $\tau : I^X \to I$ satisfying some conditions which is a generalization of Chang's fuzzy topology (*cf.* Chang [2]). In this paper, we will call the fuzzy topology introduced by Hazra, Samanta & Chattopadhyay [5] a *Hazra-Samanta-Chattopadhyay fuzzy topology.* Fuzzy almost compactness, fuzzy near compactness and fuzzy RS-compactness in fuzzy topological spaces were studied by several authors Eş [3], Kudri & Warner [6] and Mukherjee & Ghosh [7].

In this paper, we introduce the concepts of interior of a fuzzy set and several types of fuzzy compactness and fuzzy RS-compactness in a redefined fuzzy topological space and investigate their properties.

2. PRELIMINARIES

Let X be a non-empty set and $I = [0,1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of $\varnothing$ (the empty set) and X, respectively.

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A *Hazra-Samanta-Chattopadhyay fuzzy topological space* (HSCfts) (*cf.* Hazra, Samanta & Chattopadhyay [5]) is an ordered pair (X, τ), where $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$;
(O2) if $\tau(A) > 0$ and $\tau(B) > 0$, then $\tau(A \cap B) > 0$;
(O3) if $\tau(A_i) > 0$ for each $i \in J$, then $\tau(\bigcup_{i \in J} A_i) > 0$.

Then the mapping $\tau : I^X \to I$ is called a *Hazra-Samanta-Chattopadhyay fuzzy topology*(HSCft) or a *gradation of openness* on X.

If the HSCft τ on X satisfies the following condition:

(O4) $\tau(I^X) \subseteq \{0, 1\}$,

then τ corresponds in a one to one way to a fuzzy topology in Chang's sense (*cf.* Chang [2]).

A mapping $\tau^* : I^X \to I$ is called a *Hazra-Samanta-Chattopadhyay fuzzy cotopology* (HSCfc) or a *gradation of closedness* on X (*cf.* Hazra, Samanta & Chattopadhyay [5]) if the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;
(C2) if $\tau^*(A) > 0$ and $\tau^*(B) > 0$, then $\tau^*(A \cup B) > 0$;
(C3) if $\tau^*(A_i) > 0$ for each $i \in J$, then $\tau^*(\bigcap_{i \in J} A_i) > 0$.

If τ is a HSCft on X, then the mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is a HSCfc on X. Conversely, if τ^* is a HSCfc on X, then the mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is a HSCft on X (*cf.* Hazra, Samanta & Chattopadhyay [5]).

Let (X, τ) and (Y, σ) be two HSCfts's. $f : X \to Y$ is called a *gradation preserving map* (*gp-map*) (*cf.* Hazra, Samanta & Chattopadhyay [5]) if $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^Y$. $f : X \to Y$ is called a *weakly gradation preserving map* (*wgp-map*) (*cf.* Hazra, Samanta & Chattopadhyay [5]) if $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^Y$. Clearly, a gp-map is a wgp-map. $f : X \to Y$ is a gp-map if and only if $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. $f : X \to Y$ is a wgp-map if and only if $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$.

3. Properties of closure and interior

Definition 3.1 (Hazra, Samanta & Chattopadhyay [5]). Let (X,τ) be a HSCfts and $A \in I^X$. The *τ-closure* of A, denoted by $\overline{A}$, is defined by

$$\overline{A} = \cap\{K \in I^X : \tau^*(K) > 0, A \subseteq K\}.$$

Definition 3.2. Let (X,τ) be a HSCfts and $A \in I^X$. The *τ-interior* of A, denoted by A^o, is defined by

$$A^o = \cup\{K \in I^X : \tau(K) > 0, K \subseteq A\}.$$

Remark 3.3. Let (X,τ) be a HSCfts. Then

(a) In view of O3 and C3 from Definitions 3.1 and 3.2, it follows that $\tau^*(\overline{A}) > 0$ and $\tau(A^o) > 0$ for all $A \in I^X$.

(b) Let $\mathcal{T}_s = \{A \in I^X : \tau(A) > 0\}$ be the support of τ. Then $\mathcal{T}_s$ is a Chang fuzzy topology on X. For each $A \in I^X$, define the $\mathcal{T}_s$-closure and $\mathcal{T}_s$-interior of A by

$$\mathrm{cl}_s(A) = \cap\{K \in I^X : K^c \in \mathcal{T}_s, A \subseteq K\}$$
$$\mathrm{int}_s(A) = \cup\{K \in I^X : K \in \mathcal{T}_s, K \subseteq A\}.$$

Clearly, $\overline{A} = \mathrm{cl}_s(A)$ and $A^o = \mathrm{int}_s(A)$ for each $A \in I^X$.

Theorem 3.4 (Hazra, Samanta & Chattopadhyay [5]). *Let (X,τ) be a HSCfts and $A, B \in I^X$. Then*

(a) $\overline{0_X} = 0_X$,

(b) $A \subseteq \overline{A}$,

(c) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$,

(d) $\overline{(\overline{A})} = \overline{A}$,

(e) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Theorem 3.5. *Let (X,τ) be a HSCfts and $A, B \in I^X$. Then*

(a) $1_X^o = 1_X$,

(b) $A^o \subseteq A$,

(c) $A \subseteq B \Rightarrow A^o \subseteq B^o$,

(d) $(A^o)^o = A^o$,

(e) $(A \cap B)^o = A^o \cap B^o$.

Proof. (a), (b) and (c) follow directly from Definition 3.2.

(d) From (b) we have $(A^o)^o \subseteq A^o$. From Remark 3.3(a) we have $\tau(A^o) > 0$. From Definition 3.2 $A^o \subseteq (A^o)^o$ because $A^o \subseteq A^o$. Hence $(A^o)^o = A^o$.

(e) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, from (c) we have $(A \cap B)^o \subseteq A^o$ and $(A \cap B)^o \subseteq B^o$. Hence $(A \cap B)^o \subseteq A^o \cap B^o$. From (b) we have $A^o \cap B^o \subseteq A \cap B$.

Since $\tau(A^o) > 0$ and $\tau(B^o) > 0$ by Remark 3.3(a), $\tau(A^o \cap B^o) > 0$. From Definition 3.2 we have $A^o \cap B^o \subseteq (A \cap B)^o$. Hence $(A \cap B)^o = A^o \cap B^o$. □

Theorem 3.6. *Let (X, τ) be a HSCfts and $A \in I^X$. Then*

(a) $\tau^*(A) > 0 \Leftrightarrow A = \overline{A}$,

(b) $\tau(A) > 0 \Leftrightarrow A = A^o$.

Proof. The proof is straightforward. □

Theorem 3.7. *Let (X, τ) be a HSCfts and $A \in I^X$. Then*

(a) $(A^o)^c = \overline{(A^c)}$,

(b) $A^o = (\overline{(A^c)})^c$,

(c) $(\overline{A})^c = (A^c)^o$,

(d) $\overline{A} = ((A^c)^o)^c$.

Proof. The proof is straightforward. □

Theorem 3.8. *Let (X, τ) and (Y, σ) be two HSCfts's. Then the following are equivalent:*

(a) *$f : X \to Y$ is a wgp-map.*

(b) *$f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \in I^X$.*

(c) *$\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$ for every $A \in I^Y$.*

(d) *$f^{-1}(A^o) \subseteq (f^{-1}(A))^o$ for every $A \in I^Y$.*

Proof. (a) $\Rightarrow$ (b). In view of Remark 3.3(a) for every $A \in I^X$, we have $\sigma^*(\overline{f(A)}) > 0$ and hence from (a), $\tau^*\big(f^{-1}(\overline{f(A)})\big) > 0$. Hence by Theorem 3.6(a),

$$f^{-1}(\overline{f(A)}) = \overline{f^{-1}(\overline{f(A)})} \supseteq \overline{f^{-1}(f(A))} \supseteq \overline{A}.$$

Hence $f(\overline{A}) \subseteq \overline{f(A)}$.

(b) $\Rightarrow$ (c). For every $A \in I^Y$, from (b) we have

$$f(\overline{f^{-1}(A)}) \subseteq \overline{f(f^{-1}(A))} \subseteq \overline{A}.$$

Hence $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$.

(c) $\Rightarrow$ (d). For every $A \in I^Y$, from (c) and Theorem 3.7 we have

$$\begin{aligned}((f^{-1}(A))^o)^c &= \overline{(f^{-1}(A))^c} = \overline{f^{-1}(A^c)} \subseteq f^{-1}((\overline{A^c})) \\ &= f^{-1}((A^o)^c) = (f^{-1}(A^o))^c.\end{aligned}$$

Hence $f^{-1}(A^o) \subseteq (f^{-1}(A))^o$.

(d) $\Rightarrow$ (a). Let $\sigma(A) > 0$ for $A \in I^Y$. Then $A = A^o$ by Theorem 3.6. From (d) we have

$$f^{-1}(A) = f^{-1}(A^o) \subseteq (f^{-1}(A))^o.$$

Thus $f^{-1}(A) = (f^{-1}(A))^o$ by Theorem 3.5. Hence $\tau(f^{-1}(A)) > 0$ by Theorem 3.6. Thus f is a wgp-map. □

Definition 3.9. Let (X, τ) and (Y, σ) be two HSCfts's.

(a) $f : X \to Y$ is called a *gradation carrier map* (*gc-map*) if $\tau(A) \leq \sigma(f(A))$ for every $A \in I^X$.
(b) $f : X \to Y$ is called a *weakly gradation carrier map* (*wgc-map*) if $\tau(A) > 0 \Rightarrow \sigma(f(A)) > 0$ for every $A \in I^X$.

Note that a gc-map is a wgc-map.

Theorem 3.10. *Let (X, τ) and (Y, σ) be two HSCfts's. Then the following are equivalent:*

(a) *$f : X \to Y$ is a wgc-map.*
(b) *$f(A^o) \subseteq (f(A))^o$ for every $A \in I^X$.*

Proof. (a) $\Rightarrow$ (b). For every $A \in I^X$, we have $\tau(A^o) > 0$ by Remark 3.3(a) and hence from (a), $\sigma(f(A^o)) > 0$. Hence by Theorem 3.6(b), $f(A^o) = (f(A^o))^o \subseteq (f(A))^o$.

(b) $\Rightarrow$ (a). Let $\tau(A) > 0$ for $A \in I^X$. Then $A = A^o$ by Theorem 3.6. From (a) we have

$$f(A) = f(A^o) \subseteq (f(A))^o.$$

Thus $f(A) = (f(A))^o$ by Theorem 3.5. Hence $\sigma(f(A)) > 0$ by Theorem 3.6. Thus f is a wgc-map. □

4. SEVERAL TYPES OF FUZZY RS-COMPACTNESS

Azad [1] and Mukherjee & Ghosh [7] introduced the concepts of fuzzy semiopen set, fuzzy regular open set, fuzzy regular closed set and fuzzy regular semiopen set in a fuzzy topological space. In the following definition we introduce those concepts in a HSCfts.

Definition 4.1. Let (X, τ) be a HSCfts and $A \in I^X$.

(a) A is called *fuzzy semiopen* if there exists $U \in I^X$ with $\tau(U) > 0$ such that $U \subseteq A \subseteq \overline{U}$.
(b) A is called *fuzzy regular open* if $A = (\overline{A})^o$.
(c) A is called *fuzzy regular closed* if $A = \overline{(A^o)}$.
(d) A is called *fuzzy regular semiopen* if there exists a fuzzy regular open set U such that $U \subseteq A \subseteq \overline{U}$.

Note that A is fuzzy regular open $\Leftrightarrow$ A^c is fuzzy regular closed and that fuzzy regular open set $\Rightarrow$ fuzzy regular semiopen set $\Rightarrow$ fuzzy semiopen set.

Theorem 4.2. *Let (X, τ) be a HSCfts and $A \in I^X$. If A is a fuzzy regular semiopen set, then*

(a) *A^c is fuzzy regular semiopen,*
(b) $A^o = (\overline{A})^o$,
(c) $\overline{A} = \overline{(A^o)}$.

Proof. (a) Let A be a fuzzy regular semiopen set. Then there exists a fuzzy regular open set U such that $U \subseteq A \subseteq \overline{U}$. Since U^c is fuzzy regular closed, $(U^c)^o$ is a fuzzy regular open set such that $(U^c)^o \subseteq A^c \subseteq U^c = \overline{(U^c)^o}$. Thus A^c is fuzzy regular semiopen.

(b) Let A be a fuzzy regular semiopen set. Then there exists a fuzzy regular open set U such that $U \subseteq A \subseteq \overline{U}$. Hence $\overline{A} = \overline{U}$.
Since $(\overline{A})^o = U$,

$$(\overline{A})^o = U \subseteq A^o \subseteq (\overline{U})^o = (\overline{A})^o.$$

Thus $A^o = (\overline{A})^o$.

(c) Let A be a fuzzy regular semiopen set. Then A^c is also a fuzzy regular semiopen set from (a). From (b) we have $(A^c)^o = (\overline{(A^c)})^o$. Hence $(\overline{A})^c = (\overline{(A^o)})^c$ by Theorem 3.7. Thus $\overline{A} = \overline{(A^o)}$. □

Note that A^o and $\overline{A}$ of a fuzzy regular semiopen set A are fuzzy regular open set and fuzzy regular closed set, respectively.

Definition 4.3. (a) A HSCfts (X, τ) is called *fuzzy compact* if for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} A_i = 1_X$.

(b) A HSCfts (X, τ) is called *fuzzy nearly compact* if for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} (\overline{(A_i)})^o = 1_X$.

(c) A HSCfts (X, τ) is called *fuzzy almost compact* if for every family $\{A_i : i \in J\}$ in $\{A \in I^X : \tau(A) > 0\}$ covering X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} \overline{(A_i)} = 1_X$.

(d) A HSCfts (X, τ) is called *fuzzy S-closed* if for every fuzzy semiopen cover $\{A_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} \overline{(A_i)} = 1_X$.

Note that fuzzy compactness $\Rightarrow$ fuzzy near compactness $\Rightarrow$ fuzzy almost compactness and that fuzzy S-closedness $\Rightarrow$ fuzzy almost compactness.

Example 4.4. Let $X = \mathbb{N}$, the set of all natural numbers and let $P_n = \{1, 2, \ldots, n\}$ and $A_n = \chi_{P_n}$ for each $n \in \mathbb{N}$. Define $\tau : I^X \to I$ by

$$\tau(0_X) = \tau(1_X) = 1,$$
$$\tau(A_n) = \frac{n}{n+1} \quad \text{for each } n \in \mathbb{N},$$
$$\tau(A) = 0 \quad \text{for all other } A \in I^X.$$

Then clearly, τ is a HSCft on X. Note that $\overline{(A_n)} = 1_X$, so $(\overline{(A_n)})^o = 1_X$ for each $n \in \mathbb{N}$. Hence (X, τ) is fuzzy nearly compact. But $\cup_{n=1}^{\infty} A_n = 1_X$, *i. e.*, $\{A_n : n \in \mathbb{N}\}$ covers X and there exists no finite subset $\mathbb{N}_0$ of $\mathbb{N}$ such that $\bigcup_{i \in \mathbb{N}_0} A_i = 1_X$. Thus (X, τ) is not fuzzy compact.

Definition 4.5. (a) A HSCfts (X, τ) is called *fuzzy RS-compact* if for every fuzzy regular semiopen cover $\{A_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} (A_i)^o = 1_X$.

(b) A HSCfts (X, τ) is called *fuzzy nearly RS-compact* if for every fuzzy regular semiopen cover $\{A_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} (\overline{(A_i)})^o = 1_X$.

(c) A HSCfts (X, τ) is called *fuzzy almost RS-compact* if for every fuzzy regular semiopen cover $\{A_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} \overline{(A_i)} = 1_X$.

Note that fuzzy RS-compactness $\Leftrightarrow$ fuzzy near RS-compactness, fuzzy RS-compactness $\Rightarrow$ fuzzy almost RS-compactness and that fuzzy RS-compactness $\Rightarrow$ fuzzy near compactness.

Ghosh [4] introduced the concept of a fuzzy extremally disconnected fuzzy topological space. In the following definition we introduce the concept of a fuzzy extremally disconnected HSCfts.

Definition 4.6. A HSCfts (X, τ) is called *fuzzy extremally disconnected* if $\tau(\overline{A}) > 0$ for every $A \in I^X$ with $\tau(A) > 0$.

Theorem 4.7. *A HSCfts (X, τ) is fuzzy RS-compact if and only if for each family $\{A_i : i \in J\}$ of fuzzy regular semiopen sets of X such that $\bigcap_{i \in J} A_i = 0_X$, there exists a finite subset J_0 of J such that $\bigcap_{i \in J_0} \overline{(A_i)} = 0_X$.*

Proof. Suppose that (X, τ) is fuzzy RS-compact. Let $\{A_i : i \in J\}$ be a family of fuzzy regular semiopen sets of X such that $\bigcap_{i \in J} A_i = 0_X$. Then by Theorem 4.2, $\{(A_i)^c : i \in J\}$ is a family of fuzzy regular semiopen sets of X such that $\bigcup_{i \in J}(A_i)^c = (\bigcap_{i \in J} A_i)^c = 1_X$. Since (X, τ) is fuzzy RS-compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0}((A_i)^c)^o = 1_X$. Hence $\bigcap_{i \in J_0} \overline{(A_i)} = \left(\bigcup_{i \in J_0}((A_i)^c)^o\right)^c = 0_X$.

Converse follows by reversing the previous arguments. □

Theorem 4.8. *Let (X, τ) be a HSCfts. Then the following are equivalent:*

(a) *(X, τ) is fuzzy RS-compact.*

(b) *For each family $\{A_i : i \in J\}$ of fuzzy regular open sets of X such that $\bigcap_{i \in J} A_i = 0_X$, there exists a finite subset J_0 of J such that $\bigcap_{i \in J_0} \overline{(A_i)} = 0_X$.*

(c) *For each fuzzy regular closed cover $\{A_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0}(A_i)^o = 1_X$.*

Proof. (a) $\Rightarrow$ (b): Since every fuzzy regular open set is fuzzy regular semiopen, it follows directly from Theorem 4.7.

(b) $\Rightarrow$ (a): Let $\{A_i : i \in J\}$ be a family of fuzzy regular semiopen sets of X such that $\cap_{i \in J} A_i = 0_X$. Since A_i is a fuzzy regular semiopen set for each $i \in J$, $\overline{(A_i)} = \overline{((A_i)^o)}$ for each $i \in J$ by Theorem 4.2. Since $\{(A_i)^o : i \in J\}$ is a family of fuzzy regular open sets of X such that $\bigcap_{i\in J}(A_i)^o = 0_X$, by (b) there exists a finite subset J_0 of J such that

$$\bigcap_{i\in J_0} \overline{(A_i)} = \bigcap_{i\in J_0} \overline{((A_i)^o)} = 0_X.$$

Thus (X,τ) is fuzzy RS-compact by Theorem 4.7.

(b) $\Leftrightarrow$ (c): It is obvious. □

Theorem 4.9. *A fuzzy extremally disconnected and fuzzy compact space is fuzzy RS-compact.*

Proof. Suppose that (X,τ) is a fuzzy extremally disconnected and fuzzy compact space. Let $\{A_i : i \in J\}$ be a fuzzy regular semiopen cover of X. Then there exists a fuzzy regular open set U_i such that $U_i \subseteq A_i \subseteq \overline{(U_i)}$ for each $i \in J$. Since (X,τ) is fuzzy extremally disconnected and $U_i = (\overline{(U_i)})^o$ for each $i \in J$, $A_i = (A_i)^o$ for each $i \in J$. Hence (X,τ) is fuzzy RS-compact since (X,τ) is fuzzy compact. □

Definition 4.10. Let (X,τ) and (Y,σ) be two HSCfts's. $f : X \to Y$ is called *fuzzy weakly open* if $f(A) \subseteq (f(\overline{A}))^o$ for every $A \in I^X$ with $\tau(A) > 0$.

Note that wgc-map $\Rightarrow$ fuzzy weakly open map.

Theorem 4.11. *Let (X,τ) and (Y,σ) be two HSCfts's. If $f : X \to Y$ is a fuzzy weakly open and wgp-map, then $f^{-1}(A)$ is a fuzzy regular open set in X for every fuzzy regular open set A in Y and also $f^{-1}(A)$ is a fuzzy regular closed set in X for every fuzzy regular closed set A in Y.*

Proof. Let A be a fuzzy regular open set in Y. From Definition 4.1 and Theorem 3.6 we have $\sigma(A) > 0$. Since f is a wgp-map, $\tau(f^{-1}(A)) > 0$.
Hence

$$f^{-1}(A) = (f^{-1}(A))^o \subseteq (\overline{f^{-1}(A)})^o$$

by Theorem 3.5 and Theorem 3.6.

Since f is fuzzy weakly open, $f((\overline{f^{-1}(A)})^o) \subseteq \left(f(\overline{f^{-1}(A)})\right)^o$. Since f is a wgp-map, $\left(f(\overline{f^{-1}(A)})\right)^o \subseteq \left(f(f^{-1}(\overline{A}))\right)^o \subseteq (\overline{A})^o = A$ by Theorem 3.8. Hence $\left(\overline{f^{-1}(A)}\right)^o \subseteq f^{-1}(A)$. Thus $f^{-1}(A)$ is a fuzzy regular open set in X.

Let A be a fuzzy regular closed set in Y. Then A^c is a fuzzy regular open set in Y. By the previous result $f^{-1}(A^c) = (f^{-1}(A))^c$ is a fuzzy regular open set in X. Hence $f^{-1}(A)$ is a fuzzy regular closed set in X. □

Theorem 4.12. *Let (X,τ) and (Y,σ) be two HSCfts's and let $f : X \to Y$ be a surjective, fuzzy weakly open and wgp-map. If (X,τ) is fuzzy extremally disconnected, then so is (Y,σ).*

Proof. Let $A \in I^Y$ with $\sigma(A) > 0$. Then $A = A^o$ by Theorem 3.6. Hence $\overline{A}$ is a fuzzy regular closed set in Y. By Theorem 4.11, $f^{-1}(\overline{A})$ is a fuzzy regular closed set in X *i. e.*, $f^{-1}(\overline{A}) = \overline{(f^{-1}(\overline{A}))^o}$. Since X is fuzzy extremally disconnected and $\tau\big((f^{-1}(\overline{A}))^o\big) > 0$, $\tau\big(\overline{(f^{-1}(\overline{A}))^o}\big) > 0$. From the surjectivity and fuzzy weak openness of f we have

$$\begin{aligned}\overline{A} = f(f^{-1}(\overline{A})) = f\big(\overline{(f^{-1}(\overline{A}))^o}\big) &\subseteq \big(f\big(\overline{\overline{(f^{-1}(\overline{A}))^o}}\big)\big)^o \\ &= \big(f\big(\overline{(f^{-1}(\overline{A}))^o}\big)\big)^o = \big(f(f^{-1}(\overline{A}))\big)^o = (\overline{A})^o.\end{aligned}$$

Hence $\overline{A} = (\overline{A})^o$ and so $\sigma(\overline{A}) > 0$ by Theorem 3.6. Thus (Y,σ) is fuzzy extremally disconnected. □

Theorem 4.13. *Let (X,τ) and (Y,σ) be two HSCfts's and let $f : X \to Y$ be a surjective, fuzzy weakly open and wgp-map. If (X,τ) is fuzzy RS-compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a fuzzy regular closed cover of Y. By Theorem 4.11, $\{f^{-1}(A_i) : i \in J\}$ is a fuzzy regular closed cover of X. Since X is fuzzy RS-compact, by Theorem 4.8 there exists a finite subset J_0 of J such that $\bigcup_{i\in J_0}\big(f^{-1}(A_i)\big)^o = 1_X$. From the surjectivity and fuzzy weak openness of f we have

$$\begin{aligned}1_Y = f\Big(\bigcup_{i\in J_0}(f^{-1}(A_i))^o\Big) = \bigcup_{i\in J_0} f((f^{-1}(A_i))^o) &\subseteq \bigcup_{i\in J_0}\big(f(\overline{(f^{-1}(A_i))^o})\big)^o \\ &= \bigcup_{i\in J_0}\big(f(f^{-1}(A_i))\big)^o = \bigcup_{i\in J_0}(A_i)^o.\end{aligned}$$

Hence $\bigcup_{i\in J_0}(A_i)^o = 1_Y$. Thus (Y,σ) is fuzzy RS-compact by Theorem 4.8. □

Theorem 4.14. *A HSCfts (X,τ) is fuzzy almost RS-compact if and only if for each family $\{A_i : i \in J\}$ of fuzzy regular semiopen sets of X such that $\bigcap_{i\in J} A_i = 0_X$, there exists a finite subset J_0 of J such that $\bigcap_{i\in J_0}(A_i)^o = 0_X$.*

Proof. Suppose that (X,τ) is fuzzy almost RS-compact. Let $\{A_i : i \in J\}$ be a family of fuzzy regular semiopen sets of X such that $\bigcap_{i\in J} A_i = 0_X$. Then $\{(A_i)^c : i \in J\}$ be a family of fuzzy regular semiopen sets of X such that $\bigcup_{i\in J}(A_i)^c = (\bigcap_{i\in J} A_i)^c = 1_X$

by Theorem 4.2. Since (X,τ) is fuzzy almost RS-compact, there exists a finite subset J_0 of J such that $\bigcup_{i\in J_0}\overline{(A_i)^c} = 1_X$.
Hence

$$\bigcap_{i\in J_0}(A_i)^o = \big(\bigcup_{i\in J_0}\overline{(A_i)^c}\big)^c = 0_X.$$

Converse can be proved similarly. □

Theorem 4.15. *Let (X,τ) be a HSCfts. Then the following are equivalent:*

(a) *(X,τ) is fuzzy almost RS-compact.*

(b) *For each family $\{A_i : i \in J\}$ of fuzzy regular open sets of X such that $\bigcap_{i\in J} A_i = 0_X$, there exists a finite subset J_0 of J such that $\bigcap_{i\in J_0} A_i = 0_X$.*

(c) *For each fuzzy regular closed cover $\{A_i : i \in J\}$ of X, there exists a finite subset J_0 of J such that $\bigcup_{i\in J_0} A_i = 1_X$.*

Proof. (a) $\Rightarrow$ (b). This follows directly from Theorem 4.14.

(b) $\Rightarrow$ (a). Let $\{A_i : i \in J\}$ be a family of fuzzy regular semiopen sets of X such that $\bigcap_{i\in J} A_i = 0_X$. Since A_i is a fuzzy regular semiopen set for each $i \in J$, $(A_i)^o$ is a fuzzy regular open set for each $i \in J$.

Hence $\{(A_i)^o : i \in J\}$ is a family of fuzzy regular open sets of X such that $\bigcap_{i\in J}(A_i)^o = 0_X$. By (b), there exists a finite subset J_0 of J such that $\bigcap_{i\in J_0}(A_i)^o = 0_X$. By Theorem 4.13, (X,τ) is fuzzy almost RS-compact.

(b) $\Leftrightarrow$ (c). This is straightforward. □

Theorem 4.16. *A HSCfts (X,τ) is fuzzy almost RS-compact if and only if (X,τ) is fuzzy S-closed.*

Proof. Let (X,τ) be fuzzy S-closed. Since every fuzzy regular semiopen set is fuzzy semiopen set, (X,τ) is fuzzy almost RS-compact.

Conversely, suppose that (X,τ) is fuzzy almost RS-compact. Let $\{A_i : i \in J\}$ be a fuzzy semiopen cover of X. Then there exists $U_i \in I^X$ with $\tau(U_i) > 0$ such that $U_i \subseteq A_i \subseteq \overline{(U_i)}$ for each $i \in J$. We can easily show that $\overline{(U_i)}$ is fuzzy regular closed for each $i \in J$.

Since $U_i \subseteq A_i \subseteq \overline{(U_i)}$ for each $i \in J$, $\overline{(U_i)} \subseteq \overline{(A_i)} \subseteq \overline{(\overline{(U_i)})} = \overline{(U_i)}$ for each $i \in J$. Thus $\overline{(A_i)} = \overline{(U_i)}$ for each $i \in J$. Thus $\{\overline{(A_i)} : i \in J\}$ is a fuzzy regular closed cover of X. Since (X,τ) is fuzzy almost RS-compact, by Theorem 4.15 there exists a finite subset J_0 of J such that $\bigcup_{i\in J_0}\overline{(A_i)} = 1_X$.

Hence (X,τ) is fuzzy S-closed. □

Theorem 4.17. *A fuzzy extremally disconnected and fuzzy almost compact space is fuzzy almost RS-compact.*

Proof. Let (X, τ) be a fuzzy extremally disconnected and fuzzy almost compact space and let $\{A_i : i \in J\}$ be a fuzzy regular semiopen cover of X. Then there exists a fuzzy regular open set U_i such that $U_i \subseteq A_i \subseteq \overline{(U_i)}$ for each $i \in J$. Since (X, τ) is fuzzy extremally disconnected and $U_i = (\overline{(U_i)})^o$ for each $i \in J$, $A_i = (A_i)^o$ for each $i \in J$. Thus $\{A_i : i \in J\}$ is a family in $\{A \in I^X : \tau(A) > 0\}$ covering X. Since (X, τ) is fuzzy almost compact, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} \overline{(A_i)} = 1_X$.

Hence (X, τ) is fuzzy almost RS-compact. □

Theorem 4.18. *Let (X, τ) and (Y, σ) be two HSCfts's and let $f : X \to Y$ be a surjective, fuzzy weakly open and wgp-map. If (X, τ) is fuzzy almost RS-compact, then so is (Y, σ).*

Proof. Let $\{A_i : i \in J\}$ be a fuzzy regular closed cover of Y. By Theorem 4.11, $\{f^{-1}(A_i) : i \in J\}$ is a fuzzy regular closed cover of X. Since (X, τ) is fuzzy almost RS-compact, by Theorem 4.15 there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} f^{-1}(A_i) = 1_X$. From the surjectivity of f we have

$$1_Y = f(\bigcup_{i \in J_0} f^{-1}(A_i)) = \bigcup_{i \in J_0} f(f^{-1}(A_i)) = \bigcup_{i \in J_0} A_i.$$

Hence $\bigcup_{i \in J_0} A_i = 1_Y$. Thus (Y, σ) is fuzzy almost RS-compact by Theorem 4.15. □

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