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NOOR ITERATIONS FOR NONLINEAR LIPSCHITZIAN STRONGLY ACCRETIVE MAPPINGS

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ABSTRACT. In this paper, we suggest and analyze Noor (three-step) iterative scheme for solving nonlinear strongly accretive operator equation $Tx = f$. The results obtained in this paper represent an extension as well as refinement of previous known results.

1. INTRODUCTION

Let X be a real Banach space with norm $\|\cdot\|$ and dual X^*. An operator T with domain $D(T)$ and range $R(T)$ in X is said to be *accretive* (*cf.* Browder [1], Kato [12]) if the inequality

$$\|x-y\| \leq \|x-y+t(Tx-Ty)\|$$

holds for each x and y in $D(T)$ and for all $t \geq 0$. T is accretive if and only if for any $x, y \in D(T)$, there exists $j \in J(x-y)$ such that $\langle Tx-Ty, j\rangle \geq 0$, where

$$J(x) = \{f^* \in X^* : \|f^*\|^2 = \langle x, f^*\rangle = \|x\|^2\}, \quad x \in X,$$

is the normalized duality mapping of X and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*.

A fundamental result, due to Browder [2], in the theory of accretive operators states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \quad u(0) = u_0,$$

is solvable if T is a locally Lipschitzian and accretive operator on X.

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Definition 1.1. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to X$ is said to be *strongly accretive* if there exists a real number $k > 0$ such that for every $x, y \in K$,

$$\langle Tx - Ty, j\rangle \geq k\|x - y\|^2$$

holds for some $j \in J(x - y)$.

Without loss of generality we assume that $k \in (0, 1)$. This class of mappings has been investigated by many authors (see Browder [2], Gwinner [9], Morales [14]).

In particular, Morales [14] proved that if $T : X \to X$ is continuous and strongly accretive, then T maps X onto X, that is, for each f in X, the equation $Tx = f$ has a solution in X.

Definition 1.2. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to X$ is said to be *strictly pseudocontractive* if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1.1}$$

holds for all x, y in K and $r > 0$.

Strictly pseudocontractive mappings have been studied by various authors (see Chidume [3, 4]).

The objective of this paper is to study the iterative solutions to the equation $Tx = f$ in the case when T is Lipschitzian and strongly accretive and X is L_p(or l_p) with $p \geq 2$. For this purpose, let us first recall the following three iteration processes due to Mann [13], Ishikawa [11] and Noor [15, 16], Noor, Rassias & Huang [17], respectively.

Mann Iteration. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ in K by the iterative scheme

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, \quad n \geq 0,$$

where $\{c_n\}_{n=0}^{\infty}$ is a real sequence satisfying $c_0 = 1$, $0 < c_n \leq 1$ for all $n \geq 1$ and $\sum_{n=0}^{\infty} c_n = \infty$,

Ishikawa Iteration. For a given $x_0 \in K$, compute the sequence $\{x_n\}_{n=0}^{\infty}$ in K by the iterative scheme

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0, \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in [0, 1] satisfying the conditions: $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n, $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n\beta_n = \infty$,

Noor Iteration. For a given $x_0 \in K$, compute sequences $\{x_n\}_{n=0}^{\infty}$ in K by iterative scheme

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \end{aligned}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying some certain conditions.

It is well known that three-step iteration processes were suggested and analyzed by Noor [15, 16], Noor, Rassias & Huang [17] for variational inclusions(inequalities) in a Hilbert space by using techniques of updating the solution and the auxiliary principle. These three-step iterative schemes are also called Noor iterations, see, for example, Rhoades & Soltuz [18]. Clearly Mann and Ishikawa iterations are special cases of Noor iterations. We would like to mention that Noor iterations are similar to those of the so-called θ-schemes of Glowinski & Le Tallec [8] for finding a zero of the sum of two (or more) maximal monotone operators by using the Lagrange multiplier method. Glowinski & Le Tallec [8] used three-step iterative schemes to find the approximate solutions of the elasto-viscoplasticity, liquid crystal theory and eigenvalue problems.

They have shown that the three-step approximations perform better than the two-step and one-step iterative methods. Haubruge, Nguyen & Strodiot [10] have studied the convergence analysis of the three-step schemes of Glowinski & Le Tallec [8] and applied these three-step iteration processes to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They have also proved that three-step iteration processes lead to highly parallelized algorithms under certain conditions. It has been shown in Haubruge, Nguyen & Strodiot [10], Noor [15, 16] that three-step schemes are a natural generalization of the splitting methods for solving partial differential equations (inclusions). On the other hand, there are no such three-step schemes for solving nonlinear operator equations in L_P (or l_p) space.

In this paper, we consider and analyze Noor iteration process in L_p(or l_p) space. We prove that the Noor iteration process converges strongly to the unique solution of the equation $Tx = f$ in case T is a Lipschitzian and strongly accretive operator from L_p (or l_p) into itself, or to the unique fixed point of T in case T is a Lipschitzian and pseudo-contractive mapping from a bounded closed convex subset K of L_p(or l_p) into itself. Our results can be viewed as an extension of three-step and two-step iterative schemes of Glowinski & Le Tallec [8], Noor [15, 16], Ishikawa [11], Chidume [4] and Lei Deng [7].

2. Main Results

In this section, we study the convergence properties of the Noor iterative schemes. For this purpose, we need the following result.

Lemma 2.1 (Chidume [3, 4]). *Let $X = L_p$ (or l_p), $2 \le p < \infty$. For any $x, y \in X$, we have*

$$\|x+y\|^2 \le (p-1)\|x\|^2 + \|y\|^2 + 2\langle x, j(y)\rangle, \quad \forall j \in J(x+y). \tag{2.1}$$

Theorem 2.1. *Let $X = L_p$ (or l_p), $2 \le p < \infty$, and $T : X \to X$ be a Lipschitzian and strongly accretive map with the Lipschitz constant $L\,(\ge 1)$. Define $S : X \to X$ by $Sx = f - Tx + x$. For arbitrary $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n \tag{2.2}$$

$$y_n = (1-\beta_n)x_n + \beta_n S z_n \tag{2.3}$$

$$z_n = (1-\gamma_n)x_n + \gamma_n S x_n, \quad n \ge 0, \tag{2.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in [0, 1] satisfying:

(i)

$$\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n,$$

(ii)

$$\sum_{n=0}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of $Tx = f$.

Proof. We first observe that the equation $Tx = f$ has a unique solution which is denoted by q. In fact, the existence and the uniqueness of a solution to $Tx = f$ follow from Morales [14] and the strong accretiveness of T. Observe that S is Lipschitzian with the same Lipschitz constant L and q is a fixed point of S. And it follows from (2.4) that

$$\|x_n - z_n\| = \gamma_n \|x_n - S x_n\|$$

$$
\begin{aligned}
&= \gamma_n \|Tx_n - Tq\| \\
&\leq \gamma_n L \|x_n - q\| \\
&\leq L\|x_n - q\| \qquad (2.5)
\end{aligned}
$$

Using (2.3) we obtain

$$
\begin{aligned}
\|x_n - y_n\| &= \beta_n \|x_n - Sz_n\| \\
&\leq \beta_n (\|Sz_n - Sx_n\| + \|Sx_n - x_n\|) \\
&\leq \beta_n (L\|z_n - x_n\| + \|Sx_n - x_n\|) \\
&\leq \beta_n (L^2 \gamma_n \|x_n - q\| + \|Tq - Tx_n\|) \\
&\leq L(L\gamma_n + 1)\beta_n \|x_n - q\| \\
&\leq L(L+1)\beta_n \|x_n - q\|.
\end{aligned}
$$

Since the operator T is strongly accretive with constant $k > 0$ and Lipschitz continuous with constant $L(\geq 1)$, we have

$$
\begin{aligned}
\langle z_n - q, j(x_n - q)\rangle &= -\gamma_n \langle Tx_n - Tq, j(x_n - q)\rangle \\
&\qquad + \langle x_n - q, j(x_n - q)\rangle \\
&\leq -k\gamma_n \|x_n - q\|^2 + \|x_n - q\|^2 \\
&= (1 - k\gamma_n)\|x_n - q\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\langle Sz_n - Sq, j(x_n - q)\rangle &= \langle -Tz_n + Tq + z_n - q, j(x_n - q)\rangle \\
&= \langle Tx_n - Tz_n, j(x_n - q)\rangle \\
&\qquad - \langle Tx_n - Tq, j(x_n - q)\rangle + \langle z_n - q, j(x_n - q)\rangle \\
&\leq L\|x_n - z_n\| \|x_n - q\| - k\|x_n - q\|^2 \\
&\qquad + (1 - k\gamma_n)\|x_n - q\|^2 \\
&\leq (L^2\gamma_n - k + 1 - k\gamma_n)\|x_n - q\|^2 \\
&= [(L^2 - k)\gamma_n + 1 - k]\|x_n - q\|^2 \\
&\leq (L^2 - k + 1 - k)\|x_n - q\|^2. \qquad (2.6)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\langle y_n - q, j(x_n - q)\rangle &= (1 - \beta_n)\langle x_n - q, j(x_n - q)\rangle + \beta_n \langle Sz_n - q, j(x_n - q)\rangle \\
&= (1 - \beta_n)\|x_n - q\|^2 + \beta_n \langle Sz_n - q, j(x_n - q)\rangle
\end{aligned}
$$

$$\begin{aligned}
&\leq \|x_n - q\|^2 + \beta_n\langle Sz_n - Sq, j(x_n - q)\rangle \\
&\leq \|x_n - q\|^2 + [(L^2 - k)\gamma_n + 1 - k]\beta_n\|x_n - q\|^2 \\
&\leq \|x_n - q\|^2 + (L^2 - k + 1 - k)\beta_n\|x_n - q\|^2,
\end{aligned}$$

$$\begin{aligned}
&\langle Sy_n - Sq, j(x_n - q)\rangle \\
&= \langle Tx_n - Ty_n, j(x_n - q)\rangle - \langle Tx_n - Tq, j(x_n - q)\rangle \\
&\qquad + \langle y_n - q, j(x_n - q)\rangle \\
&\leq Tx_n - Ty_n\|\|x_n - q\| - k\|x_n - q\|^2 \\
&\qquad + \langle y_n - q, j(x_n - q)\rangle \\
&\leq L\|x_n - y_n\|\|x_n - q\| - k\|x_n - q\|^2 \\
&\qquad + \langle y_n - q, j(x_n - q)\rangle \\
&\leq L^2(L+1)\beta_n\|x_n - q\|^2 - k\|x_n - q\|^2 \\
&\qquad + \|x_n - q\|^2 + (L^2 - k + 1 - k)\beta_n\|x_n - q\|^2 \\
&= [1 - k + (L^2(L+2) - k + 1 - k)\beta_n]\|x_n - q\|^2, \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
\langle Sx_n - Sq, j(x_n - q)\rangle &= -\langle Tx_n - Tq, j(x_n - q)\rangle + \langle x_n - q, j(x_n - q)\rangle \\
&\leq (1 - k)\|x_n - q\|^2. \qquad (2.8)
\end{aligned}$$

Thus, from Lemma 2.1 and from (2.4), (2.8) for all $n \in N$, $n \geq 0$,

$$\begin{aligned}
\|z_n - q\|^2 &= \|\gamma_n(Sx_n - Sq) + (1 - \gamma_n)(x_n - q)\|^2 \\
&\leq (p-1)\gamma_n^2\|Sx_n - Sq\|^2 \\
&\qquad + (1 - \gamma_n)^2\|x_n - q\|^2 + 2\gamma_n(1 - \gamma_n)\langle Sx_n - Sq, j(x_n - q)\rangle \\
&\leq (p-1)\gamma_n^2 L^2\|x_n - q\|^2 + (1 - \gamma_n)^2\|x_n - q\|^2 \\
&\qquad + 2\gamma_n(1 - \gamma_n)(1 - k)\|x_n - q\|^2 \\
&\leq (w\gamma_n^2 + 1 - \gamma_n^2)\|x_n - q\|^2; w = (p-1)L^2 \\
&= [1 + (w - 1)\gamma_n^2]\|x_n - q\|^2 \\
&\leq w\|x_n - q\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - q\|^2 &= \|\beta_n(Sz_n - Sq) + (1 - \beta_n)(x_n - q)\|^2 \\
&\leq (p-1)\beta_n^2\|Sz_n - Sq\|^2 + (1 - \beta_n)^2\|x_n - q\|^2
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad + 2\beta_n(1-\beta_n)\langle Sz_n - Sq, j(x_n - q)\rangle \\
&\le w\beta_n^2\|z_n - q\|^2 + (1-\beta_n)^2\|x_n - q\|^2 \\
&\qquad\qquad\qquad + 2\beta_n(1-\beta_n)(L^2 - k + 1 - k)\|x_n - q\|^2 \\
&\le [w^2\beta_n^2 + (1-\beta_n)^2 + 2\beta_n(1-\beta_n)(L^2 - k + 1 - k)]\|x_n - q\|^2 \\
&\le [(w^2 - 1)\beta_n^2 + 1 + 2(L^2 - k)]\|x_n - q\|^2 \\
&\le (w^2 + 2(L^2 - k))\|x_n - q\|^2. \qquad (2.9)
\end{aligned}
$$

In a similar way, from Lemma 2.1 and from (2.8), (2.9), we obtain that for all $n \in N$, $n \ge 0$,

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q) + \alpha_n(Sy_n - Sq)\|^2 \\
&\le (p-1)\alpha_n^2\|Sy_n - Sq\|^2 + (1-\alpha_n)^2\|x_n - q\|^2 \\
&\qquad\qquad + 2\alpha_n(1-\alpha_n)\langle Sy_n - Sq, j(x_n - q)\rangle \\
&\le (p-1)L^2\alpha_n^2\|y_n - q\|^2 \\
&\qquad + (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(1-\alpha_n)\langle Sy_n - Sq, j(x_n - q)\rangle \\
&\le w\alpha_n^2(w^2 + 2(L^2 - k))\|x_n - q\|^2 + (1-\alpha_n)^2\|x_n - q\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)[1 - k + (L^2(L+2) - k + 1 - k)\beta_n]\|x_n - q\|^2 \\
&\le [1 - 2k\alpha_n + [(w(w^2 + 2(L^2 - k)) + 1)\alpha_n \\
&\qquad + (L^2(L+2) - k + 1 - k)\beta_n]\alpha_n]\|x_n - q\|^2. \qquad (2.10)
\end{aligned}
$$

By condition (i), we have

$$
\begin{aligned}
\alpha_n &\le \frac{k}{2(w(w^2 + 2(L^2 - k)) + 1)}, \\
\beta_n &\le \frac{k}{4(L^2(L+2) - k + 1 - k)}. \qquad (2.11)
\end{aligned}
$$

Now with the help of condition (ii) and from the above relations, we obtain

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - k\alpha_n)\|x_n - q\|^2 \\
&\le \exp(-k\alpha_n)\|x_n - q\|^2 \\
&\le \exp(-k\sum_{i=1}^{n}\alpha_i)\|x_1 - q\|^2,
\end{aligned}
$$

which shows that $\lim_{n\to\infty}\|x_n - q\| = 0$. □

We now turn to consider approximating fixed points of pseudo-contractive mappings via Noor iteration process.

We now turn to consider approximating fixed points of pseudo-contractive mappings via Noor iteration process.

Theorem 2.2. *Let $X = L_p$ (or l_p), $2 \le p < \infty$. Suppose that K is a nonempty closed and convex subset of X and $T : K \to K$ is a Lipschitz strictly pseudo-contractive mapping with Lipschitz constant $L(\ge 1)$. Let the sequence $\{x_n\}_{n=0}^{\infty}$ be defined by*

$$x_0 \in K$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \tag{2.12}$$

$$y_n = (1-\beta_n)x_n + \beta_n T z_n \tag{2.13}$$

$$z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \ge 0, \tag{2.14}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying the conditions (i) and (ii) of Theorem 2.1. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Proof. The existence of a fixed point follows from Deimling [6]. Let q denotes a fixed point of T. We will show that q is the unique fixed point of T. Suppose there exists $p \in F(T)$, where $F(T)$ is the fixed point set of T. Since T is strictly pseudo-contractive, $(I-T)$ is a strongly accretive map Chidume [3]. Thus

$$\operatorname{Re}\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge s\|x-y\|^2, \tag{2.15}$$

where $s = \frac{t-1}{t}$. Hence

$$\begin{aligned} \|p-q\|^2 &= \langle p-q, j(p-q)\rangle \\ &= \langle Tp - Tq, j(p-q)\rangle \\ &= -\langle (I-T)p - (I-T)q, j(p-q)\rangle + \langle p-q, j(p-q)\rangle \\ &\le (1-s)\|p-q\|^2. \end{aligned}$$

Since $s \in (0,1)$, it follows that $\|p-q\|^2 \le 0$, which implies the uniqueness. It follows from (2.11) that

$$\begin{aligned} \langle Tx_n - Tq, j(x_n-q)\rangle &= -\langle (I-T)x_n - (I-T)q, j(x_n-q)\rangle \\ &\qquad + \langle x_n - q, j(x_n-q)\rangle \\ &\le (1-s)\|x_n-q\|^2, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\|x_n - z_n\| &\leq \gamma_n(\|x_n - q\| + \|q - Tx_n\|)\\ &\leq \gamma_n(1+L)\|x_n - q\|\\ &\leq (1+L)\|x_n - q\|. \end{aligned} \tag{2.17}$$

Thus, using Lemma 2.1, (2.15) and (2.16), we have

$$\begin{aligned}\|z_n - q\|^2 &= \|\gamma_n(Tx_n - Tq) + (1-\gamma_n)(x_n - q)\|^2\\ &\leq (p-1)\gamma_n^2\|Tx_n - Tq\|^2 + (1-\gamma_n)^2\|x_n - q\|^2\\ &\quad + 2\gamma_n(1-\gamma_n)\langle Tx_n - Tq, j(x_n - q)\rangle\\ &\leq [w\gamma_n^2 + (1-\gamma_n)^2 + 2\gamma_n(1-\gamma_n)(1-s)]\|x_n - q\|^2\\ &\leq (1 + (w-1)\gamma_n^2)\|x_n - q\|^2\\ &\leq w\|x_n - q\|^2. \end{aligned} \tag{2.18}$$

Using (2.13) we obtain

$$\begin{aligned}\|x_n - y_n\| &= \beta_n\|x_n - Tz_n\|\\ &\leq \beta_n(\|Tz_n - Tq\| + \|x_n - q\|)\\ &\leq \beta_n(L\|z_n - q\| + \|x_n - q\|)\\ &\leq \beta_n(L\sqrt{w} + 1)\|x_n - q\|. \end{aligned} \tag{2.19}$$

From (2.15) and (2.16), we have

$$\begin{aligned}\langle Tz_n - Tq, j(x_n - q)\rangle &= \langle Tz_n - Tx_n, j(x_n - q)\rangle + \langle Tx_n - Tq, j(x_n - q)\rangle\\ &\leq \|Tz_n - Tx_n\|\|x_n - q\| + (1-s)\|x_n - q\|^2\\ &\leq L\|z_n - x_n\|\|x_n - q\| + (1-s)\|x_n - q\|^2\\ &\leq L(1+L)\|x_n - q\|^2 + (1-s)\|x_n - q\|^2\\ &= [L(1+L) + 1 - s]\|x_n - q\|^2. \end{aligned} \tag{2.20}$$

In a similar way, we obtain

$$\begin{aligned}\langle Ty_n - Tq, j(x_n - q)\rangle &= \langle Ty_n - Tx_n, j(x_n - q)\rangle + \langle Tx_n - Tq, j(x_n - q)\rangle\\ &\leq L\|y_n - x_n\|\|x_n - q\| + (1-s)\|x_n - q\|^2\\ &\leq L(L\sqrt{w} + 1)\beta_n\|x_n - q\|^2 + (1-s)\|x_n - q\|^2\\ &= [L(L\sqrt{w} + 1)\beta_n + 1 - s]\|x_n - q\|^2. \end{aligned} \tag{2.21}$$

Next we make an estimation for $\|y_n - q\|^2$. From (2.17) and (2.18) we have

$$\begin{aligned}
\|y_n - q\|^2 &= \|\beta_n(Tz_n - Tq) + (1-\beta_n)(x_n - q)\|^2 \\
&\le (p-1)\beta_n^2\|Tz_n - Tq\|^2 + (1-\beta_n)^2\|x_n - q\|^2 \\
&\qquad + 2\beta_n(1-\beta_n)\langle Tz_n - Tq, j(x_n - q)\rangle \\
&\le w\beta_n^2\|z_n - q\|^2 + (1-\beta_n)^2\|x_n - q\|^2 \\
&\qquad + 2\beta_n(1-\beta_n)[L(1+L) + 1 - s]\|x_n - q\|^2 \\
&\le [w^2\beta_n^2 + (1-\beta_n)^2 + 2\beta_n(1-\beta_n)[L(1+L) + 1 - s]]\|x_n - q\|^2 \\
&\le [(w^2-1)\beta_n^2 + 1 + 2L(1+L)]\|x_n - q\|^2 \\
&\le [w^2 + 2L(1+L)]\|x_n - q\|^2. \qquad (2.22)
\end{aligned}$$

Thus, from Lemma 2.1 and from (2.17), (2.19), we obtain that for all $n \ge 0$,

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 \\
&\le (p-1)\alpha_n^2\|Ty_n - Tq\|^2 + (1-\alpha_n)^2\|x_n - q\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)\langle Ty_n - Tq, j(x_n - q)\rangle \\
&\le w\alpha_n^2\|y_n - q\|^2 + (1-\alpha_n)^2\|x_n - q\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)\langle Ty_n - Tq, j(x_n - q)\rangle \\
&\le w\alpha_n^2[w^2 + 2L(1+L)]\|x_n - q\|^2 + (1-\alpha_n)^2\|x_n - q\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)[L(Lx_n - q\|^2 \\
&\le [1 - 2s\alpha_n + [(w(w + 2L(1+L)) + 1)\alpha_n + 2L(L\sqrt{w})
\end{aligned}$$

By condition (i), we have

$$\begin{aligned}
\alpha_n &\le \frac{k}{2(w(w + 2L(1+L)) + 1)}, \qquad (2.23) \\
\beta_n &\le \frac{k}{4L(L\sqrt{w} + 1)}.
\end{aligned}$$

With the help of condition (ii) and (2.20), we get

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le (1 - s\alpha_n)\|x_n - q\|^2 \\
&\le \exp(-s\alpha_n)\|x_n - q\|^2 \\
&\le \exp(-s\sum_{i=1}^{n}\alpha_i)\|x_1 - q\|^2,
\end{aligned}$$

which shows that $\lim_{n\to\infty}\|x_n - q\| = 0$. □

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