COMMON FIXED POINT IN FUZZY METRIC SPACES

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ABSTRACT. In this paper we prove common fixed point theorems for three mappings under the condition of weak compatible mappings, without taking any function continuous in fuzzy metric space and then extend this result to fuzzy 2-metric space and fuzzy 3-metric space.

1. INTRODUCTION

The notion of fuzzy sets was introduced by Zadeh [33] in 1965. Since then many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [6], Erceg [8], Kaleva & Seikkala [18], Kramosil & Michalek [20] have introduced the concept of fuzzy metric spaces in different ways. There are many view, points of the notion of a metric space in fuzzy topology, we can divide them into two groups.

The first group is formed by those results in which a fuzzy metric on a set X is treated as a map $d : X \times X \to R^+$ where $X \subset I^x$ Erceg [8] or $X =$ the totality of all fuzzy points of a set Bose & Sahani [1], and Hu [16] satisfying some collection of axioms or that are analogous of the ordinary metric axioms. Thus in such an approach numerical distances are set up between fuzzy objects.

We keep in the second group results in which the distance between objects is fuzzy, the objects themselves may be fuzzy or not.

Many authors have studied common fixed point theorems in fuzzy metric spaces. The most interesting references in this direction are Cho [5], George & Veeramani [13], Grabiec [14], Kaleva [19], Kramosil & Michálek [20], Mishra, Sharma & Singh [22], Sharma [25, 27], Sharma & Bagwan [28], Sharma & Deshpande [29, 30, 31] and fuzzy mappings Bose & Sahani [1], Butnariu [2], Chang [3], Chang, Cho, Lee & Lee

[4], Heilpern [15], Lee, Cho & Jung [21], Sharma [26]. Gahler in a series of papers Gáhler [10, 11, 12] investigated 2-metric spaces. It is to be remarked that Sharma, Sharma & Iseki [24] studied for the first time contraction type mappings in 2-metric spaces.

Recently Wenzhi [32] and many others initiated the study of Probabilistic 2-metric spaces. We know that 2-metric space is a real valued function of a point triples on a set X, whose abstract properties were suggested by the area function in Euclidean spaces. Now it is natural to expect 3-metric space which is suggested by the volume function. The method of introducing this is naturally different from 2-metric space theory, here we have to use simplex theory from algebraic topology.

In this paper we improve results of Sharma [27]. We prove common fixed point theorems in fuzzy metric space, fuzzy 2-metric space and fuzzy 3-metric space without taking any function continuous.

2. PRELIMINARIES

Now we begin with some definitions.

Definition 2.1 (Schweizer & Sklar [23]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Example of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2 (Kramosil & Michálek [20]). The 3-tuple $(X, M, *)$ is called a *fuzzy metric space (shortly, FM-space)* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times x[0, \infty)$ satisfying the following conditions: for all $x, y, z \in$X and $s, t > 0$,

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, 1) \to [0, 1]$ is left continuous.

In what follows, $(X, M, *)$ will denote a fuzzy metric space. Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞. In the following example, we know that every metric induces a fuzzy metric.

Example 2.1 (George & Veeramani [13]). *Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all $x, y \in X$ and $t > 0$,*

$$M(x, y, t) = \frac{t}{t + d(x, y)} \tag{1.a}$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Lemma 2.1 (Grabiec [14]). *For all $x, y \in X$, $M(x, y, .)$ is nondecreasing.*

Definition 2.3 (Grabiec [14]). Let $(X, M, *)$ be a fuzzy metric space:

(1) A sequence $\{x_n\}$ in X is said to be *convergent to a point $x \in X$,* (denoted by $\lim\limits_{n \to \infty} x_n = x$), if

$$\lim_{n \to \infty} M(X_n, x, t) = 1$$

for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if

$$\lim_{n \to \infty} M(X_{n+p}, x_n, t) = 1$$

for all $t > 0$ and $p > 0$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be *complete.*

Remark 2.1. Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined.

Let $(X, M, *)$ be a fuzzy metric space with the following condition:

(FM-6) $\lim\limits_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$.

Lemma 2.2 (Cho [5]). *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition (FM-6). If there exists a number $k \in (0, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t) \tag{1.b}$$

for all $t > 0$ and $n = 1, 2, \ldots$ then $\{y_n\}$ is a Cauchy sequence in X.

Proof. For $t > 0$ and $k \in (0, 1)$, we have

$M(y_2, y_3, kt) \geq M(y_1, y_2, t) \geq M(y_0, y_1, t/k)$ or $M(y_2, y_3, t) \geq M(y_0, y_1, t/k^2)$.

By simple induction with the condition (1.b) Grabiec [14], we have for all $t > 0$ and $n = 0, 1, 2, \ldots$

$$M(y_{n+1}, y_{n+2}, t) \geq M(y_1, y_2, t/k^n) \tag{1.c}$$

Thus by (1.c) and (FM-4), for any positive integer p and real number $t > 0$, we have

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * \overset{p\text{-times}}{\cdots} * M(y_{n+p-1}, y_{n+p}, t/p)$$

$$\geq M(y_1, y_2, t/pk^{n-1}) * \overset{p\text{-times}}{\cdots} * M(y_1, y_2, t/pk^{n+p-2})$$

Therefore, by (FM-6), we have

$$\lim_{n \to \infty} M(y_n, y_{n+p}, t) \geq 1 * \overset{p\text{-times}}{\cdots} * 1 \geq 1,$$

which implies that $\{y_n\}$ is a Cauchy sequence in X. This completes the proof. $\square$

Lemma 2.3 (Mishra, Sharma & Singh [22]). *If for all $x, y \in X$, $t > 0$ and for a number $k \in (0, 1)$,*

$$M(x, y, kt) \geq M(x, y, t)$$

then $x = y$.

Lemmas 2.1, 2.2, 2.3 and Remark 2.1 hold for fuzzy 2-metric spaces and fuzzy 3-metric spaces also.

Definition 2.4 (Jungck & Rhoades [17]). A pair of mappings A and S is said to be *weakly compatible in fuzzy metric space* if they commute at coincidence points.

Example 2.2. *Define $A, S : [0, 3] \to [0, 3]$ by*

$$Ax = \begin{cases} x & \text{if} \quad x \in [0, 1) \\ 3 & \text{if} \quad x \in [1, 3) \end{cases} \quad \text{and} \quad Sx = \begin{cases} 3 - x & \text{if} \quad x \in [0, 1) \\ 3 & \text{if} \quad x \in [1, 3) \end{cases}$$

Then for any $x \in [1, 3]$, $ASx = SAx$, showing that A, S are weakly compatible maps on $[0, 3]$.

Example 2.3. *Let $X = R$ and define $A, S : R \to R$ by $Ax = \frac{x}{3}$, $x \in R$ and $Sx = x_2$, $x \in R$. Hence 0 and $\frac{1}{3}$ are two coincidence points for the maps A and S. Note that A and S commute at 0, i.e., $AS(0) = SA(0) = 0$, but $AS(\frac{1}{3}) = A(\frac{1}{9}) = \frac{1}{27}$ and $SA(\frac{1}{3}) = S(\frac{1}{9}) = \frac{1}{81}$ and so A and S are not weakly compatible maps on R.*

Remark 2.2. Weakly compatible maps need not be compatible. Let $X = [2, 20]$ and d be the usual metric on X. Define mappings $A, S : X \to X$ by $Ax = x$ if $x = 2$ or > 5, $Ax = 6$ if $2 < x \leq 5$, $Sx = x$ if $x = 2$, $Sx = 12$ if $2 < x \leq 5$, $Sx = x - 3$ if $x > 5$. The mappings A and S are non-compatible since sequence $\{x_n\}$ defined by $x_n = 5 + (1/n), n \geq 1$.

Then $\lim_{n \to \infty} Sx_n = 2$, $\lim_{n \to \infty} SAx_n = 2$, and $\lim_{n \to \infty} ASx_n = 6$. But they are weakly compatible since they commute at coincidence point at $x = 2$.

Definition 2.5. A binary operation $* : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in $[0,1]$.

Definition 2.6. The 3-tuple $(X, M, *)$ is called a *fuzzy 2-metric space* if X is an arbitrary set, $*$ is a continuous *t*-norm and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$.

(FM'-1) $M(x, y, z, 0) = 0$,

(FM'-2) $M(x, y, z, t) = 1$, $t > 0$ and when at least two of the three points are equal,

(FM'-3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$,

      (Symmetry about three variables)

(FM'-4) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$

      (This corresponds to tetrahedron inequality in 2-metric space).

The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.

(FM'-5) $M(x, y, z, .) : [0, 1) \to [0, 1]$ is left continuous.

Definition 2.7. Let $(X, M, *)$ be a fuzzy 2-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be *convergent to a point* $x \in X$, if

$$\lim_{n \to \infty} M(x_n, x, a, t) = 1$$

for all $a \in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a *Cauchy sequence*, if

$$\lim_{n \to \infty} M(x_{n+p}, x_n, a, t) = 1$$

for all $a \in X$ and $t > 0$, $p > 0$.

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be *complete*.

Definition 2.8. A pair of mappings A and S is said to be *weakly compatible* in fuzzy 2-metric space if they commute at coincidence points.

Definition 2.9. A binary operation $* : [0,1]^4 \to [0,1]$ is called a *continuous t-norm* if $([0,1], *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ and $d_1 \leq d_2$ for all a_1, a_2, b_1, b_2, c_1, c_2 and d_1, d_2 are in $[0,1]$.

Definition 2.10. The 3-tuple $(X, M, *)$ is called a *fuzzy 3-metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^4 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z, w, u \in X$ and $t_1, t_2, t_3, t_4 > 0$.

(FM''-1) $M(x, y, z, w, 0) = 0$,

(FM''-2) $M(x, y, z, w, t) = 1$ for all $t > 0$,

         Only when the three simplex $< x, y, z, w >$ degenerate)

(FM''-3) $M(x, y, z, w, t) = M(x, w, z, y, t) = M(y, z, w, x, t) = M(z, w, x, y, t) = \ldots$

(FM''-4) $M(x, y, z, w, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2)$

$$* M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$$

(FM''-5) $M(x, y, z, w, .) : [0, 1) \to [0, 1]$ is left continuous.

Definition 2.11. Let $(X, M, *)$ be a fuzzy 3-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be *convergent to a point* $x \in X$, if

$$\lim_{n \to \infty} M(x_n, x, a, b, t) = 1$$

for all $a, b \in X$ and $t > 0$.

(2) A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a *Cauchy sequence*, if

$$\lim_{n \to \infty} M(x_{n+p}, x_n, a, b, t) = 1$$

for all $a, b \in X$ and $t > 0$, $p > 0$.

(3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be *complete*.

Definition 2.12. A pair of mappings A and S is said to be *weakly compatible* in fuzzy 3-metric space if they commute at coincidence points.

Fisher [9] proved the following theorem for three mappings in complete metric space:

Theorem A. *Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and satisfy:*

(i) $$d(Ax, Ay) \leq \alpha d(Sx, Ty)$$

for all $x, y \in X$ and $0 < \alpha < 1$. Indeed S, T and A have a unique common fixed point.

Sharma [27] extended Theorem A to fuzzy metric space, fuzzy 2-metric space and fuzzy 3-metric space and proved the following.

Theorem B. *Let $(X, M, *)$ be a complete fuzzy metric space with the condition (FM-6) and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and*

(i) $M(Ax, Ay, qt) \geq \min \{M(Ty, Ay, t), M(Sx, Ax, t), M(Sx, Ty, t)\}$
for all $x, y \in X$, $t > 0$ and $0 < q < 1$. Then S, T and A have a unique common fixed point.

Theorem C. *Let $(X, M, *)$ be a complete fuzzy 2-metric space and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and*

(i) $M(Ax, Ay, a, qt) \geq \min \{M(Ty, Ay, a, t), M(Sx, Ax, a, t), M(Sx, Ty, a, t)\}$ *for all x, y, a in X, $t > 0$ and $0 < q < 1$,*

(ii) $\lim_{n \to \infty} M(x, y, z, t) = 1$ *for all x, y, z in X. Then S, T and A have a unique common fixed point.*

Theorem D. *Let $(X, M, *)$ be a complete fuzzy 3-metric space and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists continuous mapping A of X into $S(X) \cap T(X)$ which commute with S and T and*

(i) $M(Ax, Ay, a, b, qt)$
$$\geq \min\{M(Ty, Ay, a, b, t), M(Sx, Ax, a, b, t), M(Sx, Ty, a, b, t)\}$$
for all x, y, a, b in $X, t > 0$ and $0 < q < 1$,

(ii) $\lim_{n \to \infty} M(x, y, z, w, t) = 1$ *for all x, y, z, w in X. Then S, T and A have a unique common fixed point.*

3. Main Results

We extend Theorem B and prove the following:

Theorem 3.1. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition (FM-6). Let A, S and T be mappings from X into itself such that*

(3.1) $S(X) \cup T(X) \subset A(X)$

(3.2) *the pairs $\{A, T\}$ and $\{A, S\}$ are weakly compatible,*

(3.3) *there exists a number $k \in (0, 1)$ such that*

$$M(Sx, Ty, kt) \geq M(Ax, Ay, t) * M(Sx, Ax, t) * M(Ay, Ty, t)$$
$$* M(Ay, Sx, t) * M(Ax, Ty, (2 - \alpha)t)$$

for all $x, y \in X, \alpha \in (0, 2), t > 0$. Then S, T and A have a unique common fixed point.

Proof. We define a sequence $\{x_n\}$ and $\{y_n\}$ such that

$$y_{2n} = Ax_{2n} = Tx_{2n-1} \quad \text{and} \quad y_{2n+1} = Ax_{2n+1} = Sx_{2n}, \; n = 1, 2, \ldots$$

We shall prove that $\{y_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.3), for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$, we write

$$M(Sx_{2n}, Tx_{2n+1}, kt) \geq M(Ax_{2n}, Ax_{2n+1}, t) * M(Sx_{2n}, Ax_{2n}, t)$$
$$* M(Ax_{2n+1}, Tx_{2n+1}, t) * M(Ax_{2n+1}, Sx_{2n}, t) * M(Ax_{2n}, Tx_{2n+1}, (2 - \alpha)t)$$

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t)$$
$$* M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+1}, t) * M(y_{2n}, y_{2n+2}, (1 + q)t)$$

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$$
$$* M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, qt) \qquad \text{(by FM-4)}$$

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, qt)$$

Since the t-norm $*$ is continuous and $M(x, y, .)$ is left continuous letting $q \to 1$, we have

(3.4) $M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$

Similarly, we have also

(3.5) $M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t)$

Thus by (3.4) and (3.5), it follows that

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t)$$

for $n = 1, 2, \ldots$ and so for positive integers n, p

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t/k^p).$$

Thus since $M(y_{n+1}, y_{n+2}, t/k^p) \to 1$ as $p \to \infty$ we have

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t)$$

By Lemma 2.2, $\{y_n\}$ is a Cauchy sequence in X. Since the space X is complete, the sequence converges to some point z in X. Also the subsequences $\{Ax_{2n}\}, \{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ also converge to z.

Since $S(X) \subset A(X)$, there exists a point $u \in X$ such that $Au = z$. Then using (3.3), with $\alpha = 1$, we write

$$M(Su, Tx_{2n+1}, kt) \geq M(Au, Ax_{2n+1}, t) * M(Su, Au, t) * M(Ax_{2n+1}, Tx_{2n+1}, t)$$
$$* M(Ax_{2n+1}, Su, t) * M(Au, Tx_{2n+1}, t)$$

$$M(Su, Tx_{2n+1}, kt) \geq M(z, Ax_{2n+1}, t) * M(Su, z, t) * M(Ax_{2n+1}, Tx_{2n+1}, t)$$
$$* M(Ax_{2n+1}, Su, t) * M(z, Tx_{2n+1}, t)$$

Taking the limit $n \to \infty$, we have

$$M(Su, z, kt) \geq M(Su, z, t)$$

Therefore by Lemma 2.3, we have $Su = z$. Since $Au = z$. Therefore $Su = z = Au$. Similarly since $T(X) \subset A(X)$, there exists a point $v \in X$ such that $Av = z$. Then using (3.3), with $\alpha = 1$, we write

$$M(z, Tv, kt) \geq M(z, Av, t) * 1 * M(Av, Tv, t) * M(Av, z, t) * M(z, Tv, t)$$

This gives

$$M(z, Tv, kt) \geq M(z, Tv, t)$$

Therefore by Lemma 2.3, we have $Tv = z$. Thus $Tv = z = Av$.

Since the pair $\{A, S\}$ is weakly compatible therefore A and S commute at their coincidence point *i. e.*, $ASu = SAu$ *i. e.*, $Az = Sz$. Now, we prove that $Az = z$. By (3.3) with $\alpha = 1$, we have

$$M(Sz, Tx_{2n+1}, kt) \geq M(Az, Ax_{2n+1}, t) * M(Sz, Az, t) * M(Ax_{2n+1}, Tx_{2n+1}, t)$$
$$* M(Ax_{2n+1}, Sz, t) * M(Az, Tx_{2n+1}, t)$$

$$M(Sz, Tx_{2n+1}, kt) \geq M(Sz, Ax_{2n+1}, t) * M(Sz, Sz, t) * M(Ax_{2n+1}, Tx_{2n+1}, t)$$
$$* M(Ax_{2n+1}, Sz, t) * M(Sz, Tx_{2n+1}, t)$$

Taking the limit $n \to \infty$, we have

$$M(Sz, z, kt) \geq M(Sz, z, t)$$

Therefore by Lemma 2.3, we have $Sz = z$. Therefore $Sz = z = Az$.

Similarly since the pair $\{A, T\}$ is weakly compatible therefore A and T commute at their coincidence point *i. e.*, $ATv = TAv$ *i. e.*, $Az = Tz$. Since we have already proved $Az = z$. Therefore $Az = Tz = z$. Hence $Az = Sz = Tz = z$. Thus z is a common fixed point of A, S and T.

For uniqueness of fixed point let $w(w \neq z)$ be another common fixed point of S, T and A. Then by (3.3) with $\alpha = 1$, we write

$$M(Sw, Tz, kt) \geq M(Aw, Az, t) * M(Sw, Aw, t) * M(Az, Tz, t)$$
$$* M(Az, Sw, t) * M(Aw, Tz, t)$$

$$M(w, z, kt) \geq M(w, z, t) * M(w, w, t) * M(z, z, t) * M(z, w, t) * M(w, z, t)$$

which implies that

$$M(w, z, kt) \geq M(w, z, t)$$

Therefore by Lemma2.3, we write $z = w$.

This completes the proof of Theorem 3.1.   □

Now we prove Theorem 3.1 for fuzzy 2-metric space. We prove the following:

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy 2-metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the conditions (3.1) and (3.2). Let A, S and T be mappings from X into itself such that*

(3.6) *there exists a number $k \in (0, 1)$ such that*
$$M(Sx, Ty, a, kt) \geq M(Ax, Ay, a, t) * M(Sx, Ax, a, t) * M(Ay, Ty, a, t)$$
$$* M(Ay, Sx, a, t) * M(Ax, Ty, a, (2 - \alpha)t)$$

for all $x, y, a \in X$, $\alpha \in (0, 2)$, $t > 0$.

(3.7) $\lim_{t \to \infty} M(x, y, z, t) = 1$ *for all x, y, z in X.*

Then S, T and A have a unique common fixed point.

Proof. We define sequences as we defined in Theorem 3.1. We shall prove that $\{y_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.6), for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$, we write

$$M(Sx_{2n}, Tx_{2n+1}, a, kt) \geq M(Ax_{2n}, Ax_{2n+1}, a, t)$$
$$* M(Sx_{2n}, Ax_{2n}, a, t) * M(Ax_{2n+1}, Tx_{2n+1}, a, t)$$
$$* M(Ax_{2n+1}, Sx_{2n}, a, t) * M(Ax_{2n}, Tx_{2n+1}, a, (2 - \alpha)t)$$

$$M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t)$$
$$* M(y_{2n+1}, y_{2n}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t)$$
$$* M(y_{2n+1}, y_{2n+1}, a, t) * M(y_{2n}, y_{2n+2}, a, (1+q)t)$$

On the lines of Sharma [27], using FM′-4, we have

$$M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t)$$
$$* M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n}, y_{2n+2}, y_{2n+1}, qt)$$
$$* M(y_{2n}, y_{2n+1}, a, t/2) * M(y_{2n+1}, y_{2n+2}, a, t/2)$$

Since the t-norm $*$ is continuous and $M(x, y, z, .)$ is left continuous letting $q \to 1$, we have

$$M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t)$$
$$* M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n}, y_{2n+2}, y_{2n+1}, t)$$
$$* M(y_{2n}, y_{2n+1}, a, t/2) * M(y_{2n+1}, y_{2n+2}, a, t/2)$$

Again using FM′-4 and (1.c), we have

$$M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t)$$
$$* M(y_0, y_2, y_1, t/3k^{2n}) * M(y_0, y_1, y_1, t/3k^{2n}) * M(y_1, y_2, y_1, t/3k^{2n})$$
$$* M(y_{2n}, y_{2n+1}, a, t/2) * M(y_{2n+1}, y_{2n+2}, a, t/2)$$

Thus since $M(y_0, y_2, y_1, t/3k^{2n}) \to 1$ as $n \to \infty$.

Thus we have

(3.8) $M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t)$

Similarly, we have also

(3.9) $M(y_{2n+2}, y_{2n+3}, a, kt) \geq M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n+2}, y_{2n+3}, a, t)$

Thus by (3.8) and (3.9), it follows that

$$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t)$$

for $n = 1, 2, \cdots$ and so for positive integers n, p

$$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t/k^p).$$

Thus since $M(y_{n+1}, y_{n+2}, a, t/k^p) \to 1$ as $p \to \infty$, we have

$$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t)$$

By Lemma 2.2, $\{y_n\}$ is a Cauchy sequence in X. Since the space X is complete, the sequence converges to some point z in X. Also the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ also converge to z.

Similarly as in the proof of the Theorem 3.1, since $S(X) \subset A(X)$, there exists a point $u \in X$ such that $Au = z$. Then using (3.6), with $\alpha = 1$, we write

$$M(Su, Tx_{2n+1}, a, kt) \geq M(Au, Ax_{2n+1}, a, t) * M(Su, Au, a, t)$$
$$* M(Ax_{2n+1}, Tx_{2n+1}, a, t) * M(Ax_{2n+1}, Su, a, t) * M(Au, Tx_{2n+1}, a, t)$$

Taking the limit $n \to \infty$, we have

$$M(Su, z, a, kt) \geq M(Su, z, a, t)$$

Therefore by Lemma 2.3, we have $Su = z$. Since $Au = z$. Therefore $Su = z = Au$. Similarly since $T(X) \subset A(X)$, there exists a point $v \in X$ such that $Av = z$.

Then using (3.6), with $\alpha = 1$, we write

$$M(z, Tv, a, kt) \geq M(z, Av, a, t) * 1 * M(Av, Tv, a, t)$$
$$* M(Av, z, a, t) * M(z, Tv, a, t)$$

This gives

$$M(z, Tv, a, kt) \geq M(z, Tv, a, t)$$

Therefore by Lemma 2.3, we have $Tv = z$. Thus $Tv = z = Av$. Since the pair $\{A, S\}$ is weakly compatible therefore A and S commute at their coincidence point i.e., $ASu = SAu$ i.e., $Az = Sz$. Now, we prove that $Az = z$. By (3.6) with $\alpha = 1$, we have

$$M(Sz, Tx_{2n+1}, a, kt) \geq M(Az, Ax_{2n+1}, a, t) * M(Sz, Az, a, t)$$
$$* M(Ax_{2n+1}, Tx_{2n+1}, a, t) * M(Ax_{2n+1}, Sz, a, t) * M(Az, Tx_{2n+1}, a, t)$$

Taking the limit $n \to \infty$, we have

$$M(Sz, z, a, kt) \geq M(Sz, z, a, t)$$

Therefore by Lemma 2.3, we have $Sz = z$. Therefore $Sz = z = Az$. Similarly by definition of weak compatibility it follows that $ATv = TAv$ and so $Az = ATv = TAv = Tz$. We have already proved that $Az = z$. Thus $Az = Tz = z$. Hence $Az = Tz = Sz = z$.

Thus z is a common fixed point of A, S and T.

We can prove uniqueness of the fixed point in similar manner as we have proved in Theorem 3.1. □

Theorem 3.3. *Let $(X, M, *)$ be a complete fuzzy 3-metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the conditions (3.1) and (3.2). Let A, S and T be mappings from X into itself such that*

(3.10) there exists a number $k \in (0, 1)$ such that
$$M(Sx, Ty, a, b, kt) \geq M(Ax, Ay, a, b, t) * M(Sx, Ax, a, b, t)$$
$$*M(Ay, Ty, a, b, t) * M(Ay, Sx, a, b, t) * M\big(Ax, Ty, a, b, (2 - \alpha)t\big)$$
for all $x, y, a, b \in X$, $(0, 2)$, $t > 0$.

(3.11) $\lim_{t \to \infty} M(x, y, z, w, t) = 1$ for all x, y, z, w in X.

Then S, T and A have a unique common fixed point. Theorem 3.3 can be proved in the similar manner as Theorem 3.2.

4. ACKNOWLEDGEMENT

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