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INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we introduce the fundamental concepts of intuitionistic fuzzy Q-neighborhood, intuitionistic Q-first axiom of countability, intuitionistic first axiom of countability, intuitionistic fuzzy closure operator, intuitionistic fuzzy boundary point and intuitionistic fuzzy accumulation point and investigate some of their properties.

0. INTRODUCTION

After the introduction of the concept of fuzzy sets by Zadeh [10], Chang [2] was the first to introduce the concept of a fuzzy topology on a set X. After that several researchers (*e. g.*, Pu & Liu [7], Wang [8], Weiss [9], etc.) have investigated many properties for a fuzzy topology.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues (Çoker [3], Çoker & Haydar Eş [4], Gürçay, Çoker & Eş [5]) introduced intuitionistic fuzzy topological spaces using intuitioinstic fuzzy sets. Moreover, S. J. Lee & E. P. Lee [6] introduced the concept of intuitionistic fuzzy points and intuitionistic fuzzy neighborhoods and investigated the properties of continuous, open and closed mappings in the intuitionistic fuzzy topological spaces.

In Section 1, we introduce the concept of "quasi-coincident" and investigate some of it's properties. Furthermore, we study some properties of the image and inverse image for a mapping.

In Section 2, we constuct some intuitionistic fuzzy topologies.

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In Section 3, we introduce the concepts of intuitionistic fuzzy Q-neighborhood, intuitionistic fuzzy local base, intuitionistic Q-first axiom of countabilityintuitionistic and first axiom of countability and investigate some of their properties.

In Section 4, we introduce the concepts of intuitionistic fuzzy closure operator, intuitionistic fuzzy boundary point and intuitionistic fuzzy accumulation point and study some of their properties.

1. INTUITIONISTIC FUZZY SETS AND INTUITIONISTIC FUZZY POINTS

We will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 1.1 (Atanassov [1]). Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set*(in short, *IFS*) in X if $\mu_A : X \to I$ and $\nu_A : X \to I$ are mappings, and $\mu_A + \nu_A \leq 1$.

In this case, μ_A and ν_A denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively.

We will denote the set of all the IFSs in X as $IFS(X)$.

It is clear that if $\mu_A \in I^X$, then $(\mu_A, \mu_A^c) \in IFS(X)$.

Definition 1.2 (Atanassov [1]). Let X be a nonempty set and let $A, B \in IFS(X)$.

(1) $A \subset B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ if and only if $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\ \ ]A = (\mu_A, 1 - \mu_A)$.

(7) $<> A = (1 - \nu_A, \nu_A)$.

Definition 1.3 (Çoker [3]). Let $\{A_\alpha\}_{\alpha \in \Gamma}$ be an arbitrary family of IFSs in X. Then

(a) $\bigcap A_\alpha = (\bigwedge \mu_{A_\alpha}, \bigvee \nu_{A_\alpha})$.

(b) $\bigcup A_\alpha = (\bigvee \mu_{A_\alpha}, \bigwedge \nu_{A_\alpha})$.

Definition 1.4 (Çoker [3]). $0_\sim = (0, 1)$ and $1_\sim = (1, 0)$.

Result 1.A (Çoker [3] Corollary 2.8). Let $A, B, C, D \in IFS(X)$. Then

(1) $A \subset B$ and $C \subset D \Rightarrow A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$.
(2) $A \subset B$ and $A \subset C \Rightarrow A \subset B \cap C$.
(3) $A \subset C$ and $B \subset C \Rightarrow A \cup B \subset C$.
(4) $A \subset B$ and $B \subset C \Rightarrow A \subset C$.
(5) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.
(6) $A \subset B \Rightarrow B^c \subset A^c$.
(7) $(A^c)^c = A$.
(8) $1_\sim^c = 0$, $0_\sim^c = 1_\sim$.

Proposition 1.5. *Let $A, B, C \in IFS(X)$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset IFS(X)$.*
(1) *(Idempotent laws)* $A \cap A = A, \quad A \cup A = A$.
(2) *(Commutative laws)* $A \cap B = B \cap A, \quad A \cup B = B \cup A$.
(3) *(Associative laws)* $A \cap (B \cap C) = (A \cap B) \cap C$, $A \cup (B \cup C) = (A \cup B) \cup C$.
(4) *(Distributive laws)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(4′) *(Generalized distributive laws)* $A \cap (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma}(A \cap A_\alpha)$,
$A \cup (\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma}(A \cup A_\alpha)$.
(5) *(Absorptive laws)* $A \cap (A \cup B) = A, \quad A \cup (A \cap B) = A$.
(6) *(DeMorgan's laws)* $(\bigcap_{\alpha \in \Gamma} A_\alpha)^c = \bigcup_{\alpha \in \Gamma} A_\alpha^c$, $(\bigcup_{\alpha \in \Gamma} A_\alpha)^c = \bigcap A_\alpha^c$.

Proof. The proofs of (1), (2) and (3) are obvious from Definition 1.2.

$$\begin{aligned}(4)\ A \cap (B \cup C) &= A \cap (\mu_B \vee \mu_C, \nu_B \wedge \nu_C)\\ &= (\mu_A \wedge (\mu_B \vee \mu_C), \nu_A \vee (\nu_B \wedge \nu_C))\\ &= ((\mu_A \wedge \mu_B) \vee (\mu_A \wedge \mu_C), (\nu_A \vee \nu_B) \wedge (\nu_A \vee \nu_C))\\ &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B) \cup (\mu_A \wedge \mu_C, \nu_A \vee \nu_C)\\ &= (A \cap B) \cup (A \cap C).\end{aligned}$$

By the similar arguments, we can see that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$$\begin{aligned}(4')\ A \cup (\textstyle\bigcap_{\alpha \in \Gamma} A_\alpha) &= A \cup (\textstyle\bigwedge_{\alpha \in \Gamma} \mu_{A_\alpha}, \bigvee_{\alpha \in \Gamma} \nu_{A_\alpha})\\ &= (\mu_A \vee (\textstyle\bigwedge_{\alpha \in \Gamma} \mu_{A_\alpha}), \nu_A \wedge (\bigvee_{\alpha \in \Gamma} \nu_{A_\alpha}))\\ &= (\textstyle\bigwedge_{\alpha \in \Gamma}(\mu_A \vee \mu_{A_\alpha}), \bigvee_{\alpha \in \Gamma}(\nu_A \wedge \nu_{A_\alpha}))\\ &= \textstyle\bigcap_{\alpha \in \Gamma}(\mu_A \vee \mu_{A_\alpha}, \nu_A \wedge \nu_{A_\alpha})\\ &= \textstyle\bigcap_{\alpha \in \Gamma}(A \cap A_\alpha).\end{aligned}$$

By the similar arguments, we can see that $A \cap (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma}(A \cap A_\alpha)$.

$$\begin{aligned}(6)\ (\textstyle\bigcap_{\alpha \in \Gamma} A_\alpha)^c &= (\textstyle\bigwedge_{\alpha \in \Gamma} \mu_{A_\alpha}, \bigvee_{\alpha \in \Gamma} \nu_{A_\alpha})^c = (\bigvee_{\alpha \in \Gamma} \nu_{A_\alpha}, \bigwedge_{\alpha \in \Gamma} \mu_{A_\alpha})\\ &= \textstyle\bigcup_{\alpha \in \Gamma}(\nu_{A_\alpha}, \mu_{A_\alpha}) = \bigcup_{\alpha \in \Gamma} A_\alpha^c.\end{aligned}$$

Similarly, we can see that $(\bigcup_{\alpha \in \Gamma} A_\alpha)^c = \bigcap_{\alpha \in \Gamma} A_\alpha^c$. □

Definition 1.6 (S. J. Lee & E. P. Lee [6]). Let $\lambda, \mu \in I$ and $\lambda + \mu \leq 1$ and let $A \in IFS(X)$. Then A is called an *intuitionistic fuzzy point* (in short IFP)*with the support* $x \in X$ *and the value* λ *and the nonvalued* μ if for each $y \in X$

$$A(y) = \begin{cases} (\lambda, \mu) & \text{if } y = x, \\ (0,1) & \text{if } y \neq x. \end{cases}$$

In this case, we write $A = x_{(\lambda,\mu)}$.

An IFP $x_{(\lambda,\mu)}$ is said to *belong to* an IFS A in X, denoted by $x_{(\lambda,\mu)} \in A$, if $\lambda \leq \mu_A(x)$ and $\mu \geq \nu_A(x)$.

It is clear that $x_{(\lambda,\mu)} = (x_\lambda, 1 - x_{1-\mu})$.

We will denote the set of all IFPs in X as $IF_P(X)$.

Result 1.B (S. J. Lee & E. P. Lee [6], Theorem 2.2). Let $A \in IFS(X)$. Then $x_{(\lambda,\mu)} \in A$ if and only if $x_\lambda \in \mu_A$ and $x_{1-\mu} \in \nu_A^c$.

Result 1.C (S. J. Lee & E. P. Lee [6], Theorem 2.3). Let $A, B = IFS(X)$. Then $A \subset B$ if and only if $x_{(\lambda,\mu)} \in A$ implies $x_{(\lambda,\mu)} \in B$ for each $x_{(\lambda,\mu)} \in IF_P(X)$.

Result 1.D (S. J. Lee & E. P. Lee [6], Theorem 2.4). Let $A \in IFS(X)$. Then $A = \bigcup\{x_{(\lambda,\mu)} : x_{(\lambda,\mu)} \in A\}$.

Proposition 1.7. *Let* $A, B \in IFS(X)$ *and let* $\{A_\alpha\}_{\alpha\in\Gamma} \subset IFS(X)$.

(1) *If* $x_{(\lambda,\mu)} \in A$ *or* $x_{(\lambda,\mu)} \in B$, *then* $x_{(\lambda,\mu)} \in A \cup B$.

(1′) *If there exists an* $\alpha \in \Gamma$ *such that* $x_{(\lambda,\mu)} \in A_{\alpha_0}$, *then* $x_{(\lambda,\mu)} \in \bigcup_{\alpha\in\Gamma} A_\alpha$.

(2) $x_{(\lambda,\mu)} \in A \cap B$ *if and only if* $x_{(\lambda,\mu)} \in A$ *and* $x_{(\lambda,\mu)} \in B$.

(2′) $x_{(\lambda,\mu)} \in \bigcap_{\alpha\in\Gamma} A_\alpha$ *if and only if* $x_{(\lambda,\mu)} \in A_\alpha$ *for each* $\alpha \in \Gamma$.

Proof. (1) Suppose $x_{(\lambda,\mu)} \in A$ or $x_{(\lambda,\mu)} \in B$. Then, by Result 1.B, $\lambda \leq \mu_A(x)$ and $\nu_A(x) \leq \mu$ or $\lambda \leq \mu_B(x)$ and $\nu_B(x) \leq \mu$. Thus $\lambda \leq \mu_A(x) \vee \mu_B(x)$ and $\nu_A(x) \wedge \nu_B(x) \leq \mu$. So $X \leq \mu_{A\cup B}(x)$ and $\nu_{A\cup B}(x) \leq \mu$. Hence $x_{\lambda,\mu} \in A \cup B$.

(1′) Suppose there exists an $\alpha_0 \in \Gamma$ such that $x_{(\lambda,\mu)} \in A_{\alpha_0}$. Then, by Result 1.B, $\lambda \leq \mu_{A_{\alpha_0}}(x)$ and $\nu_{A_{\alpha_0}}(x) \leq \mu$. Thus $\lambda \leq \bigvee_{\alpha\in\Gamma} \mu_{A_\alpha}(x)$ and $\bigwedge_{\alpha\in\Gamma} \nu_{A_\alpha}(x) \leq \mu$. Hence $x_{(\lambda,\mu)} \in \bigvee_{\alpha\in\Gamma} A_\alpha$.

(2) $x_{(\lambda,\mu)} \in A \cap B$

if and only if $x_{(\lambda,\mu)} \in (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$

if and only if $x_\lambda \in \mu_A \wedge \mu_B, x_{1-\mu} \in (\nu_A \vee \nu_B)^c$ (By Result 1.B)

if and only if $\lambda \leq \mu_A(x) \wedge \mu_B(x)$ and $\mu \geq \nu_A(x) \vee \nu_B(x)$

if and only if $\lambda \leq \mu_A(x)$, $\lambda \leq \mu_B(x)$ and $\mu \geq \nu_A(x)$, $\mu \geq \nu_B(x)$

if and only if $\lambda \leq \mu_A(x), \mu \geq \nu_A(x)$ and $\lambda \leq \mu_B(x), \mu \geq \nu_B(x)$
if and only if $x_{(\lambda,\mu)} \in A$ and $x_{(\lambda,\mu)} \in B$. (By Result 1.B)

(2′) $x_{(\lambda,\mu)} \in \bigcap_{\alpha\in\Gamma} A_\alpha$
if and only if $x_{(\lambda,\mu)} \in (\bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}, \bigvee_{\alpha\in\Gamma} \nu_{A_\alpha})$
if and only if $x_\lambda \in \bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}$ and $x_{1-\mu} \in (\bigvee_{\alpha\in\Gamma} \nu_{A_\alpha})^c$ (By Result 1.B)
if and only if $\lambda \leq \bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}(x)$ and $\mu \geq \bigvee_{\alpha\in\Gamma} \nu_{A_\alpha}(x)$
if and only if $\lambda \leq \mu_{A_\alpha}(x)$ and $\mu \geq \nu_{A_\alpha}(x)$ for each $\alpha \in \Gamma$
if and only if $x_\lambda \in \mu_{A_\alpha}$ and $x_{1-\mu} \in {\nu_{A_\alpha}}^c$ for each $\alpha \in \Gamma$
if and only if $x_{(\lambda,\mu)} \in A_\alpha$ for each $\alpha \in \Gamma$.

In general, the converses of (1) and (1)′ do not hold. □

Example 1.8. Let $X = \{x\}$ and let A and B be two intuitionistic fuzzy sets in X defined as follows :

$$A(x) = (0.7, 0.3) \quad \text{and} \quad B(x) = (0.5, 0.1).$$

Then clearly $(A \cup B)(x) = (0.7, 0.1)$. Thus $x_{(0.6,0.2)} \in A \cup B$. But $x_{(0.6,0.2)} \notin A$ and $x_{(0.6,0.2)} \notin B$.

Definition 1.9. Let $A \in IFS(X)$ and let $x_{(\lambda,\mu)} \in IF_p(X)$. Then $x_{(\lambda,\mu)}$ is said to be *quasi-coincident with* A, denoted by $x_{(\lambda,\mu)} qA$ if $x_{(\lambda,\mu)} \notin A^c$, *i. e.*, $\lambda > \nu_A(x)$ or $\mu < \mu_A(x)$.

Remark 1.10. Let $\mu_A \in I^X$. Then $x_{(\lambda,\mu)} q(\mu_A, \mu_A^c)$ if and only if $x_\lambda q \mu_A$ or $\mu < \mu_A(x)$.

Definition 1.11. Let $A, B \in IFS(X)$. Then A is said to be *quasi-coincident with* B, denoted by AqB, if there exists an $x \in X$ such that $\nu_B(x) < \mu_A(x)$ or $\mu_B(x) > \nu_A(x)$.

Proposition 1.12. *Let $A, B \in IFS(X)$. Then $A \subset B$ if and only if $A\bar{q}B^c$. In particular, $x_{(\lambda,\mu)} \in A$ if and only if $x_{(\lambda,\mu)}\bar{q}A^c$.*

Proof. $A\bar{q}B^c$
if and only if $\sim (\exists x \in X$ such that $\mu_B(x) < \mu_A(x)$ or $\nu_B(x) > \nu_A(x))$
if and only if $\forall x \in X$, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$
if and only if $A \subset B$. □

Corollary 1.13. *For any $A \in IFS(X)$. $A\bar{q}A^c$.*

Proposition 1.14. *Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset IFS(X)$ and let $A, B \in IFS(X)$. Then:*

(1) $A \subset B$ *if and only if* $x_{(\lambda,\mu)}qB$ *for each* $x_{(\lambda,\mu)}qA$
if and only if $x_{(\lambda,\mu)}\bar{q}A$ *for each* $x_{(\lambda,\mu)}\bar{q}B$.
(2) *If there exists an* $\alpha_0 \in \Gamma$ *such that* $x_{(\lambda,\mu)}qA_{\alpha_0}$, *then* $x(\lambda,\mu)q(\bigcup_{\alpha\in\Gamma})$.
(3) AqB *if and only if there exists an* $x_{(\lambda,\mu)} \in A$ *such that* $x_{(\lambda,\mu)}qB$.

Proof. (1)($\Rightarrow$): Suppose $A \subset B$ and let $x_{(\lambda,\mu)}qA$. Then $\lambda > \nu_A(x)$ or $\mu < \mu_A(x)$. Since $A \subset B$, $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$. Thus $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$. So $\lambda > \nu_B(x)$ or $\mu < \mu_B(x)$. Hence $x_{(\lambda,\mu)}qB$.

($\Leftarrow$): Suppose the necessary condition holds. Assume that $A \not\subset B$. Then there exists a $y \in X$ such that $\mu_A(y) > \mu_B(y)$ or $\nu_A(y) < \nu_A(y)$. Let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$ such that $\nu_A(y) < \lambda < \nu_B(y)$ or $\mu_B(y) < \mu < \mu_A(y)$. Then $y_{(\lambda,\mu)}qA$. By the hypothesis, $y_{(\lambda,\mu)}qB$. Thus $\lambda > \nu_B(y)$ or $\mu < \mu_B(y)$. This contradicts the fact that $\lambda < \nu_B(y)$ or $\mu_B(y) < \mu$. Hence $A \subset B$.

It is clear that $x_{(\lambda,\mu)}qB$ for each $x_{(\lambda,\mu)}qA$ if and only if $x_{(\lambda,\mu)}\bar{q}A$ for each $x_{(\lambda,\mu)}\bar{q}B$.

(2) Suppose there exists an $\alpha_0 \in \Gamma$ such that $x_{(\lambda,\mu)}qA_{\alpha_0}$. Then $\lambda > \nu_{A_{\alpha_0}}(x)$ or $\mu < \mu_{A_{\alpha_0}}(x)$. Thus $\lambda > \bigwedge_{\alpha\in\Gamma}\nu_{A_\alpha}(x)$ or $\mu < \bigvee_{\alpha\in\Gamma}\mu_{A_\alpha}(x)$.

(3)($\Rightarrow$): Suppose AqB. Then there exists an $x \in X$ such that $\nu_B(x) < \mu_A(x)$ or $\mu_B(x) < \nu_A(x)$.

Case (i): Suppose $\nu_B(x) < \mu_A(x)$. Then clearly there exists a $\lambda \in I$ such that $\nu_B(x) < \lambda < \mu_A(x)$. Since $\mu_A(x) + \nu_A(x) \leq 1$, $\lambda < 1 - \nu_A(x)$. Then there exists a $\mu \in I$ such that $\lambda < 1 - \mu < 1 - \nu_A(x)$. Thus $\mu > \nu_A(x)$. So $x_{(\lambda,\mu)} \in A$. Since $\lambda > \nu_B(x)$, $x_{(\lambda,\mu)}qB$.

Case (ii): Suppose $\mu_B(x) > \nu_A(x)$. By the similar arguments as the proof of Case(i), we can see that the necessary condition holds.

($\Leftarrow$): It is obvious from Definition 1.11. □

Definition 1.15 (Çoker [3]). Let X and Y be nonempty sets and let $f : X \to Y$ a mapping. Let $A \in IFS(X)$ and $B \in IFS(Y)$.

(1) The *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$ and $f^{-1}(\nu_B) = \nu_B \circ f$.

(2) The *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by

$$f(A) = (f(\mu_A), f(\nu_B)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \Big(\bigvee_{x\in f^{-1}(y)} \mu_A(x), \bigwedge_{x\in f^{-1}(y)} \nu_A(x)\Big) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0_\sim & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Result 1.E (Çoker [3], Corollary 2.10). Let $A \in IFS(X), \{A_\alpha\}_{\alpha\in\Gamma} \subset IFS(X)$, let $B \in IFS(Y)$, $\{B_\alpha\}_{\alpha\in\Gamma} \subset IFS(Y)$ and let $f : X \to Y$ a mapping. Then

(1) $A_\alpha \subset A_\beta \Rightarrow f(A_\alpha) \subset f(A_\beta)$.
(2) $B_\alpha \subset B_\beta \Rightarrow f^{-1}(B_\alpha) \subset f^{-1}(B_\beta)$.
(3) $A \subset f^{-1}(f(A))$. If f is injective, then $A = f^{-1}(f(A))$.
(4) $f(f^{-1}(B)) \subset B$. If f is surjective, then $f(f^{-1}(B)) = B$.
(5) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha)) = \bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)$.
(6) $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha)) = \bigcap_{\alpha\in\Gamma} f^{-1}(B_\alpha)$.
(7) $f(\bigcup_{\alpha\in\Gamma} A_\alpha)) = \bigcup_{\alpha\in\Gamma} f(A_\alpha)$.
(8) $f(\bigcap_{\alpha\in\Gamma} A_\alpha)) \subset \bigcap_{\alpha\in\Gamma} f(A_\alpha)$.
If f is injective, then $f(\bigcap A_\alpha)) = \bigcap f(A_\alpha)$.
(9) $f^{-1}(1_\sim) = 1_\sim$, $f^{-1}(0_\sim) = 0_\sim$.
(10) $f(1_\sim) = 1_\sim$ if f is surjective.
(11) $f(0_\sim) = 0_\sim$.
(12) $[f(A)]^c \subset f(A^c)$ if f is surjective.
(13) $f^{-1}(B^c) = [f^{-1}(B)]^c$.

Proposition 1.16. *Let $A \in IFS(X)$ and let $f : X \to Y$ a mapping. If f is bijective, then $[f(A)]^c = f(A^c)$.*

Proof.
$$\begin{aligned} [f(A)]^c &= f\big(f^{-1}[f(A)]^c\big) && \text{(by Result 1.E(4))} \\ &= f\big(f^{-1}(f(A))\big)^c && \text{(by Result 1.E(13))} \\ &= f(A^c) && \text{(by Result 1.E(3)).} \end{aligned}$$
□

Proposition 1.17. *Let $f : X \to Y$ be a mapping and let $A \in IFS(X)$, $B \in IFS(Y)$. Then*

(1) *$f(x_{(\lambda,\mu)}) \in IF_P(Y)$ and $f(x_{(\lambda,\mu)}) = [f(x)]_{(\lambda,\mu)}$ for each $x_{(\lambda,\mu)} \in IF_P(X)$.*
(2) *If f is injective and $y_{(\lambda,\mu)} \in IF_P(f(X))$, then*
$$f^{-1}(y_{(\lambda,\mu)}) = f^{-1}(y)_{(\lambda,\mu)} \in IF_P(X).$$
In general, $f^{-1}(y_{(\lambda,\mu)})$ needs not be an IFP in X for each $y_{(\lambda,\mu)} \in IF_P(Y)$.
(3) *If $x_{(\lambda,\mu)} \in A$, then $[f(x)]_{(\lambda,\mu)} \in f(A)$.*
(4) *If $[f(x)]_{(\lambda,\mu)} \in B$, then $x_{(\lambda,\mu)} \in f^{-1}(B)$.*

Proof. (1) Let $y \in Y$. Then:

$$\mu_{f(x_{(\lambda,\mu)})}(y) = f(\mu_{x_{(\lambda,\mu)}})(y) = \bigvee_{z \in f^{-1}(y)} \mu_{x_{(\lambda,\mu)}}(z) = \begin{cases} \lambda & \text{if } z = x, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu_{f(x_{(\lambda,\mu)})}(y) = f(\nu_{x_{(\lambda,\mu)}})(y) = \bigwedge_{z \in f^{-1}(y)} \nu_{x_{(\lambda,\mu)}}(z) = \begin{cases} \mu & \text{if} z = x, \\ 1 & \text{otherwise.} \end{cases}$$

Hence $f(x_{(\lambda,\mu)}) = [f(x)]_{(\lambda,\mu)}$.

(2) By the hypothesis, there is a unique $x \in X$ such that $y \in f(x)$. Let $z \in X$. Then

$$\mu_{f^{-1}(y_{(\lambda,\mu)})}(z) = f^{-1}(\mu_{y_{(\lambda,\mu)}})(z) = \mu_{y_{(\lambda,\mu)}}(f(z)) = \begin{cases} \lambda & \text{if } f(z) = y = f(x), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\nu_{f^{-1}(y_{(\lambda,\mu)})}(z) = f^{-1}(\nu_{y_{(\lambda,\mu)}})(z) = \nu_{y_{(\lambda,\mu)}}(f(z)) = \begin{cases} \mu & \text{if } f(z) = y = f(x), \\ 1 & \text{otherwise.} \end{cases}$$

Hence $f^{-1}(y_{(\lambda,\mu)}) = x_{(\lambda,\mu)} = f^{-1}(y)_{(\lambda,\mu)}$.

The proofs of (3) and (4) are obvious. □

The following is the immediate result of Definition 1.15:

Proposition 1.18. *Let $f : X \to Y$ and $g : Y \to Z$ be mappings.*

(1) *If $B \in IFS(Z)$, then $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$.*

(2) *If $A \in IFS(X)$, then $(g \circ f)(A) = g(f(A))$.*

Proposition 1.19. *Let $f : X \to Y$ be a mapping, let $x_{(\lambda,\mu)} \in IF_P(X)$ and let $A \in IFS(X)$, $B \in IFS(Y)$.*

(1) *If $f(x_{(\lambda,\mu)})qB$, then $x_{(\lambda,\mu)}qf^{-1}(B)$.*

(2) *If $x_{(\lambda,\mu)}qA$, then $f(x_{(\lambda,\mu)})qf(A)$, i.e., $[f(x)]_{(\lambda,\mu)}qf(A)$.*

Proof. (1) By Definition 1.8, $\lambda > \nu_B(f(x))$ or $\mu < \mu_B(f(x))$. Thus, by Definition 1.15(1), $\lambda > f^{-1}(\nu_B)(x)$ or $\mu < f^{-1}(\mu_B)(x)$. Hence, by Definition 1.9, $x_{(\lambda,\mu)} q f^{-1}(B)$.

(2) By Definition 1.8, $\lambda > \nu_A(x)$ or $\mu < \mu_A(x)$. We will show that $\lambda > f(\nu_A)(f(x))$ or $\mu < f(\mu_A)(f(x))$. By Definition 1.15(2),

$$f(\nu_A)(f(x)) = \bigwedge_{x \in f^{-1}(f(x))} \nu_A(x) = \nu_A(x)$$

and

$$f(\mu_A)(f(x)) = \bigvee_{x \in f^{-1}(f(x))} \mu_A(x) = \mu_A(x).$$

Thus $\lambda > f(\nu_A)(f(x))$ or $\mu < f(\mu_A)(f(x))$. Hence $f(x_{(\lambda,\mu)}) = [f(x)]_{(\lambda,\mu)} q f(A)$. □

Definition 1.20. Let $A \in IFS(X)$ and let $B \in IFS(Y)$. Then the *product* of A and B denoted by $A \times B$, is an IFS in $X \times Y$, defined as follows: for each $(x, y) \in X \times Y$,

$$\mu_{A\times B}(x,y) = \mu_A(x) \wedge \mu_B(y) \quad \text{and} \quad \nu_{A\times B}(x,y) = \nu_A(x) \vee \nu_B(y).$$

Hence $A \times B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

The *product mapping* $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = \big(f_1(x_1), f_2(x_2)\big)$ for each $(x_1, x_2) \in X_1 \times X_2$. And, for a mapping $f : X \to Y$, the *graph* $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$ for each $x \in X$.

Proposition 1.21. *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be mappings.*

(1) *If $A_i \in IFS(X_i)$, then $(f_1 \times f_2)(A_1 \times A_2) \subset f_1(A_1) \times f_2(A_2)$.*

(2) *If $B_i \in IFS(Y_i)$, then $(f_1 \times f_2)^{-1}(B_1 \times B_2) \subset f_1^{-1}(B_1) \times f_2^{-1}(B_2)$.*

Proof. Since

$$(f_1 \times f_2)(A_1 \times A_2) = \big((f_1 \times f_2)(\mu_{A_1\times A_2}), (f_1 \times f_2)(\nu_{A_1\times A_2})\big)$$

and

$$f_1(A_1) \times f_2(A_2) = \big(f_1(\mu_{A_1}) \times f_2(\mu_{A_2}), f_1(\nu_{A_1}) \times f_2(\nu_{A_2})\big),$$

it is enough to show that

$$(f_1 \times f_2)(\mu_{A_1\times A_2}) \leq f_1(\mu_{A_1}) \times f_2(\mu_{A_2})$$

and

$$(f_1 \times f_2)(\nu_{A_1\times A_2}) \geq f_1(\nu_{A_1}) \times f_2(\mu_{A_2}).$$

Let $(y_1, y_2) \in Y_1 \times Y_2$ and suppose $(f_1 \times f_2)^{-1}(y_1, y_2) \neq \varnothing$. Then:

$$\begin{aligned}(f_1 \times f_2)(\mu_{A_1 \times A_2})(y_1, y_2) &= \bigvee_{(x_1,x_2)\in(f_1\times f_2)^{-1}(y_1,y_2)} \mu_{A_1\times A_2}(x_1, x_2) \\ &= \bigvee_{(x_1,x_2)\in(f_1\times f_2)^{-1}(y_1,y_2)} [\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)] \\ &\leq \left[\bigvee_{x_1\in f_1^{-1}(y_1)} \mu_{A_1}(x_1)\right] \wedge \left[\bigvee_{x_2\in f_2^{-1}(y_2)} \mu_{A_2}(x_2)\right] \\ &= f_1(\mu_{A_1})(y_1) \wedge f_2(\mu_{A_2})(y_2) \\ &= [f_1(\mu_{A_1}) \times f_2(\mu_{A_2})](y_1, y_2).\end{aligned}$$

Thus $(f_1 \times f_2)(\mu_{A_1\times A_2}) \leq f_1(\mu_{A_1}) \times f_2(\mu_{A_2})$. On the other hand,

$$\begin{aligned}(f_1 \times f_2)(\nu_{A_1 \times A_2})(y_1, y_2) &= \bigwedge_{(x_1,x_2)\in(f_1\times f_2)^{-1}(y_1,y_2)} \nu_{A_1\times A_2}(x_1, x_2) \\ &= \bigwedge_{(x_1,x_2)\in(f_1\times f_2)^{-1}(y_1,y_2)} [\nu_{A_1}(x_1) \vee \nu_{A_2}(x_2)] \\ &\geq \left[\bigwedge_{x_1\in f_1^{-1}(y_1)} \nu_{A_1}(x_1)\right] \vee \left[\bigwedge_{x_2\in f_2^{-1}(y_2)} \nu_{A_2}(x_2)\right] \\ &= f_1(\nu_{A_1})(y_1) \vee f_2(\nu_{A_2})(y_2) \\ &= [f_1(\nu_{A_1}) \times f_2(\nu_{A_2})](y_1, y_2).\end{aligned}$$

Thus $(f_1 \times f_2)(\nu_{A_1\times A_2}) \geq f_1(\nu_{A_1}) \times f_2(\nu_{A_2})$. This completes the proof. □

Proposition 1.22. *Let $g : X \to X \times Y$ be the graph of a mapping $f : X \to Y$. If $A \in IFS(X)$ and $B \in IFS(Y)$, then $g^{-1}(A \times B) = A \cap f^{-1}(B)$.*

Proof. Let $x \in X$. Then

$$\begin{aligned}g^{-1}(A \times B)(x) &= (A \times B)[g(x)] = (A \times B)(x, f(x)) \\ &= A(x) \wedge B(f(x)) = A(x) \wedge f^{-1}(B)(x) \\ &= (A \cap f^{-1}(B))(x).\end{aligned}$$

Hence $g^{-1}(A \times B) = A \cap f^{-1}(B)$. □

Proposition 1.23. *Let $A \in IFS(X)$ and let $B \in IFS(Y)$. Then*

$$(A \times B)^c = (A^c \times Y) \cup (X \times B^c).$$

Proof. $(A^c \times Y) \cup (X \times B^c) = (\nu_A \wedge \mu_Y, \mu_A \vee \nu_Y) \cup (\mu_X \wedge \nu_B, \nu_X \vee \mu_B)$

$$= (\nu_A \wedge 1, \mu_A \vee 0) \cup (1 \wedge \nu_B, 0 \vee \mu_B)$$
$$= (\nu_A, \mu_A) \cup (\nu_B, \mu_B)$$
$$= (\nu_A \vee \nu_B, \mu_A \wedge \mu_B)$$
$$= (A \times B)^c.$$

□

Definition 1.24. Let $A \in IFS(X)$. Then the set

$$\{x \in X : \mu_A(x) > 0 \quad \text{and} \quad \nu_A(x) < 1\}$$

is called the *support* of A and denoted by $S(A)$ or $A_{0_\sim}$.

Proposition 1.25. *Let* $\{A_\alpha\}_{\alpha\in\Gamma} \subset IFS(X)$. *Then*

(1) $S(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} S(A_\alpha)$.

(2) $S(\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma} S(A_\alpha)$.

Proof. (1) $x \in S(\bigcup_{\alpha\in\Gamma} A_\alpha)$

if and only if $\mu_{\bigcup_{\alpha\in\Gamma} A_\alpha}(x) > 0$ and $\nu_{\bigcup_{\alpha\in\Gamma} A_\alpha}(x) < 1$

if and only if $\bigvee_{\alpha\in\Gamma} \mu_{A_\alpha}(x) > 0$ and $\bigwedge_{\alpha\in\Gamma} \nu_{A_\alpha}(x) < 1$

(Since $\mu_{A_\alpha}(x) + \nu_{A_\alpha}(x) \leq 1$)

if and only if there exists an $\alpha \in \Gamma$ such that $\mu_{A_\alpha}(x) > 0$ and $\nu_{A_\alpha}(x) < 1$

if and only if there exists an $\alpha \in \Gamma$ such that $x \in S(A_\alpha)$

if and only if $x \in \bigcup_{\alpha\in\Gamma} S(A_\alpha)$.

(2) $x \in S(\bigcap_{\alpha\in\Gamma} A_\alpha)$

if and only if $\mu_{\bigcap_{\alpha\in\Gamma} A_\alpha}(x) > 0$ and $\nu_{\bigcap_{\alpha\in\Gamma} A_\alpha}(x) < 1$

if and only if $\bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}(x) > 0$ and $\bigvee_{\alpha\in\Gamma} \nu_{A_\alpha}(x) < 1$

if and only if $\mu_{A_\alpha}(x) > 0$ and $\nu_{A_\alpha}(x) < 1$, for each $\alpha \in \Gamma$

if and only if $x \in S(A_\alpha)$ for each $\alpha \in \Gamma$

if and only if $x \in \bigcap_{\alpha\in\Gamma} S(A_\alpha)$. □

2. Some properties of intuitionistic fuzzy topological spaces

Definition 2.1 (Çoker [3]). Let X be a nonempty set and let $T \subset IFS(X)$. Then T is called an *intuitionistic fuzzy topology* (in short, *IFT*) on X if it satisfies the following axioms :

(T_1) $0_\sim, 1_\sim \in T$.

(T_2) $G_1 \cap G_2 \in T$ for any $G_1, G_2 \in T$.

(T_3) $\bigcup G_\lambda \in T$ for any $\{G_\lambda\}_{\lambda \in \Gamma} \subset T$.

In this case, the pair (X, T) is called an *intuitionistic fuzzy topological space*(in short, *IFTS*) and each member G of T is called an *intuitionistic fuzzy open set*(in short, *IFOS*) in X.

It is clear that if (X, T_C) is an fts in the sense of Chang, then (X, T) is an IFTS, where $T = \{A = (\mu_A, \mu_A^c) : \mu_A \in T_C\}$ (See Example 3.2 in Çoker & Eş [4]).

Example 2.2. (1) Let X be a non-empty set and let $\mathcal{I} = \{0_\sim, 1_\sim\}$. Then clearly $\mathcal{I}$ is an IFT on X. In this case, $\mathcal{I}$ is called the *intuitionistic fuzzy indiscrete topology*(in short, *IFIT*) on X and the pair $(X, \mathcal{I})$ is called an *intuitionistic fuzzy indiscrete space*(in short, *IFIS*).

(2) Let X be a nonempty set and let $\mathcal{D} = IFS(X)$. Then $\mathcal{D}$ is an IFT on X. In this case, $\mathcal{D}$ is called the *intuitionistic fuzzy discrete topology*(in short, *IFDT*) on X and the pair $(X, \mathcal{D})$ is called an *intuitionistic fuzzy discrete space*(in short, *IFDS*).

(3) Let (X, T_C) be an fts in the sense of Chang such that T_C is not indiscrete. Then we can construct two IFTs on X as follows:

(a) $T^1 = \{0_\sim, 1_\sim\} \cup \{(\mu_G, 0) : \mu_{G_\alpha} \in T_C\}$,

(b) $T^2 = \{0_\sim, 1_\sim\} \cup \{(0, \mu_G^c) : \mu_{G_\alpha} \in T_C\}$,

where $T_C = \{0, 1\} \cup \{\mu_{G_\alpha} : \alpha \in \Gamma\}$ (See Example 3.5 in Çoker [3]).

Result 2.A (Çoker [3], Proposition 3.6). Let (X, T) be an IFTS. Then we can also construct two IFTs on X in the following way:

(a) $T_{c,1} = \{[\,]G : G \in T\} = \{(\mu_G, \mu_G^c) : G \in T\}$.

(b) $T_{c,2} = \{\langle\rangle G : G \in T\} = \{(\nu_G^c, \nu_G) : G \in T\}$.

Remark 2.3 (Çoker [3], Remark 3.7). Let (X, T) be an IFTS. Then:

(a) $T_1 = \{\mu_G : G \in T\}$ is a fuzzy topology on X in the sense of Chang.

(b) $T_2 = \{\nu_G^c : G \in T\}$ is a fuzzy topology on X in the sense of Chang.

Hence we may conclude that (X, T_1, T_2) is a fuzzy bi-topological space.

Definition 2.4 (Çoker [3]). Let (X, T) be an IFTS. Then (X, T) is called an *IFTS in the sense of Lowen* if for any $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$, $C_{\lambda,\mu} = (\lambda, \mu) \in T$.

It is clear that if (X, T) be an IFTS in the sense of Lowen, then we can construct two fuzzy topologies T_1 and T_2 on X in the sense of Lowen as follows (See Example 3.11 in Çoker [3]):

(a) $T_1 = \{\mu_G : G \in T\}$.
(b) $T_2 = \{\nu_G^c : G \in T\}$.

Definition 2.5 (Çoker [3]). Let (X, T) be an IFTS and let $A \in IFS(X)$. Then A is called an *intuitionistic fuzzy closed set* (in short, *IFCS*) in X if $A^c \in T$.

Notation 2.6. Let X be an IFTS. Then:
(a) $IFO(X)$ denotes the set of all IFOSs in X.
(b) $IFC(X)$ denotes the set of all IFCSs in X.

Proposition 2.7. *Let X be an IFTS. Then the following conditions hold :*
(1) $1_\sim,\ 0_\sim \in IFC(X)$
(2) *If* $A_1, A_2 \in IFC(X)$, *then* $A_1 \cup A_2 \in IFC(X)$.
(3) *If* $\mathcal{A} \subset IFC(X)$, *then* $\cap\mathcal{A} \in IFC(X)$.

Proof. (1) $1_\sim^c = (0,1) = 0_\sim \in IFO(X)$ and $0_\sim^c = (1,0) = 1_\sim \in IFO(X)$. So $1_\sim, 0_\sim \in IFC(X)$.

(2) Suppose $A_1, A_2 \in IFC(X)$. Then $A_1^c, A_2^c \in IFO(X)$. Then

$$A_1^c \cap A_2^c = (A_1 \cup A_2)^c \in IFO(X).$$

So $A_1 \cup A_2 \in IFC(X)$.

(3) Suppose $\mathcal{A} \subset IFC(X)$. Then $A \in IFC(X)$ for each $A \in \mathcal{A}$. Then

$$A^c \in IFO(X)$$

for each $A \in \mathcal{A}$. So $\bigcup_{A\in\mathcal{A}} A^c \in IFO(X)$ and $\bigcup_{A\in\mathcal{A}} A^c = (\bigcap_{A\in\mathcal{A}} A)^c = (\bigcap \mathcal{A})^c$. Hence $\bigcap \mathcal{A} \in IFC(X)$. □

Proposition 2.8. *Let X be an infinite set and let $T = \{U \in IFS(X) : U = 0_\sim$ or U^c is finite$\}$, where $A \in IFS(X)$ is finite if and only if $S(A) = \{x \in X : \mu_A(x) > 0$ and $\nu_A(x) < 1\}$ is finite. Then T is an IFT on X. In this case, T is called the intuitionistic fuzzy complement topology or intuitionistic fuzzy cofinite topology (in short, $IFCFT$) on X and denoted by $Cof_{IF}(X)$.*

Proof. From the definition of T, it is clear that $0_\sim, 1_\sim \in T$. Let $U, V \in T$.

Case (i): Suppose $U = 0_\sim$ or $V = 0_\sim$. Then clearly $U \wedge V = 0_\sim$. So $U \cap V \in T$.

Case (ii): Suppose $U \neq 0_\sim$ and $V \neq 0_\sim$. Then U^c and V^c are finite.

Thus $S(U^c)$ and $S(V^c)$ are finite. Thus $S(U^c) \cup S(V^c)$ is finite. By Proposition 1.25(1) and Proposition 1.5(6), $S(U^c \cup V^c) = S[(U \cap V)^c]$ is finite. So $(U \cap V)^c$ is finite. Hence $U \cap V \in T$. Let $\{U_\alpha\}_{\alpha\in\Gamma} \subset T$.

Case (i): Suppose $U_\alpha = 0_\sim$ for each $\alpha \in \Gamma$. Then clearly $\bigcup_{\alpha\in\Gamma} U_\alpha = 0_\sim$. Thus $\bigcup_{\alpha\in\Gamma} U_\alpha \in T$.

Case (ii): Suppose there exists an $\alpha \in \Gamma$ such that $U_\alpha \neq \emptyset$. Let

$$\Gamma' = \{\alpha \in \Gamma : U_\alpha \neq 0_\sim\}.$$

Then clearly U_α^c is finite for each $\alpha \in \Gamma'$. Thus $S(U_\alpha^c)$ is finite for each $\alpha \in \Gamma'$. So $\bigcap_{\alpha\in\Gamma'} S(U_\alpha^c)$ is finite. By Proposition 1.25(2), and Proposition 1.5(6), $S(\bigcap_{\alpha\in\Gamma'} U_\alpha^c) = S[(\bigcup_{\alpha\in\Gamma'} U_\alpha)^c]$ is finite. Thus $(\bigcup_{\alpha\in\Gamma'} U_\alpha)^c$ is finite. So

$$\bigcup_{\alpha\in\Gamma'} U_\alpha \in T$$

and hence $\bigcup_{\alpha\in\Gamma} U_\alpha \in T$. This completes the proof. □

By the similar arguments as proof of Proposition 2.8, we can easily show the following result:

Proposition 2.9. *Let X be an infinite set and let $T = \{U \in IFS(X) : U = 0_\sim$ or U^c is countable$\}$, where $A \in IFS(X)$ is countable if and only if $S(A) = \{x \in X : \mu_A(x) > 0$ and $\nu_A(x) < 1\}$ is countable. Then T is an IFT on X.*

In this case, T is called the *intuitionistic fuzzy countable complement topology* or *intuitionistic fuzzy cocountable topology* (in short, it IFCCT) on X and denoted by $Coc_{IF}(X)$.

It is clear that if X is a finite set, then $Cof_{IF}(X) = Coc_{IF}(X) = \mathcal{D}$. Moreover, $Coc_{IF}(\mathbf{N}) = \mathcal{D}$.

Proposition 2.10. *Let X be a non-empty set, let $x_{(\lambda,\mu)} \in IF_P(X)$ and let*

$$T_{x_{(\lambda,\mu)}} = \{U \in IFS(X) : U = 0_\sim \text{ or } x_{(\lambda,\mu)} \in U\}.$$

Then $T_{x_{(\lambda,\mu)}}$ is an IFT on X. In this case, $T_{x_{(\lambda,\mu)}}$ is called the included intuitionistic fuzzy point $x_{(\lambda,\mu)}$ topology on X.

Proof. From the definition of $T_{x_{(\lambda,\mu)}}$, it is clear that $0_\sim, 1_\sim \in T_{x_{(\lambda,\mu)}}$. Let

$$U, V \in T_{x_{(\lambda,\mu)}}.$$

Case (i): Suppose $U = 0_\sim$ or $V = 0_\sim$. Then $U \cap V = 0_\sim$. Thus $U \cap V \in T_{x_{(\lambda,\mu)}}$.

Case (ii): Suppose $U \neq 0_\sim$ and $V \neq 0_\sim$. Then $x_{(\lambda,\mu)} \in U$ and $x_{(\lambda,\mu)} \in V$. Thus, by Proposition 1.7(2), $x_{(\lambda,\mu)} \in U \cap V$. So $U \cap V \in T_{x_{(\lambda,\mu)}}$.

Let $\{U_\alpha\}_{\alpha\in\Gamma} \subset T_{x_{(\lambda,\mu)}}$.

Case (i): Suppose $U_\alpha = 0_\sim$ for each $\alpha \in \Gamma$. Then $\bigcup_{\alpha\in\Gamma} U_\alpha = 0_\sim$. Thus

$$\bigcup_{\alpha\in\Gamma} U_\alpha \in T_{x_{(\lambda,\mu)}}.$$

Case (ii): Suppose there exists an $\alpha \in \Gamma$ such that $U_\alpha \neq 0_\sim$. Then $x_{(\lambda,\mu)} \in U_\alpha$ for some $\alpha \in \Gamma$. By Proposition 1.7(1′), $x_{(\lambda,\mu)} \in \bigcup_{\alpha\in\Gamma} U_\alpha$. So $\bigcup_{\alpha\in\Gamma} U_\alpha \in T_{x_{(\lambda,\mu)}}$. This completes the proof. □

Definition 2.11. Let X contain only two elements. Then the included intuitionistic fuzzy point topology on X is called an *intuitionistic fuzzy Sierpinski topology.* Clearly it is the generalization of the ordinary one.

Proposition 2.12. *Let X be a non-empty set, let $x_{(\lambda,\mu)} \in IF_P(X)$ and let*

$$T_{\overline{x}_{(\lambda,\mu)}} = \{U \in IFS(X) : U = 1_\sim \text{ or } x_{(\lambda,\mu)} \in U^c\}.$$

Then $T_{\overline{x}_{(\lambda,\mu)}}$ is an IFT on X. In this case, $T_{\overline{x}_{(\lambda,\mu)}}$ is called the excluded intuitionistic fuzzy point $x_{(\lambda,\mu)}$ topology on X.

Proof. From the definition of $T_{\overline{x}_{(\lambda,\mu)}}$, it is clear that $0_\sim, 1_\sim \in T_{\overline{x}_{(\lambda,\mu)}}$. Let

$$U, V \in T_{\overline{x}_{(\lambda,\mu)}}.$$

Case (i): Suppose $U = 1_\sim$ or $V = 1_\sim$. Then $U \cap V = 1_\sim$. Thus $U \cap V \in T_{\overline{x}_{(\lambda,\mu)}}$.

Case (ii): Suppose $U \neq 1_\sim$ and $V \neq 1_\sim$. Then $x_{(\lambda,\mu)} \in U^c$ and $x_{(\lambda,\mu)} \in V^c$. By Proposition 1.7(1) and Proposition 1.5(6), $x_{(\lambda,\mu)} \in U^c \cup V^c = (U \cap V)^c$. Thus $U \cap V \in T_{\overline{x}_{(\lambda,\mu)}}$. Let $\{U_\alpha\}_{\alpha\in\Gamma} \subset T_{\overline{x}_{(\lambda,\mu)}}$.

Case (i): Suppose there exists an $\alpha \in \Gamma$ such that $U_\alpha = 1_\sim$. Then $\bigcup_{\alpha\in\Gamma} U_\alpha = 1_\sim$. Thus $\bigcup_{\alpha\in\Gamma} U_\alpha \in T_{\overline{x}_{(\lambda,\mu)}}$.

Case (ii): Suppose $U_\alpha \neq 1_\sim$ for each $\alpha \in \Gamma$. Then $x_{(\lambda,\mu)} \in U_\alpha^c$ for each $\alpha \in \Gamma$. By Proposition 1.7(2′) and Proposition 1.5(6), $x_{(\lambda,\mu)} \in \bigcap_{\alpha\in\Gamma} U_\alpha^c = (\bigcap_{\alpha\in\Gamma} U_\alpha)^c$. Thus $\bigcup_{\alpha\in\Gamma} U_\alpha \in T_{\overline{x}_{(\lambda,\mu)}}$. This completes the proof. □

We obtain the similar result as Proposition 2.10:

Proposition 2.13. *Let X be a non-empty set, let $A \in IFS(X)$ and let $T_A = \{U \in IFS(X) : U = 0_\sim$ or $A \subset U\}$. Then T_A is an IFT on X. In this case, T_A is called the induced intuitionistic fuzzy set A topology on X.*

Proof. Clearly $0_\sim, 1_\sim \in T_A$. Let $U, V \in T_A$.

Case (i): Suppose $U = 0_\sim$ or $V = 0_\sim$. Then $U \cap V = 0_\sim$. Thus $U \cap V \in T_A$.

Case (ii): Suppose $U \neq 0_\sim$ and $V \neq 0_\sim$. Then $A \subset U$ and $A \subset V$. Thus $A \subset U \cap V$. So $U \cap V \in T_A$. Let $\{U_\alpha\}_{\alpha\in\Gamma} \subset T_A$.

Case (i) Suppose $U_\alpha = 0_\sim$ for each $\alpha \in \Gamma$. Then $\bigcup_{\alpha\in\Gamma} A_\alpha = 0_\sim$. Thus

$$\bigcup_{\alpha\in\Gamma} A_\alpha \in T_A.$$

Case (i) Suppose there exists an $\alpha \in \Gamma$ such that $U_\alpha \neq 0_\sim$. Then $A \subset U_\alpha$. Thus $A \subset \bigcup_{\alpha\in\Gamma} U_\alpha$.

So $\bigcup_{\alpha\in\Gamma} U_\alpha \in T_A$. This completes the proof. □

Also we obtain the similar result as Proposition 2.12:

Proposition 2.14. *Let X be a non-empty set, let $A \in IFS(X)$ and let*

$$T_{\overline{A}} = \{U \in IFS(X) : U = 1_\sim \quad \text{or} \quad A \subset U^c\}.$$

Then $T_{\overline{A}}$ is an IFT on X. In this case, $T_{\overline{A}}$ is called the excluded intuitionistic fuzzy set A topology on X.

3. INTUITIONISTIC FUZZY NEIGHBORHOODS AND INTUITIONISTIC Q-NEIGHBORHOODS

Definition 3.1. Let (X, T) be an IFTS, let $A \in IFS(X)$ and let $x_{(\lambda,\mu)} \in IF_P(X)$.

(1) A is called an *intuitionistic fuzzy neighborhood* (in short, *IFN*) of $x_{(\lambda,\mu)}$ (*cf.* S. J. Lee & E. P. Lee [6]) if there is a $B \in T$ such that $x_{(\lambda,\mu)} \in B \subset A$. The family of all the IFNs of $x_{(\lambda,\mu)}$ is called the *system of IFNs* of $x_{(\lambda,\mu)}$ and denoted by $\mathcal{N}(x_{(\lambda,\mu)})$.

(2) A is called an *intuitionistic Q-neighborhood*(in short, *IQN*) of $x_{(\lambda,\mu)}$ if there is a $B \in T$ such that $x_{(\lambda,\mu)} q B \subset A$. The family of all the IQNs of $x_{(\lambda,\mu)}$ is called *the system of IQNs* of $x_{(\lambda,\mu)}$ and denoted by $\mathcal{N}_{IQ}(x_{(\lambda,\mu)})$.

Result 3.A (S. J. Lee & E. P. Lee [6], Theorem 2.6). Let X be an $IFTS$ and let $A \in IFS(X)$. Then $A \in IFO(X)$ if and only if $A \in \mathcal{N}(x_{(\lambda,\mu)})$ for each $x_{(\lambda,\mu)} \in IFp(X)$.

We obtain the similar one as Result 3.A using Definition 3.2:

Proposition 3.2. *Let (X, T) be an $IFTS$ and let $A \in IFS(X)$. Then $A \in T$ if and only if $A \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$ for each $x_{(\lambda,\mu)} \in IF_P(X)$ with $x_{(\lambda,\mu)} \in A$.*

The following is the immediate result of Definition 3.1:

Proposition 3.3. *Let (X,T) be an IFTS and let $x_{(\lambda,\mu)} \in IF_p(X)$.*

(1) *If $U \in \mathcal{N}(x_{(\lambda,\mu)})$[resp.$U \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$], then $x_{(\lambda,\mu)} \in U$[resp. $x_{(\lambda,\mu)}qU$].*
(2) *If $U \in \mathcal{N}(x_{(\lambda,\mu)})$ [resp. $U \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$] and $U \subset V$, then $V \in \mathcal{N}(x_{(\lambda,\mu)})$[resp. $V \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$].*
(3) *If $U,V \in \mathcal{N}(x_{(\lambda,\mu)})$[resp. $U,V \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$], then $U \cap V \in \mathcal{N}(x_{(\lambda,\mu)})$[resp. $U \cap V \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$].*
(4) *If $U \in \mathcal{N}(x_{(\lambda,\mu)})$[resp. $U \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$], then there exists a $V \in \mathcal{N}(x_{(\lambda,\mu)})$ [resp. $V \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$] such that $V \subset U$ and $V \in \mathcal{N}(y_{(\lambda',\mu')})$[resp. $V \in \mathcal{N}_{IQ}(y_{(\lambda',\mu')})$] for each $y_{(\lambda',\mu')} \in IF_P(X)$ with $y_{(\lambda',\mu')} \in V$ [resp. $y_{(\lambda',\mu')}qV$].*

Proposition 3.4. *If for each $x_{(\lambda,\mu)} \in IF_P(X)$, $\mathcal{B}(x_{(\lambda,\mu)})$[resp. $\mathcal{B}_{IQ}(x_{(\lambda,\mu)})$] is a family of IFSs in X satisfying the conditions (1),(2), (3) and (4) of Proposition 3.3, then there exits a unique IFT T on X such that for each $x_{(\lambda,\mu)} \in IF_P(X)$, $\mathcal{B}(x_{(\lambda,\mu)}) = \mathcal{N}(x_{(\lambda,\mu)})$ [resp. $\mathcal{B}_{IQ}(x_{(\lambda,\mu)}) = \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$] in (X,T).*

Proof. Let $T = \{U \in IFS(X) : U \in \mathcal{B}(x_{(\lambda,\mu)})$ for each $x_{(\lambda,\mu)} \in U\}$ [*resp.* $T = \{U \in IFS(X) : U \in \mathcal{B}_{IQ}(x_{(\lambda,\mu)})$ for each $x_{(\lambda,\mu)}qU\}$]. Then we can easily check that T is an IFT on X. Moreover, we can see that $\mathcal{B}(x_{(\lambda,\mu)}) = \mathcal{N}(x_{(\lambda,\mu)})$ [*resp.* $\mathcal{B}_{IQ}(x_{(\lambda,\mu)}) = \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$] in (X,T) for each $x_{(\lambda,\mu)} \in IF_P(X)$. □

Definition 3.5. Let (X,T) be an IFTS and let $x_{(\lambda,\mu)} \in IF_P(X)$. Then $\mathcal{B}_{x_{(\lambda,\mu)}} \subset T$ is called a *local base* at $x_{(\lambda,\mu)}$ if it satisfies the following conditions:

(1) If $B \in \mathcal{B}_{x_{(\lambda,\mu)}}$, then $x_{(\lambda,\mu)} \in B$.
(2) If $U \in T$ and $x_{(\lambda,\mu)} \in U$, then there exists a $B \in \mathcal{B}_{x_{(\lambda,\mu)}}$ such that $B \subset U$.

Definition 3.6. Let (X,T) be an IFTS and let $x_{(\lambda,\mu)} \in IF_P(X)$.

(1) $\mathcal{B}_{x_{(\lambda,\mu)}} \subset \mathcal{N}(x_{(\lambda,\mu)})$ is called an *intuitionistic fuzzy neighborhood base*(in short, *IFNB*) of $\mathcal{N}(x_{(\lambda,\mu)})$ if for each $A \in \mathcal{N}(x_{(\lambda,\mu)})$, there is a $B \in \mathcal{B}_{x_{(\lambda,\mu)}}$ such that $B \subset A$.
(1′) $\mathcal{B}_{x_{(\lambda,\mu)}IQ} \subset \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$ is called an *intuitionistic fuzzy Q-neighborhood base*(in short, *IQNB*) of $\mathcal{N}_{IQ}(x_{(\lambda,\mu)})$ if for each $A \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$ there is a $B \in \mathcal{B}_{x_{(\lambda,\mu)}IQ}$ such that $B \subset A$.
(2) (X,T) is said to *satisfy the intuitionistic Q-first axiom of countability* or to be $IQ - C_I$ if each IFP $x_{(\lambda,\mu)}$ in X has a countable IQNB.

(2′) (X,T) is said to *satisfy the intuitionistic first axiom of countability* or to be IC_I if each IFP $x_{(\lambda,\mu)}$ in X has a countable IFNB.

Proposition 3.7. *If (X,T) is IC_I, then it is $IQ-C_I$.*

Proof. Let $x_{(\lambda,\mu)} \in IF_P(X)$. Consider two sequences $\{\lambda_n\}_{n\in\mathrm{N}}$ in $[0,\lambda)$ and $\{\mu_n\}_{n\in\mathbf{N}}$ in $(\mu,1]$ such that $\lim\lambda_n = \lambda$ and $\lim\mu_n = \mu$, respectively. Then

$$x_{(\mu_n,\lambda_n)} \in IF_P(X)$$

for each $n \in \mathbf{N}$. Since X is IC_I, for each $n \in \mathbf{N}$, there exists a countable intuitionistic fuzzy open neighborhood base $\mathcal{B}_n$ of $x_{(\mu_n,\lambda_n)}$ (there is evidently no loss of generality in assuming the openness of each member of $\mathcal{B}_n$). Let $B_n \in \mathcal{B}_n$ for each $n \in \mathbf{N}$. Then $x_{(\mu_n,\lambda_n)} \in B_n$ for each $n \in \mathbf{N}$. Thus $\mu_n \leq \mu_{B_n}(x)$ and $\lambda_n \geq \nu_{B_n}(x)$ for each $n \in \mathbf{N}$. So $\mu < \mu_{B_n}(x)$ or $\lambda > \nu_{B_n}(x)$.

Hence $x_{(\lambda,\mu)} q B_n$ for each $n \in \mathbf{N}$. Let $\mathcal{B}$ be the collection consisting of all the members of all $\mathcal{B}_n$. Then clearly $\mathcal{B}$ is a family of intuitionistic fuzzy open Q-neighborhoods of $x_{(\lambda,\mu)}$. Let $A \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$. Then $\lambda > \nu_A(x)$ or $\mu < \mu_A(x)$. Since $\lambda_n \in [0,\lambda)$ and $\mu_n \in (\mu,1]$ for each $n \in \mathbf{N}$, there exists an $m \in \mathbf{N}$ such that $\mu_m \leq \mu_A(x)$ and $\lambda_m \geq \nu_A(x)$.

Thus $x_{(\mu_m,\lambda_m)} \in A$ and A is an intuitionistic fuzzy open neighborhood of $x_{(\mu_m,\lambda_m)}$. So there exists a $B \in \mathcal{B}_n \subset \mathcal{B}$ such that $B \subset A$. Moreover, $x_{(\lambda,\mu)} qB$. This shows that $\mathcal{B}$ is a countable intuitionistic Q-neighborhood base of $x_{(\lambda,\mu)}$.
Hence X is $IQ-C_I$. □

4. CLOSURES, INTERIORS AND DERIVED SETS OF SETS.

Definition 4.1 (Çoker [3]). Let (X,T) be an IFTS and let $A \in IFS(X)$. Then the *intuitionistic fuzzy interior* (in short, *IFI*), int$A$ and the *intuitionistic fuzzy closure*(in short, *IFC*), cl$A$ of A are defined by:

$$\mathrm{int}A = \bigcup\{G \in T : G \subset A\} \text{ and } \mathrm{cl}A = \bigcap\{F : F^c \in T \text{ and } A \subset F\}.$$

It is clear that int$A$ is the largest IFOS contained in A and cl$A$ is the smallest IFCS containing A. Moreover $A \in IFC(X)$ if and only if cl$A = A$.

Definition 4.2. Let (X,T) be an IFTS, let $A \in IFS(X)$ and let $x_{(\lambda,\mu)} \in A$. Then $x_{(\lambda,\mu)}$ is called an *intuitionistic fuzzy interior point*(in short, *IFIP*) of A if there is a $U \in \mathcal{N}(x_{(\lambda,\mu)})$ such that $U \subset A$.

It is clear that $\text{int}A = \bigcup\{x_{(\lambda,\mu)} \in IF_P(X) : x_{(\lambda,\mu)}$ is an IFIP of $A\}$.

The following is the immediate result of Definitions 4.1 and 4.2:

Proposition 4.3. *Let X be an IFTS and let $A \in IFS(X)$. Then $x_{(\lambda,\mu)} \in \text{int}A$ if and only if there is a $U \in \mathcal{N}(x_{(\lambda,\mu)})$ such that $U \subset A$.*

Corollary 4.4 (S. J. Lee & E. P. Lee [6], Theorem 2.6). *Let X be an IFTS and let $A \in IFS(X)$. Then $A \in IFO(X)$ if and only if $A \in \mathcal{N}(x_{(\lambda,\mu)})$ for all $x_{(\lambda,\mu)} \in A$.*

Proposition 4.5. *Let X be an IFTS and let $A \in IFS(X)$. Then $x_{(\lambda,\mu)} \in \text{cl}A$ if and only if for each $U \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$, UqA.*

Proof. $x_{(\lambda,\mu)} \in \text{cl}A$

if and only if $\lambda \le \mu_F(x)$ and $\mu \ge \nu_F(x)$ for each $F \in IFC(X)$

if and only if $\mu_U(x) \le \mu$ and $\nu_U(x) \ge \lambda$ for each $U \in IFO(X)$ with $U = F^c \subset A^c$

if and only if $U \not\subset A^c$ for each $U \in IFO(X)$ with $\mu_U(x) > \mu$ or $\nu_U(x) < \lambda$

if and only if UqA for each $U \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$(By Definition 1.9 and Proposition 1.12). □

Definition 4.6. Let X be an IFTS and let $A \in IFS(X)$. Then $x_{(\lambda,\mu)} \in IF_p(X)$ is called an *intuitionistic fuzzy adherence point*(in short, *IFAP*) of A if for each $U \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$, UqA.

Form Proposition 4.4 and Definition 4.5, it is clear that cl$A$ is the union of all the $IFAPs$ of A.

Result 4.A (Çoker [3], Proposition 3.15). Let X be an IFTS and let $A \in IFS(X)$. Then:

(a) $\text{cl}A^c = (\text{int}A)^c$

(b) $\text{int}A^c = (\text{cl}A)^c$.

Corollary 4.A. $\text{cl}A = (\text{int}A^c)^c$ *and* $\text{int}A = (\text{cl}A^c)^c$.

Result 4.B (Çoker [3], Proposition 3.16). Let X be an IFTS and let $A, B \in IFS(X)$. Then the following properties hold:

(1) $\text{int}A \subset A$.

(1′) $A \subset \text{cl}A$.

(2) If $A \subset B$ then $\text{int}A \subset \text{int}B$.

(2′) If $A \subset B$, then $\text{cl}A \subset \text{cl}B$.

(3) $\text{int}(\text{int}A) = \text{int}A$.
(3′) $\text{cl}(\text{cl}A) = \text{cl}A$.
(4) $\text{int}(A \cap B) = \text{int}A \cap \text{int}B$.
(4′) $\text{cl}(A \cup B) = \text{cl}A \cup \text{cl}B$.
(5) $\text{int}1_\sim = 1_\sim$.
(5′) $\text{cl}0_\sim = 0_\sim$.

Definition 4.7. A mapping $f : IFS(X) \to IFS(X)$ is called an *intuitionistic fuzzy closure operator* on a set X if it satisfies the following *Kuratowski closure axioms* :
(1) $f(0_\sim) = 0_\sim$.
(2) $A \subset f(A)$ for each $A \in IFS(X)$.
(3) $f(f(A)) = f(A)$ for each $A \in IFS(X)$.
(4) $f(A \cup B) = f(A) \cup f(B)$ for any $A, B \in IFS(X)$.

It is clear that in an IFTS X, the mapping $f : IFS(X) \to IFS(X)$ defined by $f(A) = \text{cl}A$ is an intuitionistic fuzzy closure operator on X.

Proposition 4.8. *Let f be an intuitionistic fuzzy closure operator on a set X, let $\mathcal{F} = \{A \in IFS(X) : f(A) = A\}$ and let $T = \{U^c : U \in \mathcal{F}\}$. Then T is an IFT on X such that $f(A) = \text{cl}_T A$ for each $A \in IFS(X)$. In this case, T is called the IFT induced by f.*

Proof. Clearly $f(0_\sim) = 0_\sim$. Thus $1_\sim = 0_\sim^c \in T$. Also $f(1_\sim) = 1_\sim$. Thus $0_\sim = 1_\sim^c \in T$. Suppose $A, B \in T$. Then $A^c, B^c \in \mathcal{F}$. Thus $f(A^c) = A^c$ and $f(B^c) = B^c$. So $f((A \cap B)^c) = f(A^c \cup B^c) = f(A^c) \cup f(B^c) = A^c \cup B^c = (A \cap B)^c \in \mathcal{F}$. Hence $A \cap B \in T$.

Now suppose $\{A_\alpha\}_{\alpha \in \Gamma} \subset T$. Then $A_\alpha^c \in \mathcal{F}$ for each $\alpha \in \Gamma$. Thus $f(A_\alpha^c) = A_\alpha^c$ for each $\alpha \in \Gamma$. Since f is an intuitionistic fuzzy closure operator on X, $\cap A_\alpha^c \subset f(\cap A_\alpha^c)$. Moreover $f(\cap A_\alpha^c) \subset \cap A_\alpha^c$. Thus $f(\cap A_\alpha^c) = A_\alpha^c$. So $\cap A_\alpha^c \in \mathcal{F}$ and thus $\cup A_\alpha \in T$. Hence T is an IFT on X. Futhermore, we can see that for each $A \in IFS(X)$, $f(A) = \text{cl}_T A$. This completes the proof. □

Definition 4.9. Let X be an IFTS and let $A \in IFS(X)$. Then $x_{(\lambda,\mu)} \in IF_P(X)$ is called an *intuitionistic fuzzy boundary point* (in short, IFBP) of A if $x_{(\lambda,\mu)} \in \text{cl}A \cap \text{cl}A^c$. The union of all the IFBPs of A is called a *boundary* of A and dented by $\text{b}(A)$.

It is clear that $\text{b}(A) = \text{cl}A \cap \text{cl}A'$.

The following is the immediate result of Definition 4.9.

Proposition 4.10. *For each $A \in IFS(X)$, $A \cup \mathrm{b}(A) \subset \mathrm{cl}A$.*

However, the inclusion cannot be replaces by an equality by Example 4.11.

Example 4.11. Let $x \in X$, let $T = \{1_\sim, 0_\sim, x_{(\frac{1}{2},\frac{1}{2})}\}$ and let $A = x_{(\frac{2}{3},\frac{1}{3})}$ consider the IFP $x_{(\frac{3}{4},\frac{1}{4})}$ in X. Then $\mathcal{N}_{IQ}(x_{(\frac{3}{4},\frac{1}{4})}) = \{1_\sim, x_{(\frac{1}{2},\frac{1}{2})}\}$. Moreover, $1_\sim qA$ and $x_{(\frac{1}{2},\frac{1}{2})}qA$. So, by Proposition 4.4, $x_{(\frac{3}{4},\frac{1}{4})} \in \mathrm{cl}A$. On the other hand, $x_{(\frac{3}{4},\frac{1}{4})} \notin A$ and $x_{(\frac{1}{2},\frac{1}{2})}\bar{q}A^c$. Thus $x_{(\frac{3}{4},\frac{1}{4})} \notin \mathrm{b}(A)$. Hence $x_{(\frac{3}{4},\frac{1}{4})} \notin A \cup \mathrm{b}(A)$.

Definition 4.12. Let X be an IFTS and let $A \in IFS(X)$. Then $x_{(\lambda,\mu)} \in IF_p(X)$ is called an *intuitionistic fuzzy accumulation point* of A if it satisfies the following conditions :

(i) $x_{(\lambda,\mu)}$ is an IFAP of A.
(ii) If $x_{(\lambda,\mu)} \in A$, then for each $U \in \mathcal{N}_{IQ}(x_{(\lambda,\mu)})$, U and A are quasi-coincident at some point $y \in X$ such that $y \neq x$.

The union of all the intutionistic fuzzy accumulation points of A is called the *derived set* of A and denoted by $\mathrm{d}(A)$.

It is clear that $\mathrm{d}(A) \subset \mathrm{cl}A$.

Proposition 4.13. $\mathrm{cl}A = A \cup \mathrm{d}(A)$.

Proof. Let $\Omega = \{x_{(\lambda,\mu)} : x_{(\lambda,\mu)}$ is an IFAP of $A\}$. Then, by Corollary 5.4, $\mathrm{cl}A = \cup\Omega$. Since $x_{(\lambda,\mu)} \in \Omega$, either $x_{(\lambda,\mu)} \in A$ or $x_{(\lambda,\mu)} \notin A$. If $x_{(\lambda,\mu)} \notin A$ then by Definition 4.11, $x_{(\lambda,\mu)} \in \mathrm{d}(A)$. So $\mathrm{cl}A \subset A \cup \mathrm{d}(A)$. By Definition 4.11, it is clear that $A \cup \mathrm{d}(A) \subset \mathrm{cl}A$. Hence $\mathrm{cl}A = A \cup \mathrm{d}(A)$. □

Corollary 4.14. *Let $A \in IFS(X)$. Then $A \in IFC(X)$ if and only it $\mathrm{d}(A) \subset A$.*

Proof. $A \in IFC(X)$ if and only if $A = \mathrm{cl}A$ if and only if $\mathrm{d}(A) \subset A$ by Proposition 4.13. □

Proposition 4.15. *Let X be an IFTS and let $A = x_{(\lambda,\mu)}$. Then:*

(1) *For each $x \neq y \in X$. $(\mathrm{cl}A)(y) = (\mathrm{d}(A))(y)$.*
(2) *If $(\mathrm{cl}A)(x) > (\lambda,\mu)$, then $(\mathrm{cl}A)(x) = (\mathrm{d}(A))(x)$.*
(3) *$(\mathrm{cl}A)(x) = (\lambda,\mu)$ if and only it $(\mathrm{d}(A))(x) = 0_\sim$.*

Proof. 1) By Proposition 4.12, $\text{cl}A = A \cup \text{d}(A)$. Thus $\mu_{\text{cl}A} = \mu_A \vee \mu_{\text{d}(A)}$ and $\nu_{\text{cl}A} = \nu_A \wedge \nu_{\text{d}(A)}$. Since $A = x_{(\lambda,\mu)}$ and $y \neq x$, $A(y) = (0,1)$, *i. e.*, $\mu_A(y) = 0$ and $\nu_A(y) = 1$. Then:

$$\mu_{\text{cl}A}(y) = (\mu_A \vee \mu_{\text{d}(A)})(y) = \mu_A(y) \vee \mu_{\text{d}(A)}(y) = 0 \vee \mu_{\text{d}(A)}(y) = \mu_{\text{d}(A)}(y)$$

and

$$\nu_{\text{cl}A}(y) = (\nu_A \wedge \nu_{\text{d}(A)})(y) = \nu_A(y) \wedge \nu_{\text{d}(A)}(y) = 1 \wedge \nu_{\text{d}(A)}(y) = \nu_{\text{d}(A)}(y).$$

Hence $(\text{cl}A)(y) = (\text{d}(A))(y)$.

(2) Suppose $(\text{cl}A)(x) > (\alpha, \beta)$. Then $\mu_{\text{cl}A}(x) > \alpha$ and $\nu_{\text{cl}A}(x) < \beta$. By Proposition 4.12, Since $\text{cl}A = A \cup \text{cl}(A)$, $\mu_{\text{cl}A} = \mu_A \vee \mu_{\text{d}(A)}$ and $\nu_{\text{cl}A} = \nu_A \wedge \nu_{\text{d}(A)}$. Since $A = x_{(\lambda,\mu)}$, $\mu_A(x) = \alpha$ and $\nu_A(x) = \beta$. Thus

$$\mu_{\text{cl}A}(x) = \mu_A(x) \vee \mu_{\text{d}(A)}(x) = \alpha \vee \mu_{\text{d}(A)}(x) \text{ and}$$
$$\nu_{\text{cl}A}(x) = \nu_A(x) \wedge \nu_{\text{d}(A)}(x) = \beta \wedge \nu_{\text{d}(A)}(x).$$

So $\mu_{\text{cl}A}(x) = \mu_{\text{d}(A)}(x)$ and $\nu_{\text{cl}A}(x) = \nu_{\text{d}(A)}(x)$. Hence $(\text{cl}A)(x) = [\text{d}(A)](x)$.

(3)($\Leftarrow$): Suppose $[d(A)](x) = 0_\sim$. Then it is obvious from Proposition 4.12.

($\Rightarrow$): Suppose $(\text{cl}A)(x) = (\lambda, \mu)$. If $(\lambda', \mu') > (\lambda, \mu)$, then $x_{(\alpha',\beta')} \notin \text{cl}A$ and hence $x_{(\lambda',\mu')} \notin \text{d}(A)$. If $(\lambda', \mu') > (\lambda, \mu)$, then $x_{(\alpha',\beta')} \in A$. But any IQ-neighborhood of $x_{(\alpha',\beta')}$ and A can not be quasi-coincident at a point different form x.

So $x_{(\alpha',\beta')} \notin \text{d}(A)$. Hence $[d(A)](x) = 0_\sim$. □

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