J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 11, Number 4 (November 2004), Pages 267–274

RELATED FIXED POINT THEOREM FOR SET VALUED MAPPINGS ON TWO METRIC SPACES

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ABSTRACT. A related fixed point theorem for set valued mappings on two complete metric spaces is obtained.

Let (X, d) be a complete metric space and let $B(X)$ be the set of all nonempty subsets of X. As in Fisher [2] we define the function $\delta(A, B)$ with A and B in $B(X)$ by $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. If A consists of a single point a we write $\delta(A, B) = \delta(a, B)$. If B also consists of single point b we write $\delta(A, B) = \delta(a, B) = d(a, b)$. It follows immediately that $\delta(A, B) = \delta(B, A) \geq 0$, and $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for all A, B and C in $B(X)$.

If now $\{A_n : n = 1, 2, \ldots\}$ is a sequence of sets in $B(X)$, we say that it converges to the closed set A in $B(X)$ if

(i) each point $a \in A$ is the limit of some convergent sequence

$$\{a_n \in A_n : n = 1, 2, \ldots\},$$

(ii) for arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subset A_\varepsilon$ for $n > N$, where A_ε is the union of all open spheres with centres in A and radius ε.

The set A is then said to be the limit of the sequence $\{A_n\}$.

Now let F be a mapping of X into $B(X)$. We say that the mapping F is continuous at a point x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is continuous mapping of X into $B(X)$ if F is continuous at each point x in X. We say that a point z in X is a *fixed point* of F if z is in Fz. If A is in $B(X)$ we define the set $FA = \bigcup_{a \in A} Fa$.

Received by the editors August 24, 2004 and, in revised form, November 5, 2004.
2000 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Set valued mapping, complete metric space, fixed point.

In the following, we give a new related fixed point theorem. The first related fixed point theorem was the following (see Fisher [3]).

Theorem 1. *Let (X,d_1) and (Y,d_2) be complete metrics spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$d_2(Tx,TSy) \le c\max\{d_1(x,Sy),d_2(y,Tx),d_2(y,TSy)\},$$
$$d_1(Sy,STx) \le c\max\{d_2(y,Tx),d_1(x,Sy),d_1(x,STx)\}$$

for all x in X and y in Y, where $0 \le c < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Related fixed point theorems were later extended to two pairs of mappings on metric spaces, see for example Fisher & Murthy [4]. The next generalization was to consider related fixed point theorems for set valued mappings, see for example Chourasia & Fisher [1] and Fisher & Turkoglu [5].

The following theorem was proved in Fisher & Turkoglu [5].

Theorem 2. *Let (X,d_1) and (Y,d_2) be complete metrics spaces, let F be mapping of X into $B(Y)$ and G be mapping of Y into $B(X)$ satisfying the inequalities*

$$d_2(Tx,TSy) \le c\max\{d_1(x,Sy),d_2(y,Tx),d_2(y,TSy)\},$$
$$d_1(Sy,STx) \le c\max\{d_2(y,Tx),d_1(x,Sy),d_1(x,STx)\}$$

for all x,x' in X and y,y' in Y, where $0 \le c < 1$. If F is continuous, then GF has a unique fixed point z in X and FG has a unique fixed point w in Y.

We now prove the following related fixed point theorem for set valued mappings.

Theorem 3. *Let (X,d_1) and (Y,d_2) be complete metrics spaces, let F be a mapping of X into $B(Y)$ and let G be a mapping of Y into $B(X)$ satisfying the inequalities*

$$\begin{aligned}&\delta_1(Gy,Gy')\delta_1(GFx,GFx')\\ &\qquad\le c\max\big\{[d_1(x,x')]^2,\ d_1(x,x')\delta_2(Fx,Fx'),\ d_1(x,x')\delta_1(Gy,Gy'),\\ &\qquad\qquad\delta_2(Fx,Fx')\delta_1(Gy,Gy')\big\}, \qquad (1)\end{aligned}$$

$$\begin{aligned}&\delta_2(Fx,Fx')\delta_2(FGy,FGy')\\ &\qquad\le c\max\big\{[d_2(y,y')]^2,\ d_2(y,y')\delta_1(Gy,Gy'),\ d_2(y,y')\delta_2(Fx,Fx'),\\ &\qquad\qquad\delta_1(Gy,Gy')\delta_2(Fx,Fx')\big\} \qquad (2)\end{aligned}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. If either F or G is continuous, then GF has a unique fixed point z in X and FG has a unique fixed point w in Y. Further, $Fz = \{w\}$ and $Gw = \{z\}$.

Proof. Let x_1 be an arbitrary point in X. Define sequences $\{x_n\}$ in X and $\{y_n\}$ in Y as follows. Choose a point y_1 in Fx_1 and then a point x_2 in Gy_1. In general, having chosen x_n in X and y_n in Y, choose a point x_{n+1} in Gy_n and then a point y_{n+1} in Fx_{n+1} for $n = 1, 2, \ldots$.

We will first of all suppose that $d_1(x_n, x_{n+1}) = 0$ for some n. Then putting $x_n = x_{n+1} = z$, we have

$$z = x_{n+1} \in Gy_n \subseteq GFx_n \subseteq GFz \tag{3}$$

and so z is a fixed point of GF.

Now,

$$y_n,\ y_{n+1} \in Fz \subseteq FGy_n \tag{4}$$

and on putting $y_n = w$, we get

$$w \in Fz \subseteq FGw, \tag{5}$$

showing that w is a fixed point of FG. Then from (3) we have

$$z \in Gw \subseteq GFz. \tag{6}$$

Similarly, if $d_2(y_n, y_{n+1}) = 0$ for some n, then there exist z in X and w in Y satisfying (5) and (6).

Applying inequality (1) and (5) and (6), we get

$$\begin{aligned}\delta_1(Gw, z)\delta_1(GFz, z) &\leq \delta_1(Gw, Gw)\delta_1(GFz, GFz)\\ &\leq c\delta_1(Gw, Gw)\delta_2(Fz, Fz)\end{aligned}$$

and so either

$$Gw = \{z\} \tag{7}$$

or

$$\delta_1(GFz, GFz) \leq c\delta_2(Fz, Fz) \leq c\delta_2(FGw, FGw). \tag{8}$$

Similarly, using inequality (2) and (5) and (6), we can prove that either

$$Fz = \{w\} \tag{9}$$

or

$$\delta_2(FGw, FGw) \leq c\delta_1(Gw, Gw) \leq c\delta_1(GFz, GFz). \tag{10}$$

Now (7) and (9) immediately imply that

$$Gw = \{z\}, \quad Fz = \{w\}, \quad GFz = \{z\}, \quad FGw = \{w\}. \tag{11}$$

Next, (7) and (10), together with (5), imply that $FGw = \{w\}$ and equations (11) follow.

Similarly, (8) and (9) imply equations (11).

Finally, (8) and (10) imply that

$$\delta_1(GFz, GFz) \le c\delta_2(Fz, Fz) \le c\delta_2(FGw, FGw) \le c^2\delta_1(GFz, GFz) = 0,$$

since $c < 1$. Equations (11) again follow.

It follows similarly that if $d_2(y_n, y_{n+1}) = 0$ for some n then equations (11) will again hold.

We will now suppose that $d_1(x_n, x_{n+1}) \ne 0 \ne d_2(y_n, y_{n+1})$ for all n. Then using inequality (1), we have

$$\begin{aligned} &d_1(x_n, x_{n+1})d_1(x_{n+1}, x_{n+2}) \\ &\qquad \le \delta_1(Gy_{n-1}, Gy_n)\delta_1(GFx_n, GFx_{n+1}) \\ &\qquad \le c\max\big\{[d_1(x_n, x_{n+1})]^2,\ d_1(x_n, x_{n+1})\delta_2(Fx_n, Fx_{n+1}), \\ &\qquad\qquad d_1(x_n, x_{n+1})\delta_1(Gy_{n-1}, Gy_n),\ \delta_2(Fx_n, Fx_{n+1})\delta_1(Gy_{n-1}, Gy_n)\big\} \\ &\qquad \le c\delta_1(Gy_{n-1}, Gy_n)\max\big\{\delta_1(GFx_{n-1}, GFx_n),\ \delta_2(FGy_{n-1}, FGy_n)\big\}, \end{aligned}$$

which implies that

$$\begin{aligned} d_1(x_{n+1}, x_{n+2}) &\le \delta_1(GFx_n, GFx_{n+1}) \\ &\le c\max\big\{\delta_1(GFx_{n-1}, GFx_n), \delta_2(FGy_{n-1}, FGy_n)\big\}. \end{aligned} \tag{12}$$

Similarly, using inequalities (2) and (12), we get

$$\begin{aligned} &d_2(y_{n+1}, y_{n+2}) \\ &\quad \le \delta_2(FGy_n, FGy_{n+1}) \\ &\quad \le c\max\big\{\delta_1(GFx_n, GFx_{n+1}), \delta_2(FGy_{n-1}, FGy_n)\big\} \\ &\quad \le c\max\big\{c\delta_1(GFx_{n-1}, GFx_n), c\delta_2(FGy_{n-1}, FGy_n), \delta_2(FGy_{n-1}, FGy_n)\big\} \\ &\quad \le c\max\big\{\delta_1(GFx_{n-1}, GFx_n), \delta_2(FGy_{n-1}, FGy_n)\big\} \end{aligned} \tag{13}$$

and it follows easily by induction from inequalities (12) and (13) that

$$\begin{aligned} \delta_1(GFx_n, GFx_{n+1}) &\le c^{n-1}\max\big\{\delta_1(GFx_1, GFx_2), \delta_2(FGy_1, FGy_2)\big\}, \\ \delta_2(FGy_n, FGy_{n+1}) &\le c^{n-1}\max\big\{\delta_1(GFx_1, GFx_2), \delta_2(FGy_1, FGy_2)\big\}, \end{aligned} \tag{14}$$

for $n = 1, 2, \ldots$.

It follows on using inequalities (12) and (14) that for $r = 1, 2, \ldots$

$$\begin{aligned} &d_1(x_{n+1}, x_{n+r+1}) \\ &\quad \leq \delta_1(GFx_n, GFx_{n+r}) \\ &\quad \leq \delta_1(GFx_n, GFx_{n+1}) + \delta_1(GFx_{n+1}, GFx_{n+2}) + \cdots \\ &\qquad\qquad + \delta_1(GFx_{n+r-1}, GFx_{n+r}) \\ &\quad \leq (c^{n-1} + c^n + \cdots + c^{n+r-2}) \max\{\delta_1(GFx_1, GFx_2), \delta_2(FGy_1, FGy_2)\} \\ &\quad < \varepsilon \end{aligned} \tag{15}$$

for n greater than some N, since $c < 1$. The sequence $\{x_n\}$ is therefore a Cauchy sequence in the complete metric space X and so has a limit z in X. Similarly the sequence $\{y_n\}$ is a Cauchy sequence in the complete metric space Y and so has a limit w in Y.

Now,

$$\begin{aligned} \delta_1(z, GFx_n) &\leq d_1(z, x_{m+1}) + \delta_1(x_{m+1}, GFx_n) \\ &\leq d_1(z, x_{m+1}) + \delta_1(GFx_m, GFx_n) \\ &< d_1(z, x_{m+1}) + \varepsilon, \end{aligned}$$

on using inequality (15) for $m, n > N$. Letting m tend to infinity, we see that $\delta_1(z, GFx_n) < \varepsilon$ for $n > N$ and so

$$\lim_{n\to\infty} GFx_n = \{z\} = \lim_{n\to\infty} Gy_n, \tag{16}$$

since ε is arbitrary. Similarly, we have

$$\lim_{n\to\infty} FGy_n = \{w\} = \lim_{n\to\infty} Fx_n. \tag{17}$$

Now suppose that F is continuous. Then

$$\{w\} = \lim_{n\to\infty} Fx_n = Fz. \tag{18}$$

Using inequality (1), we now have

$$\begin{aligned} &\delta_1(Gw, x_{n+1})\delta_1(GFz, x_{n+2}) \\ &\quad \leq \delta_1(Gw, Gy_n)\delta_1(GFz, GFx_{n+1}) \\ &\quad \leq c \max\{[d_1(z, x_{n+1})]^2,\ d_1(z, x_{n+1})\delta_2(Fz, Fx_{n+1}),\ d_1(z, x_{n+1})\delta_1(Gw, Gy_n), \\ &\qquad\qquad \delta_2(Fz, Fx_{n+1})\delta_1(Gw, Gy_n)\}. \end{aligned}$$

Letting n tend to infinity and using equation (18), we get

$$\delta_1(Gw, z)\delta_1(GFz, z) = 0$$

and so either

$$Gw = \{z\} \tag{19}$$

or

$$GFz = \{z\}. \tag{20}$$

It is obvious that if equations (18) and (19) or equations (18) and (20) hold, then equations (11) hold.

By the symmetry, equations (11) again hold if G is continuous instead of F.

To prove uniqueness, suppose that GF has a second fixed point z'. Then z' is in GFz' and there exists a point w' in Fz' with z' in Gw'. On using inequality (1), we have

$$\begin{aligned}\delta_1(Gw', Gw')\delta_1(GFz', GFz') &\leq c\delta_2(Fz', Fz')\delta_1(Gw', Gw')\\ &\leq c\delta_2(Fz', Fz')\delta_1(GFz', GFz'),\end{aligned}$$

which implies that either $\delta_1(Gw', Gw') = 0$, and hence

$$Gw' = \{z'\}, \tag{21}$$

or

$$\delta_1(GFz', GFz') \leq c\delta_2(Fz', Fz') \tag{22}$$

$$\leq c\delta_2(FGw', FGw'). \tag{23}$$

Similarly, using inequality (2), we get either

$$Fz' = \{w'\}, \tag{24}$$

or

$$\delta_2(FGw', FGw') \leq c\delta_1(Gw', Gw') \tag{25}$$

$$\leq c\delta_1(GFz', GFz'). \tag{26}$$

It follows easily that if (21) and (24) hold, or (21) and (25) hold, or (22) and (24), or (23) and (26) hold, then

$$Gw' = \{z'\}, \quad Fz' = \{w'\}, \quad GFz' = \{z'\}, \quad FGw' = \{w'\}. \tag{27}$$

Finally, using inequality (1) and equations (11) and (27), we get

$$\begin{aligned}[d_1(z,z')]^2 &= \delta_1(Gw,Gw')\delta_1(GFz,GFz')\\ &\leq c\max\{[d_1(z,z')]^2, d_1(z,z')\delta_2(Fz,Fz'), d_1(z,z')\delta_1(Gw,Gw'),\\ &\qquad \delta_2(Fz,Fz')\delta_1(Gw,Gw')\}\\ &= cd_1(z,z')\max\{d_1(z,z'), d_2(w,w')\}\end{aligned}$$

which implies that either $z = z'$ or $d_1(z,z') \leq cd_2(w,w')$

Similarly, applying inequality (2), we can prove that either $w = w'$ or $d_2(w,w') \leq cd_1(z,z')$ and the uniqueness of the fixed points follow. This completes the proof of the theorem. □

If we let F be a single valued mapping of X into Y and let G be a single valued mapping of Y into X, we obtain the following corollary.

Corollary 3.1. *Let (X,d_1) and (Y,d_2) be complete metric spaces, let F be a mapping of X into Y and let G be a mapping of Y into X satisfying the inequalities*

$$\begin{aligned}&d_1(Gy,Gy')d_1(GFx,GFx')\\ &\quad\leq c\max\{[d_1(x,x')]^2, d_1(x,x')d_2(Fx,Fx'), d_1(x,x')d_1(Gy,Gy'),\\ &\qquad d_2(Fx,Fx')d_1(Gy,Gy')\},\\ &d_2(Fx,Fx')d_2(FGy,FGy')\\ &\quad\leq c\max\{[d_2(y,y')]^2, d_2(y,y')d_1(Gy,Gy'), d_2(y,y')d_2(Fx,Fx'),\\ &\qquad d_1(Gy,Gy')d_2(Fx,Fx')\}\end{aligned}$$

for all x,x' in X and y,y' in Y, where $0 \leq c < 1$. If either F or G is continuous, then GF has a unique fixed point z in X and FG has a unique fixed point w in Y. Further, $Fz = w$ and $Gw = z$.

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