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K-THEORY OF CROSSED PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. We study continuous fields and K-groups of crossed products of C^*-algebras. It is shown under a reasonable assumption that there exist continuous fields of C^*-algebras between crossed products of C^*-algebras by amenable locally compact groups and tensor products of C^*-algebras with their group C^*-algebras, and their K-groups are the same under the additional assumptions.

0. INTRODUCTION

We start with recalling the following:

Notation. Let $C^*(G)$ denote the group C^*-algebra of a locally compact group G, and $L^1(G)$ be the Banach algebra of all integrable functions on G (*cf.* Dixmier [3]). For a C^*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A} \rtimes_\alpha G$ the C^*-crossed product of $\mathfrak{A}$ by G for α an action, that is, a homomorphism from G to the automorphism group of $\mathfrak{A}$, and denote by $L^1(G, \mathfrak{A})$ the Banach algebra of all $\mathfrak{A}$-valued integrable functions on G (*cf.* Pedersen [10]). Let $K_*(\mathfrak{A})$ for $* = 0, 1$ be the K-groups of a C^*-algebra $\mathfrak{A}$. See Blackadar [1], Rørdam, Larsen & Laustsen [12] and Wegge-Olsen [14] for details on the K-theory of C^*-algebras.

Our first motivation for this study is the following:

Problem. Let $\mathfrak{A}$ be a C^*-algebra, G an amenable locally compact group and $\mathfrak{A} \rtimes_\alpha G$ their crossed product for an action α. Then is it true that for $* = 0, 1$,

$$K_*(\mathfrak{A} \rtimes_\alpha G) \cong K_*(\mathfrak{A} \otimes C^*(G)) \quad ?$$

This problem is certainly reasonable since crossed products are regarded as skewed tensor products, which are certainly close to tensor products in a sense. However, the answer is false in general, but it is known that the answer is true for G any simply connected solvable Lie groups, in particular, the real group (the Connes' Thom

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isomorphism for crossed products by $\mathbb{R}$). In the process we show that there always exist continuous fields of Banach spaces between the crossed products $\mathfrak{A} \rtimes_\alpha G$ and the tensor products $\mathfrak{A} \otimes C^*(G)$. This should be nontrivial. Moreover, under a reasonable assumption on actions we obtain continuous fields of C^*-algebras between the crossed products and the tensor products. By using this we obtain a partial answer to the problem under technical assumptions on projections and unitaries of matrix algebras over the fibers. Our idea for the proofs is to use the dense parts of crossed products by amenable locally compact groups, and some techniques in the theory of C^*-algebras of continuous fields (continuous deformation of C^*-algebras). See Dixmier [3], Fell [4], Lee [6] and Lee [7] for the theory of continuous fields.

1. Continuous fields and K-groups of crossed products

First of all, we check the following:

Proposition 1.1. *Let $\mathfrak{A}$ be a C^*-algebra and G an amenable locally compact group. Then the norm equality holds for generating elements af and $a \otimes f$ (for $a \in \mathfrak{A}$ and $f \in L^1(G)$) of the dense part $L^1(G, \mathfrak{A})$ of the crossed product $\mathfrak{A} \rtimes_\alpha G$ for an action α, and $\mathfrak{A} \otimes L^1(G)$ of the tensor product $\mathfrak{A} \otimes C^*(G)$ respectively.*

Proof. Recall that any element of $\mathfrak{A} \rtimes_\alpha G$ is approximated by elements of the form: $\sum a_j f_j$ (finite sum) for $a_j \in \mathfrak{A}$ and $f_j \in L^1(G)$. Also, any element of $\mathfrak{A} \otimes C^*(G)$ is approximated by elements of the form: $\sum a_j \otimes f_j$ (finite sum) for $a_j \in \mathfrak{A}$ and $f_j \in L^1(G)$. Thus, we can always define an element-wise map Φ from $\mathfrak{A} \rtimes_\alpha G$ to $\mathfrak{A} \otimes C^*(G)$ restricted to their dense parts as follows:

$$\Phi : L^1(G, \mathfrak{A}) \ni \sum a_j f_j \mapsto \sum a_j \otimes f_j \in \mathfrak{A} \otimes L^1(G).$$

Now let π be a faithful representation of $\mathfrak{A}$ on a Hilbert space H, and λ the left regular representation of G on the Hilbert space $L^2(G)$ of all square integrable functions on G. Then let $\tilde{\pi} \times \lambda$ be the regular representation of $\mathfrak{A} \rtimes_\alpha G$ induced by the pair (π, λ) on the Hilbert space $L^2(G, H)$ of all H-valued square integrable functions on G, and defined by

$$\begin{aligned}(\tilde{\pi} \times \lambda)\big(\sum a_j f_j\big)\xi(t) &= \int_G \tilde{\pi}\big(\sum a_j f_j(s)\big)\lambda_s \xi(t) ds \\ &= \int_G \pi\Big(\alpha_{t^{-1}}\big(\sum a_j f_j(s)\big)\Big)\lambda_s \xi(t) ds\end{aligned}$$

for $a_j \in \mathfrak{A}$, $f_j \in L^1(G)$, $s,t \in G$ and $\xi \in L^2(G,H)$, where $\lambda_s\xi(t) = \xi(s^{-1}t)$. Since G is amenable, we have the following isomorphism, and set as follows:

$$\mathfrak{A} \rtimes_\alpha G \cong (\tilde{\pi} \times \lambda)(\mathfrak{A} \rtimes_\alpha G) \equiv \mathfrak{A} \rtimes_\alpha^r G$$

where $\mathfrak{A} \rtimes_\alpha^r G$ is the reduced crossed product of $\mathfrak{A} \rtimes_\alpha G$ (*cf.* Pedersen [10], Section 7.7). Furthermore, note that

$$\begin{aligned}\int_G \pi\Big(\alpha_{t^{-1}}\big(\sum a_j f_j(s)\big)\Big)\lambda_s ds &= \int_G \sum \pi\big(\alpha_{t^{-1}}(a_j)\big) f_j(s)\lambda_s ds \\ &= \sum \int_G \tilde{\pi}(a_j) f_j(s)\lambda_s ds \\ &= \sum \tilde{\pi}(a_j) \int_G f_j(s)\lambda_s ds \\ &= \sum \tilde{\pi}(a_j)\lambda(f_j)\end{aligned}$$

By the natural identification between the Hilbert spaces $L^2(G,H)$ and $H \otimes L^2(G)$, we may identify the operators $\tilde{\pi}(a_j)\lambda(f_j)$ with $\tilde{\pi}(a_j)\otimes\lambda(f_j)$ (in the following sense). Indeed, we have

$$\begin{aligned}\tilde{\pi}(a_j)\otimes\lambda(f_j)(\xi\otimes g) &= \tilde{\pi}(a_j)\xi\otimes\lambda(f_j)(g) \\ &= \tilde{\pi}(a_j)\xi\otimes(f_j * g), \\ \tilde{\pi}(a_j)\lambda(f_j)(\xi g)(t) &= \tilde{\pi}(a_j)\int_G f_j(s)\lambda_s ds\xi g(t) \\ &= \tilde{\pi}(a_j)\int_G f_j(s)\lambda_s \xi g ds(t) \\ &= \tilde{\pi}(a_j)\xi\int_G f_j(s) g(s^{-1}t) ds \\ &= \tilde{\pi}(a_j)\xi(f_j * g)(t)\end{aligned}$$

for $\xi \in H$, $g \in L^2(G)$, $\xi g \in L^2(G,H)$ and $t \in G$, where $\tilde{\pi}(a_j)\xi\otimes(f_j * g)$ of course (but somewhat confusingly) means that $\big(\tilde{\pi}(a_j)\xi\otimes(f_j*g)\big)(t) = \pi\big(\alpha_{t^{-1}}(a_j)\big)\xi\otimes(f_j*g)(t)$ for $t\in G$, and $f_j * g$ means the convolution product as used above.

Hence, we have

$$\sum \tilde{\pi}(a_j)\lambda(f_j) = \sum \tilde{\pi}(a_j)\otimes\lambda(f_j).$$

Therefore, we have the following norm equality:

$$\Big\| \sum \tilde{\pi}(a_j)\lambda(f_j)\Big\| = \Big\|\sum \tilde{\pi}(a_j)\otimes\lambda(f_j)\Big\|$$

as bounded operators on the Hilbert spaces $L^2(G,H)$ and $H\otimes L^2(G)$ respectively. □

Remark. Note that the twisted crossed product $\mathfrak{A} \rtimes_{\alpha,u} G$ of a C^*-algebra $\mathfrak{A}$ by an amenable locally compact group G with (α, u) a twisted action in the sense of Packer & Raeburn [8] has $L^1(G, \mathfrak{A})$ as a dense subspace so that the crossed product $\mathfrak{A} \rtimes_\alpha G$ in the statement can be replaced with $\mathfrak{A} \rtimes_{\alpha,u} G$.

Theorem 1.2. *Let $\mathfrak{A}$ be a C^*-algebra and G an amenable locally compact group. Then there exists a continuous field of Banach spaces between the crossed product $\mathfrak{A} \rtimes_\alpha G$ for an action α and the tensor product $\mathfrak{A} \otimes C^*(G)$ in the sense that there exists the Banach space $\Gamma([0,1], \{\mathfrak{B}_t\}_{t\in[0,1]})$ of a continuous field on $[0,1]$ with fibers $\mathfrak{B}_0 = \mathfrak{A} \otimes C^*(G)$ and $\mathfrak{B}_t = \mathfrak{A} \rtimes_\alpha G$ for $t \in (0,1]$ as Banach spaces.*

Moreover, if G is non-amenable, then one can replace $\mathfrak{A} \rtimes_\alpha G$ with the reduced crossed product $\mathfrak{A} \rtimes_\alpha^r G$, and $\mathfrak{A} \otimes C^*(G)$ with $\mathfrak{A} \otimes C_r^*(G)$ respectively, where $C_r^*(G)$ is the reduced group C^*-algebra of G.

Proof. We can define the Banach space $\Gamma([0,1], \{\mathfrak{B}_t\}_{t\in[0,1]})$ of a continuous field on the interval $[0,1]$ with those fibers in the statement to be the completion of the identity operator fields:

$$[0,1] \ni t \mapsto \begin{cases} \sum \tilde{\pi}(a_j) \otimes \lambda(f_j) \in \mathfrak{A} \otimes C^*(G), & t = 0, \\ \sum \tilde{\pi}(a_j)\lambda(f_j) \in \mathfrak{A} \rtimes_\alpha G, & t \in (0,1] \end{cases}$$

for $a_j \in \mathfrak{A}$ and $f_j \in L^1(G)$ by using continuity of the norm for those elements of the dense parts as in Proposition 1.1.

Since we are dealing with the regular representation λ on $L^2(G)$, the latter part follows the same way as above. □

Remark. This theorem should be new but it might be known to specialists. We may replace the crossed product $\mathfrak{A} \rtimes_\alpha G$ for an action α with the twisted crossed product $\mathfrak{A} \rtimes_{\alpha,u} G$ for (α, u) a twisted action, and its reduced crossed product with its reduced twisted crossed product respectively.

To obtain continuous fields of C^*-algebras we introduce the following:

Definition. Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system for $\mathfrak{A}$ a C^*-algebra, G an amenable locally compact group and α an action. Then we say that the action α is deformably trivial if there exists a family of actions α^s for $s \in [0,1]$ such that the functions: $[0,1] \ni s \mapsto \alpha_g^s(a) \in \mathfrak{A}$ for $a \in \mathfrak{A}$ and $g \in G$ are continuous, $\alpha^1 = \alpha$, and α^0 is trivial so that $\mathfrak{A} \rtimes_{\alpha^0} G \cong \mathfrak{A} \otimes C^*(G)$. We also say that the family $\{\alpha^s\}_{s\in[0,1]}$ is a continuous deformation of actions from α to the identity action.

Remark. Note that for $g \in G$, $a \in \mathfrak{A}$ and $s, t \in [0,1]$,

$$\big| \|\alpha_g^s(a)\| - \|\alpha_g^t(a)\| \big| \leq \|\alpha_g^s(a) - \alpha_g^t(a)\|.$$

If G is contractible, then α is deformably trivial. If G is a simply connected solvable Lie group, it is homeomorphic (or diffeomorphic) to the space $\mathbb{R}^n$ with $n = \dim G$, so that G is contractible and α is deformably trivial. Also, for twisted crossed products of C^*-algebras we may define a continuous deformation of twisted actions to the trivial twisted action as above.

Theorem 1.3. *Let $\mathfrak{A}$ be a C^*-algebra, G an amenable locally compact group and $\mathfrak{A} \rtimes_\alpha G$ the crossed product of $\mathfrak{A}$ by G with an action α. Suppose that there exists a continuous deformation $\{\alpha^t\}_{t\in[0,1]}$ of actions from α to the identity action. Then there exists a continuous field of C^*-algebras between the crossed product $\mathfrak{A} \rtimes_\alpha G$ for an action α and the tensor product $\mathfrak{A} \otimes C^*(G)$ in the sense that there exists a continuous field C^*-algebra $\Gamma([0,1], \{\mathfrak{B}_t\}_{t\in[0,1]})$ with fibers C^*-algebras $\mathfrak{B}_0 = \mathfrak{A} \otimes C^*(G)$, $\mathfrak{B}_1 = \mathfrak{A} \rtimes_\alpha G$ and $\mathfrak{B}_t = \mathfrak{A} \rtimes_{\alpha^t} G$ for $t \in (0,1]$.*

Moreover, if G is non-amenable, then one can replace $\mathfrak{A} \rtimes_\alpha G$ with the reduced crossed product $\mathfrak{A} \rtimes_\alpha^r G$ and $\mathfrak{A} \otimes C^*(G)$ with $\mathfrak{A} \otimes C_r^*(G)$ respectively.

Proof. The existence of the continuous deformation $\{\alpha^t\}_{t\in[0,1]}$ of actions ensures that the multiplication and involution at the fibers are continuous. By combining this point with Theorem 1.2, we obtain a continuous field of C^*-algebras as desired. □

Remark. This theorem is the key result to the following ones with additional conditions.

Definition. For a continuous field C^*-algebra on $[0,1]$ with fibers $\mathfrak{B}_t$ for $t \in [0,1]$, we say that the K-groups of the fibers are locally continuous if any class of the K-groups of $\mathfrak{B}_t$ is locally continuous under a locally continuous path connecting an element representing the class in the continuous field C^*-algebra.

Remark. If any projection (or unitary) in matrix algebras over the fibers (or its unitization) is connected to a projection (or unitary) in matrix algebras over the (other) fibers by projections (or unitaries) in matrix algebras over the continuous field C^*-algebra, then the K-groups of the fibers are locally and globally continuous. This condition on the projections and unitaries is equivalent to that any projection (or unitary) in matrix algebras over the fibers can be lifted to projections (or unitaries) in matrix algebras over the continuous field C^*-algebra.

Theorem 1.4. *Under the same situation as Theorem 1.3, it is deduced from that for* $* = 0, 1$,

$$K_*(\mathfrak{A} \rtimes_\alpha G) \cong K_*(\mathfrak{A} \otimes C^*(G))$$

under the assumption that the K-groups of $\mathfrak{A} \rtimes_{\alpha^t} G$ *for* $t \in [0, 1]$ *are locally continuous.*

Suppose that any projection (or unitary) in matrix algebras over $\mathfrak{A} \rtimes_\alpha G$ *(or its unitization) is connected to a projection (or unitary) in matrix algebras over* $\mathfrak{A} \otimes C^*(G)$ *by projections (or unitaries) in matrix algebras over the continuous field* C^**-algebra* $\Gamma([0,1], \{\mathfrak{B}_t\}_{t\in[0,1]})$ *of Theorem 1.3. Then*

$$K_*(\mathfrak{A} \rtimes_\alpha G) \cong K_*(\mathfrak{A} \otimes C^*(G))$$

for $* = 0, 1$ *without the assumption on the K-groups above.*

Furthermore, if G *is non-amenable, then one can replace* $\mathfrak{A} \rtimes_\alpha G$ *with* $\mathfrak{A} \rtimes_\alpha^r G$, *and* $\mathfrak{A} \otimes C^*(G)$ *with* $\mathfrak{A} \otimes C_r^*(G)$ *respectively.*

Proof. The existence of the continuous field C^*-algebra of Theorem 1.3 ensures that any projection (or unitary) in matrix algebras over the fibers (or their unitizations) is locally connected to a projection (or unitary) in matrix algebras over the near fibers. In fact, this follows from direct computations (or usual spectral theory) by using norm continuity of projections (or unitaries) in matrix algebras over the fibers. See the detailed proof of Theorem 2.1 below since the strategy is the same. □

Similarly as Theorem 1.4, for the equivariant K-theory we obtain.

Theorem 1.5. *Let* $(\mathfrak{A}, H, \alpha)$ *be a* C^**-dynamical system for* $\mathfrak{A}$ *a* C^**-algebra,* H *an amenable locally compact group and* α *an action, and* G *be a compact group such that* $(\mathfrak{A} \rtimes_\alpha H, G, \beta)$ *is a* C^**-dynamical system for an action* β*. Suppose that* α *is deformably trivial. Then there exists a continuous field of* C^**-algebras between the crossed product* $\mathfrak{A} \rtimes_\alpha H \rtimes_\beta G$ *and* $\mathfrak{A} \otimes C^*(H) \rtimes_{\beta^0} G$, *where* β^0 *is an action induced by a deformation from* α *to the trivial action.*

Therefore, for $* = 0, 1$,

$$\begin{aligned} K_*^G(\mathfrak{A} \rtimes_\alpha H) &\cong K_*(\mathfrak{A} \rtimes_\alpha H \rtimes_\beta G) \\ &\cong K_*(\mathfrak{A} \otimes C^*(H) \rtimes_{\beta^0} G) \\ &\cong K_*^G(\mathfrak{A} \otimes C^*(H)) \end{aligned}$$

under the assumption that the K-groups of the fibers $\mathfrak{A} \rtimes_{\alpha^t} H \rtimes_{\beta^t} G$ *for* $t \in [0,1]$ *are locally continuous, where* $\{\beta^t\}_{t\in[0,1]}$ *is a deformation between* $\beta = \beta^1$ *and* β^0 *induced by the deformation of* α.

Furthermore, suppose that any projection (or unitary) of matrix algebras over the fibers is connected by projections (or unitaries) in matrix algebras over the continuous field C^**-algebra. Then for* $* = 0, 1$,

$$K_*^G(\mathfrak{A} \rtimes_\alpha H) \cong K_*^G(\mathfrak{A} \otimes C^*(H))$$

without the assumption on the K-groups above.

Proof. See the proof of Theorem 2.3 below. □

Remark. By replacing crossed products for actions with twisted crossed products for twisted actions, we obtain the same results as Theorems 1.3 and 1,4 under the same assumptions, and also obtain the same as Theorem 1.5 by defining the equivariant K-theory for twisted dynamical systems by compact groups to be the K-theory of their twisted crossed products. Also, Theorems 1.3, 1.4 and 1.5 suggest that it is important to know continuous deformation of actions (or twisted actions), that is, homotopy classes of actions (or twisted actions). The research on this point would possibly be done elsewhere.

2. Crossed products by $\mathbb{R}$

Theorem 2.1. *Let* $(\mathfrak{A}, \mathbb{R}, \alpha)$ *be a* C^**-dynamical system of a* C^**-algebra* $\mathfrak{A}$ *by an action* α *of* $\mathbb{R}$. *Then there exists a continuous field of* C^**-algebras between the crossed product* $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ *and* $\mathfrak{A} \otimes C^*(\mathbb{R})$. *Therefore, for* $* = 0, 1$,

$$K_*(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong K_*(\mathfrak{A} \otimes C^*(\mathbb{R}))$$

under the assumption that the K-groups of the fibers $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ *for* $s \in [0,1]$ *are locally continuous, where actions* α^s *are defined by* $\alpha_t^s = \alpha_{st}$ *for* $s, t \in \mathbb{R}$.

Furthermore, suppose that any projection (or unitary) of matrix algebras over the unitizations of the fibers is connected by projections (or unitaries) in matrix algebras over the continuous field C^**-algebra. Then we have*

$$K_*(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong K_*(\mathfrak{A} \otimes C^*(\mathbb{R}))$$

for $* = 0, 1$ *without the assumption on the K-groups above.*

Moreover, the group $\mathbb{R}$ may be replaced with $\mathbb{T}$ in those (restricted) Thom isomorphisms for crossed products by $\mathbb{R}$.

Proof. We consider a deformation from the crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ to the tensor product $\mathfrak{A} \otimes C^*(\mathbb{R})$ as follows. Define actions α^s for $s \in [0,1]$ by

$$\alpha_t^s(a) = \alpha_{st}(a)$$

for $t \in \mathbb{R}$ and $a \in \mathfrak{A}$. Then $\alpha^1 = \alpha$ and α^0 is trivial. Also, we have

$$\big| \|\alpha_t^s(a)\| - \|\alpha_t^u(a)\| \big| \le \|\alpha_t^s(a) - \alpha_t^u(a)\| = \|\alpha_{st}(a) - \alpha_{ut}(a)\|.$$

Note that the crossed products $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$, $\mathfrak{A} \rtimes_{\alpha^u} \mathbb{R}$ for $s, u \in [0,1]$ are not isomorphic in general since the products of these crossed products are not the same (but they may be isomorphic for specific s, u). See Example 2.7 below. Our claim is that the actions α^s for $s \in [0,1]$ induce a deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ to $\mathfrak{A} \otimes C^*(\mathbb{R})$, that is, there exists the C^*-algebra of a continuous field on $[0,1]$ with fibers $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ for $s \in (0,1]$ and fiber $\mathfrak{A} \otimes C^*(\mathbb{R})$ at 0, denoted by $\Gamma([0,1], \{\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}\}_{s \in [0,1]})$. Indeed, the norm continuity for generating elements of the crossed products $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ and the tensor product $\mathfrak{A} \otimes C^*(\mathbb{R})$ follows from Proposition 1.1. Also, the C^*-algebra $\Gamma([0,1], \{\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}\}_{s \in [0,1]})$ may be defined as the completion of the family of all identity sections: $[0,1] \ni s \mapsto \sum a_j f_j \in \mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ for $a_j \in \mathfrak{A}$ and $f_j \in L^1(\mathbb{R})$. Therefore, any element of $\Gamma([0,1], \{\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}\}_{s \in [0,1]})$ may take values $\lim_s \sum a_j f_j$ in the fibers $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$, where $\lim_s$ for $s \in [0,1]$ mean the limits in the fibers.

Now suppose that $[p_s] - [1_l] \in K_0(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R})$ (a canonical class and a generator) for some $1 \le l \le n$, and p_s a projection of the $n \times n$ matrix algebra $M_n(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}^+)$ over the unitization $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}^+$ for some $n \ge 1$, and 1_l the $l \times l$ identity matrix, where $[\pi(p_s)] = [1_l] \in K_0(\mathbb{C})$ by definition of $K_0(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R})$ under the following exact sequence:

$$0 \longrightarrow M_n(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}) \longrightarrow M_n(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}^+) \xrightarrow{\ \pi\ } M_n(\mathbb{C}) \longrightarrow 0.$$

Then p_s is a limit of matrices s^k with $s^k = (s_{ij}^k, \lambda_{ij}^k)$, $\lambda_{ij}^k \in \mathbb{C}$ and the ij-components s_{ij}^k finite sums of $a_{ij}^k f_{ij}^k$ for $a_{ij}^k \in \mathfrak{A}$ and $f_{ij}^k \in L^1(\mathbb{R})$. Since the operator field: $[0,1] \ni t \mapsto p_t = \lim_t s^k$ is continuous, p_t for t near s are projections by standard functional calculus. We use this argument for generators of $K_0(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R})$. As for K_1-groups, we replace projections p_t with unitaries in matrix algebras over $\mathfrak{A} \rtimes_{\alpha^t} \mathbb{R}^+$. Therefore, by using connectivity of those projections or unitaries we obtain

$$K_*(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}) \cong K_*(\mathfrak{A} \rtimes_{\alpha^u} \mathbb{R})$$

for $* = 0,1$ and $s, u \in [0,1]$.

The second claim follows from the argument above about the local continuity of either projections or unitaries.

For the system $(\mathfrak{A}, \mathbb{T}, \beta)$, we define actions β^s for $s \in [0,1]$ by

$$\beta_z^s(a) = \beta_{z^s}(a)$$

for $z \in \mathbb{T}$ and $a \in \mathfrak{A}$. Thus, the last claim follows from the same argument as above. □

Remark. Similarly, we can define a deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{R}^n$ to $\mathfrak{A} \otimes C^*(\mathbb{R}^n)$ by defining actions α^s for $s \in [0,1]$ by $\alpha_t^s = \alpha_{st}$ for $t = (t_j) \in \mathbb{R}^n$, where $st = (st_j)$. Hence, under the assumptions on either the K-groups of $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}^n$ for $s \in [0,1]$ to be locally continuous, or connectivity of projections (or unitaries) we obtain

$$K_*(\mathfrak{A} \rtimes_\alpha \mathbb{R}^n) \cong K_*(\mathfrak{A} \otimes C^*(\mathbb{R}^n))$$

for $* = 0,1$. Without the assumptions above, the isomorphisms in fact do hold (see the Remark of Corollary 2.2 below). Also, those technical but certainly reasonable assumptions are not satisfied in general.

Moreover, we obtain.

Corollary 2.2. *Let $\mathfrak{A}$ be a C^*-algebra, G a simply connected solvable Lie group with $\dim G = n$ and $\mathfrak{A} \rtimes_\alpha G$ their crossed product for an action α. Then there exists a continuous field of C^*-algebras between the crossed product $\mathfrak{A} \rtimes_\alpha G$ and $\mathfrak{A} \otimes C^*(\mathbb{R}^n)$. Thus, for $* = 0,1$,*

$$K_*(\mathfrak{A} \rtimes_\alpha G) \cong K_*(\mathfrak{A} \otimes C^*(\mathbb{R}^n))$$

under the assumption that the K-groups of the fibers deformed from $\mathfrak{A} \rtimes_\alpha G$ to $\mathfrak{A} \otimes C^*(\mathbb{R}^n)$ are locally continuous.

Furthermore, suppose that any projection (or unitary) of matrix algebras over the fibers is connected by projections (or unitaries) in matrix algebras over the continuous field C^*-algebra. Then the isomorphisms of the K-groups above hold without the assumption on the K-groups above.

Similarly, there exists a continuous field of C^*-algebras between the crossed product $\mathfrak{A} \rtimes_\alpha G$ and $\mathfrak{A} \otimes C^*(G)$ so that we have

$$K_*(\mathfrak{A} \rtimes_\alpha G) \cong K_*(\mathfrak{A} \otimes C^*(G))$$

for $* = 0,1$ under the same assumptions above.

Proof. Note that G is isomorphic to a successive semi-direct product by $\mathbb{R}$:

$$G \cong (\cdots(\mathbb{R} \rtimes_{\alpha_2} \mathbb{R}) \rtimes_{\alpha_3} \mathbb{R} \cdots) \rtimes_{\alpha_n} \mathbb{R}$$

for $n = \dim G$, where α_j $(2 \le j \le n)$ are actions by $\mathbb{R}$ (Iwasawa [5]). Then we have

$$\mathfrak{A} \rtimes_\alpha G \cong (\cdots(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \rtimes_{\alpha_2} \mathbb{R} \cdots) \rtimes_{\alpha^n} \mathbb{R},$$

a successive crossed product by $\mathbb{R}$ (*cf.* Blackadar [1]). By using the method in the proof of Theorem 2.1, each α_j is deformed to the trivial action. Hence we obtain a deformation from $\mathfrak{A} \rtimes_\alpha G$ to $\mathfrak{A} \otimes C^*(\mathbb{R}^n)$. By using the method of Theorem 2.1, we obtain the conclusions.

For the last claim, note that there exists a continuous deformation of actions between α and the identity action by $\alpha_t^s = \alpha_{st}$ for $s \in [0,1]$ and $st = (st_j)_{j=1}^n \in G$ under the isomorphism of G above. □

Remark. By using the Fourier transform and the Bott periodicity, we have

$$\begin{aligned} K_*(\mathfrak{A} \rtimes_\alpha G) &\cong K_*\big(\mathfrak{A} \otimes C^*(\mathbb{R}^n)\big) \\ &\cong K_*\big(\mathfrak{A} \otimes C_0(\mathbb{R}^n)\big) \\ &\cong K_{*+n}(\mathfrak{A}) \end{aligned}$$

for $* = 0,1$, where $*+n$ means $*+n$ (mod 2), and $C_0(\mathbb{R}^n)$ is the C^*-algebra of continuous functions on $\mathbb{R}^n$ vanishing at infinity. In particular,

$$K_*(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong K_{*+1}(\mathfrak{A}).$$

This is the amazing Connes' Thom isomorphism for crossed products by $\mathbb{R}$ Connes [2] (*cf.* Blackadar [1]). The proof for this is (in part) based on the Takai duality for crossed products of C^*-algebras by abelian groups. Thus the method for our result above is quite different from that of Connes.

Moreover, our method is applicable to the case in the eqivariant K-theory as follows:

Theorem 2.3. *Let $(\mathfrak{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{R}$, and G be a compact group such that $(\mathfrak{A} \rtimes_\alpha \mathbb{R}, G, \beta)$ is a C^*-dynamical system. Then there exists a continuous field of C^*-algebras between the crossed product $\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_\beta G$ and $\mathfrak{A} \otimes C^*(\mathbb{R}) \rtimes_{\beta^0} G$, where β^0 is a deformed action from β associated with a continuous deformation from α to the trivial action. Therefore, we obtain*

$$K_*(\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_\beta G) \cong K_*(\mathfrak{A} \otimes C^*(\mathbb{R}) \rtimes_{\beta^0} G)$$

for $* = 0, 1$ *if the K-groups of the fibers* $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R} \rtimes_{\beta^s} G$ *for* $s \in [0,1]$ *are locally continuous, where* $\{\beta^s\}_{s\in[0,1]}$ *is a deformation between* β *and* β^0 *induced by the deformation of* α.

Furthermore, suppose that any projection (or unitary) of matrix algebras over the fibers is connected by projections (or unitaries) in matrix algebras over the continuous field C^**-algebra. Then for* $* = 0, 1$,

$$K_*^G(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong K_*^G(\mathfrak{A} \otimes C^*(\mathbb{R})),$$

where $K_*^G(\cdot)$ *mean the equivariant K-groups associated with the* C^**-dynamical system* $(\mathfrak{A} \rtimes_\alpha \mathbb{R}, G, \beta)$ *and a (deformed)* C^**-dynamical system* $(\mathfrak{A} \otimes C^*(\mathbb{R}), G, \beta^0)$.

Furthermore, if β^0 *is deformably trivial, there exists a continuous field of* C^**-algebras between the crossed product* $\mathfrak{A} \otimes C^*(\mathbb{R}) \rtimes_{\beta^0} G$ *and* $\mathfrak{A} \otimes C^*(\mathbb{R}) \otimes C^*(G)$*, and under the similar assumptions as above we have*

$$K_*^G(\mathfrak{A} \otimes C^*(\mathbb{R})) \cong K_*(\mathfrak{A} \otimes C^*(\mathbb{R}) \otimes C^*(G)).$$

Proof. As the same as the proof of Theorem 2.1, we consider a deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{R}$ to $\mathfrak{A} \otimes C^*(\mathbb{R})$ with fibers $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ for $s \in [0,1]$. Let (π, U) be the universal representation associated with the C^*-dynamical system $(\mathfrak{A} \rtimes_\alpha \mathbb{R}, G, \beta)$ such that $U_g \pi(a) U_g^* = \pi(\beta_g(a))$ for $a \in \mathfrak{A} \rtimes_\alpha \mathbb{R}$ and $g \in G$. Then the systems $(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}, G, \beta^s)$ can be induced from their universal covariant representations (U^s, π^s) defined by $U_g^s \pi^s(a)(U_g^s)^* = \pi^s(\beta_g^s(a))$ for $a \in \mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ and $g \in G$ since the restrictions of the actions β^s to the dense parts of $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}$ (see Proposition 1.1) are defined as the same as β. Note that for $* = 0, 1$,

$$K_*^G(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R}) \cong K_*(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R} \rtimes_{\beta^s} G)$$

for $s \in [0,1]$ (*cf.* Blackadar [1], Section 11.7). Then we assert that

$$K_*(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R} \rtimes_{\beta^s} G) \cong K_*(\mathfrak{A} \rtimes_{\alpha^t} \mathbb{R} \rtimes_{\beta^t} G)$$

for any $s, t \in [0,1]$. This follows from that there exists a deformation from $\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_\beta G$ to $(\mathfrak{A} \otimes C^*(\mathbb{R})) \rtimes_{\beta^0} G$ with fibers $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R} \rtimes_{\beta^s} G$ for $s \in [0,1]$, and the assumptions, where β^0 may be nontrivial in general. Note that since G is compact, it is amenable so that we can use Proposition 1.1 for the dense parts of the crossed products $\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R} \rtimes_{\beta^s} G$ for $s \in [0,1]$. Therefore, under the assumptions we have

$$\begin{aligned} K_*^G(\mathfrak{A} \rtimes_\alpha \mathbb{R}) &\cong K_*(\mathfrak{A} \rtimes_\alpha \mathbb{R} \rtimes_\beta G) \\ &\cong K_*(\mathfrak{A} \rtimes_{\alpha^s} \mathbb{R} \rtimes_{\beta^s} G) \end{aligned}$$

$$\cong K_*((\mathfrak{A} \otimes C^*(\mathbb{R})) \rtimes_{\beta^0} G)$$
$$\cong K_*^G(\mathfrak{A} \otimes C^*(\mathbb{R})),$$

as desired. Furthermore, we can continue this process to $\mathfrak{A} \otimes C^*(\mathbb{R}) \otimes C^*(G)$ if β^0 is deformably trivial. □

Remark. By using the Bott periodicity for the equivariant K-theory, we have

$$K_*^G(\mathfrak{A} \otimes C^*(\mathbb{R})) \cong K_*^G(\mathfrak{A} \otimes C_0(\mathbb{R})) \cong K_{*+1}^G(\mathfrak{A})$$

for $* = 0, 1$, where $* + 1$ means $* + 1 \pmod 2$.

Corollary 2.4. *With the notations and assumptions as in Theorem 2.3, we have*

$$K_*^G(\mathfrak{A} \rtimes_\alpha \mathbb{R}) \cong K_{*+1}^G(\mathfrak{A}),$$

the (restricted) Thom isomorphism for the equivariant K-theory. Moreover, for H a simply connected solvable Lie group with $n = \dim H$, we have

$$K_*^G(\mathfrak{A} \rtimes_\alpha H) \cong K_{*+n}^G(\mathfrak{A}).$$

Remark. This (restricted) formula(s) should be new.

Recall that a connected Lie group G is decomposed into a successive semi-direct product by $\mathbb{R}$ or $\mathbb{T}$:

$$G \cong H_1 \rtimes H_2 \rtimes \cdots \rtimes H_n$$

with $n = \dim G$ and $H_j \cong \mathbb{R}$ or $\mathbb{T}$ for $1 \leq j \leq n$ (*cf.* Iwasawa [5]). Thus, $\mathfrak{A} \rtimes_\alpha G$ is decomposed into a successive crossed product by $\mathbb{R}$ or $\mathbb{T}$:

$$\mathfrak{A} \rtimes_\alpha G \cong \big(\cdots\big((\mathfrak{A} \rtimes H_1) \rtimes H_2\big) \rtimes \cdots\big) \rtimes H_n.$$

Theorem 2.5. *Let $\mathfrak{A}$ be a C^*-algebra, G a connected Lie group and $\mathfrak{A} \rtimes_\alpha G$ their crossed product for an action α. Suppose the similar assumptions as in Theorems 2.1 and 2.3 for $\mathfrak{A} \rtimes_\alpha G$ decomposed into the successive crossed product as above. Then for $* = 0, 1$,*

$$K_*(\mathfrak{A} \rtimes_\alpha G) \cong K_{*+l}^{\mathbb{T}^k}(\mathfrak{A}),$$

where $\dim G = k + l$ for some $k \geq 0$, and G is decomposed into a successive semi-direct product as above involving $\mathbb{R}$ l-times and $\mathbb{T}$ k-times.

Proof. By using the method of Theorems 2.1 and 2.3 repeatedly, there exists a continuous deformation from $\mathfrak{A} \rtimes_\alpha G$ to $\mathfrak{A} \otimes C^*(H_1) \otimes \cdots \otimes C^*(H_n)$ so that we obtain the conclusion. Note that actions of $\mathbb{T}$ on the tori are always trivial. □

Finally, we give a couple of examples in the following:

Example 2.6. *Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space. Let $C_0(\mathbb{R}) \rtimes_\tau \mathbb{R}$ be the crossed product of $C_0(\mathbb{R})$ by $\mathbb{R}$ with τ the left translation. Then it is known that $C_0(\mathbb{R}) \rtimes_\tau \mathbb{R} \cong \mathbb{K}$. Therefore, for $* = 0, 1$,*

$$K_*\big(C_0(\mathbb{R}) \rtimes_\tau \mathbb{R}\big) \cong K_*(\mathbb{K}) \cong K_*(\mathbb{C}),$$

and $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$, while

$$K_*\big(C_0(\mathbb{R}) \otimes C^*(\mathbb{R})\big) \cong K_*\big(C_0(\mathbb{R}^2)\big) \cong K_*(\mathbb{C})$$

by using the Bott periodicity.

Example 2.7. *Let $C(\mathbb{T}^n) \rtimes_\Theta \mathbb{R}$ be the crossed product of $C(\mathbb{T}^n)$ by $\mathbb{R}$ with Θ the multi-rotation defined by $\Theta_t(z_j) = (e^{2\pi i\theta t} z_j)$ for $t \in \mathbb{R}$, $(z_j) \in \mathbb{T}^n$ and θ an irrational number. Since the crossed product is regarded as a foliation C^*-algebra of Connes, it is known that*

$$C(\mathbb{T}^n) \rtimes_\Theta \mathbb{R} \cong (C(\mathbb{T}^{n-1}) \rtimes_\Theta \mathbb{Z}) \otimes \mathbb{K},$$

where the crossed product $C(\mathbb{T}^{n-1}) \rtimes_\Theta \mathbb{Z}$ is a noncommutative torus (cf. Sudo [13]). Therefore, for $* = 0, 1$,

$$\begin{aligned} K_*(C(\mathbb{T}^n) \rtimes_\Theta \mathbb{R}) &\cong K_*(C(\mathbb{T}^{n-1}) \rtimes_\Theta \mathbb{Z}) \otimes \mathbb{K}) \\ &\cong K_*(C(\mathbb{T}^{n-1}) \rtimes_\Theta \mathbb{Z}) \cong \mathbb{Z}^{2^{n-1}}, \end{aligned}$$

where the last isomorphism is obtained by using Pimsner-Voiculescu exact sequence for crossed products by $\mathbb{Z}$ (cf. Blackadar [1]). On the other hand, for $* = 0, 1$,

$$K_*(C(\mathbb{T}^n) \otimes C^*(\mathbb{R})) \cong K_{*+1}(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$$

As a note, let $C(\mathbb{T}^n) \rtimes_{\Theta^s} \mathbb{R}$ be the crossed products deformed from $C(\mathbb{T}^n) \rtimes_\Theta \mathbb{R}$ with actions Θ^s defined by $\Theta_t^s(z_j) = (e^{2\pi i\theta st} z_j)$. Then they are not isomorphic in general since the crossed products $C(\mathbb{T}^{n-1}) \rtimes_{\Theta^s} \mathbb{Z}$ are so (cf. Rieffel [11]).

Example 2.8. *Let $C(\mathbb{T}) \rtimes_\tau \mathbb{T}$ be the crossed product of $C(\mathbb{T})$ by the left multiplication of $\mathbb{T}$ on $\mathbb{T}$. Then the crossed product is isomorphic to $\mathbb{K}$ the C^*-algebra of compact operators. Hence,*

$$K_*\big(C(\mathbb{T}) \rtimes_\tau \mathbb{T}\big) \cong K_*(\mathbb{K}) \cong K_*(\mathbb{C})$$

for $* = 0, 1$, while

$$K_*\big(C(\mathbb{T}) \otimes C^*(\mathbb{T})\big) \cong K_*\big(C(\mathbb{T}) \otimes C_0(\mathbb{Z})\big) \cong \oplus_{\mathbb{Z}} \mathbb{Z}$$

for $* = 0, 1$. *Therefore, this is a counterexample to the problem in the introduction. This example suggests that an obstruction to the isomorphism in the problem is that* $\mathbb{T}$ *is not contractible, but* τ *is deformably trivial in our sense, however the* K*-groups are not continuous at zero. This phenomenon would be analyzed somewhere else in the future.*

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REFERENCES

1. B. Blackadar: *K-theory for operator algebras.* Second edition. Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998. MR **99g**:46104
2. A. Connes: An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of *R*. *Adv. in Math.* **39** (1981), no. 1, 31–55. MR **82j**:46084
3. J. Dixmier: *C^*-algebras.* North-Holland, 1962.
4. J. M. G. Fell: The structure of algebras of operator fields. *Acta Math.* **106** (1961), 233–280. MR **29**#1547
5. K. Iwasawa: On some types of topological groups. *Ann. of Math. (2)* **50** (1949), 507–558. MR **10**,679a
6. R. Lee: On the C^*-algebras of operator fields. *Indiana Univ. Math. J.* **25** (1976), no. 4, 303–314. MR **53**#14150
7. ______: Full algebras of operator fields trivial except at one point. *Indiana Univ. Math. J.* **26** (1977), no. 2, 351–372. MR **55**#3812
8. J. Packer & I. Raeburn: Twisted crossed products of C^*-algebras. *Math. Proc. Cambridge Philos. Soc.* **106** (1989), no. 2, 293–311. MR **90g**:46097
9. ______: Twisted crossed products of C^*-algebras. II. *Math. Ann.* **287** (1990), no. 4, 595–612. MR **92b**:46106
10. G. K. Pedersen: *C^*-algebras and their automorphism groups.* London Mathematical Society Monographs, 14. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979. MR **81e**:46037

11. M. A. Rieffel: C^*-algebras associated with irrational rotations. *Pacific J. Math.* **93** (1981), no. 2, 415–429. MR **83b**:46087
12. M. Rørdam, F. Larsen & N. J. Laustsen: *An introduction to K-theory for C^*-algebras.* London Mathematical Society Student Texts, 49. Cambridge University Press, Cambridge, 2000. MR **2001g**:46001
13. T. Sudo: Structure of group C^*-algebras of Lie semi-direct products $\mathbb{C}^n \rtimes \mathbb{R}$. *J. Operator Theory* **46** (2001), no. 1, 25–38. MR **2002k**:46149
14. N. E. Wegge-Olsen: *K-theory and C^*-algebras.* A friendly approach. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. MR **95c**:46116

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