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INTUITIONISTIC H-FUZZY SETS

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ABSTRACT. We introduce the category **ISet**(H) of intuitionistic H-fuzzy sets and show that **ISet**(H) satisfies all the conditions of a topological universe except the terminal separator property. And we study the relation between **Set**(H) and **ISet**(H).

0. INTRODUCTION

The subject of fuzzy sets as an approach to a mathematical representation of vagueness in every day language was introduced by Zadeh [20] in 1965. He generalized the idea of the characteristic function of a subset of a set X by defining a fuzzy subset of X as a map from X into $[0, 1]$. In Goguen [6], altered this definition to the case in which $[0, 1]$ is replaced by a partially ordered set H.

There are many other categories, for instance, **Set**(H), $\mathbf{Set_f}(H)$, $\mathbf{Set_g}(H)$ and **Fuz**(H) introduced in Eytan [5], Goguen [6], Negoiţă & Ştefănescu [15], Ponasse [19], in connection with fuzzy set theory. However, the category **Set**(H) is the most useful one as the "standard'" category, because the category **Set**(H) is very suitable for describing fuzzy sets and maps between them. Until now, many authors Dubuc [4], Eytan [5], Goguen [6], Negoiţă & Ştefănescu [15], Pitts [17], Ponasse [18, 19] have investigated **Set**(H) in topos view-point. In particular, Hur [9] investigated **Set**(H) in topological universe view-point. The concept of a topological universe was introduced by Nel [16], which implies a cartesian closed and a concrete quasitopos. The notion of a topological universe has already been put to effective use for several areas of mathematics.

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In 1986, as a generalization of fuzzy sets, Atanassov [1] introduced the concept of an intuitionistic fuzzy set in X as a complex mapping from X into $[0,1] \times [0,1]$ satisfying a certain condition. After that time, Çoker [3], S. J. Lee & E. P. Lee [14] and Hur and his colleagues Hur, Kim & Ryou [11] introduced the concept of an intuitionistic fuzzy topological space and investigated its some properties. In particular, Hur and his colleagues Hur, Jun & Ryou [10] applied the notion of intuitionistic fuzzy sets to topological group.

In this paper, we introduce the category $\mathbf{ISet}(H)$ of intuitionistic H-fuzzy sets and study $\mathbf{ISet}(H)$ in the sense of a topological universe.

1. PRELIMINARIES

In this section, we will introduce some basic definitions and well-known results from Herrlich [7, 8], Hur [9], Johnstone [12], Kim, S. S. Hong, Y. H. Hong & Park [13], Nel [16] which are needed in the next section.

Definition 1.1 (Kim, S. S. Hong, Y. H. Hong & Park [13]). Let $\mathbf{A}$ be a concrete category and $((Y_i, \xi_i))_I$ a family of objects in $\mathbf{A}$ indexed by a class I. For any set X, let $(f_i : X \to Y_i)_I$ be a source of maps indexed by I. An $\mathbf{A}$-structure ξ on X is called *initial with respect to* $(X, (f_i), ((Y_i, \xi_i)))$ provided that the following conditions hold:

(1) For each $i \in I$, $f_i : (X, \xi) \to (Y_i, \xi_i)$ is an $\mathbf{A}$-morphism.
(2) If (Z, ρ) is an $\mathbf{A}$-object and $g : Z \to X$ is a map such that for each $i \in I$, the map $f_i \circ g : (Z, \rho) \to (Y_i, \xi_i)$ is an $\mathbf{A}$-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an $\mathbf{A}$-morphism. In this case, $(f_i : (X, \xi) \to (Y_i, \xi_i))_I$ is called an *initial source in* $\mathbf{A}$.

Dual notions: *final structure; final sink.*

Definition 1.2 (Kim, S. S. Hong, Y. H. Hong & Park [13]). A concrete category $\mathbf{A}$ is called *topological over* **Set** provided that for each set X, for any family $((Y_i, \xi_i))_I$ of $\mathbf{A}$-objects, and for any source $(f_i : X \to Y_i)_I$ of maps, there exists a unique $\mathbf{A}$-structure ξ on X which is initial with respect to $(X, (f_i), ((Y_i, \xi_i)))$.

Dual notions: *cotopological category.*

Result 1.A (Kim, S. S. Hong, Y. H. Hong & Park [13], Theorem 1.5). A concrete category $\mathbf{A}$ is topological if and only if $\mathbf{A}$ is cotopological.

Result 1.B (Kim, S. S. Hong, Y. H. Hong & Park [13], Theorem 1.6; Herrlich, [8] Proposition in Section 1). Let **A** be a topological category over **Set**. Then **A** is complete and cocomplete.

Definition 1.3 (Kim, S. S. Hong, Y. H. Hong & Park [13]). Let **A** be a concrete category.

(1) The **A**-*fibre* of a set X is the class of all **A**-structures on X.

(2) **A** is called *properly fibred over* **Set** provided that the following conditions hold:

  (i) (*Fibre-smallness*) For each set X, the **A**-fibre of X is a set.

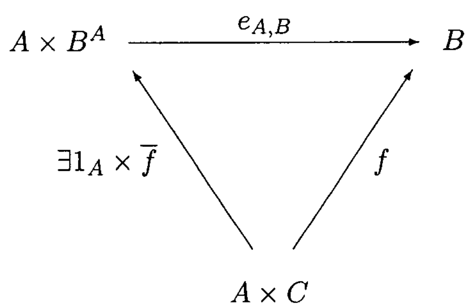
  (ii) (*Terminal separator property*) For each singleton set X, the **A**-fibre of X has precisely one element.

  (iii) If ξ and η are **A**-structures on a set X such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are **A**-morphisms, then $\xi = \eta$.

Definition 1.4 (Herrlich [7]). A category **A** is called *cartesian closed* provided that the following conditions hold:

(1) For any **A**-objects A and B, there exists a product $A \times B$ in **A**.

(2) Exponential exist in **A**, *i. e.*, for any **A**-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, *i. e.*, for any **A**-object B, there exists an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$ (called the *evaluation*) such that for any **A**-object C and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\overline{f} : C \to B^A$ such that the diagram



commutes.

Definition 1.5 (Nel [16]). A category **A** is called a *topological universe over* **Set** provided that the following conditions hold:

(1) **A** is well-structured over **Set**, *i. e.*, (i) **A** is a concrete category; (ii) **A** has the fibre-smallness condition; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final epi-sinks in **A** are preserved by pullbacks, *i. e.*, for any final epi-sink $(g_\lambda : X \to Y)_\Lambda$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final epi-sink.

Definition 1.6 (Birkhoff [2], Johnstone [12]). A lattice H is called a *complete Heyting algebra*, if H satisfies the following conditions hold:

(1) H is a complete lattice.

(2) For any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has a greatest element denoted by $a \to b$ (called *pseudo-complement of* a *and* b), *i. e.*, $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.

In particular, for each $a \in H$, $N(a) = a \to o$ is called the *negation* or the *pseudo-complement* of a.

Result 1.C (Birkhoff [2], Ex. 6 in p. 46). Let H be a complete Heyting algebra and let $a, b \in H$. Then:

(1) If $a \leq b$, then $N(b) \leq N(a)$, *i. e.*, $N : H \to H$ is an involutive order reversing operation in $(H, \leq)$.

(2) $a \leq NN(a)$.

(3) $N(a) = NNN(a)$.

(4) $N(a \vee b) = N(a) \wedge N(b)$ and $N(a \wedge b) = N(a) \vee N(b)$.

Throughout this paper, we use H as a complete Heyting algebra.

Definition 1.7 (Hur [9]). The concrete category **Set**(H) is defined by: Objects are (X, ν), called an *H-fuzzy set (or simple, a fuzzy set) on* X, where X is any set and ν any map from X to H. A morphism $f : (X, \nu) \to (Y, \eta)$ is a map from X to Y satisfying $\nu(x) \leq \eta \circ f(x)$ for each $x \in X$, where "$\leq$" means the order induced by the operation "$\wedge$ or $\vee$" in H. Every **Set**(H)-morphism will be called a **Set**(H)-*map.*

2. THE CATEGORY ISet(H)

In this section, we introduce the category **ISet**(H) of intuitionistic H-fuzzy sets and study some of it's properties.

Definition 2.1. Let X be a set. A triple (X, μ, ν) is called an *intuitionistic H-fuzzy set (in short, IHFS) on* X if the following conditions hold:

(1) $\mu, \nu \in H^X$, *i. e.*, μ and ν are H-fuzzy sets.

(2) $\mu \leq N(\nu)$, *i. e.*, $\mu(x) \leq N(\nu(x))$ for each $x \in X$, where $N : H \to H$ is an involutive order reversing operation in $(H, \leq)$.

Definition 2.2. Let (X, μ_X, ν_X) and (Y, μ_Y, ν_Y) be IHFSs. A mapping $f : X \to Y$ is called a *morphism* if $\mu_X \leq \mu_Y \circ f$ and $\nu_X \geq \nu_Y \circ f$.

The following is the immediate result of Definition 2.2:

Proposition 2.3. *Let (X, μ_X, ν_X) and (Y, μ_Y, ν_Y) and (Z, μ_Z, ν_Z) be IHFSs.*

(1) *The identity mapping $1_X : (X, \mu_X, \nu_X) \to (X, \mu_X, \nu_X)$ is a morphism.*
(2) *If $f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y)$ and $g : (Y, \mu_Y, \nu_Y) \to (Z, \mu_Z, \nu_Z)$ are morphisms, then $g \circ f : (X, \mu_X, \nu_X) \to (Z, \mu_Z, \nu_Z)$ is a morphism.*

From Definition 2.1, Definition 2.2 and Proposition 2.3, we can form a concrete category $\mathbf{ISet}(H)$ *consisting of all IHFSs and morphisms between them. In this case, each* $\mathbf{ISet}(H)$*-morphism will be called an* $\mathbf{ISet}(H)$*-mapping.*

It is clear that if $f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y)$ is an $\mathbf{ISet}(H)$*-mapping, then $f : (X, \mu_X) \to (Y, \mu_Y)$ is a* $\mathbf{Set}(H)$*-mapping (cf. Hur [9]).*

Theorem 2.4. $\mathbf{ISet}(H)$ *is topological over* $\mathbf{Set}$.

Proof. Let X be a set and let $((X_\alpha, \mu_\alpha, \nu_\alpha))_\Gamma$ any family of IHFSs indexed by a class Γ. Let $(f_\alpha : X \to (X_\alpha, \mu_\alpha, \nu_\alpha))_\Gamma$ be any source of mappings. We define two mappings $\mu, \nu : X \to H$, respectively by for each $x \in X$,

$$\mu(x) = \bigwedge_\Gamma \mu_\alpha \circ f_\alpha(x) \quad \text{and} \quad \nu(x) = \bigvee_\Gamma \nu_\alpha \circ f_\alpha(x).$$

Let $x \in X$. Since $(X_\alpha, \mu_\alpha, \nu_\alpha) \in \mathbf{ISet}(H)$ for each $\alpha \in \Gamma$, $\mu_\alpha \leq N(\nu_\alpha)$ for each $\alpha \in \Gamma$. Then:

$$N(\nu(x)) = N\Big(\bigvee_\Gamma \nu_\alpha \circ f_\alpha(x)\Big) = \bigwedge_\Gamma N\Big(\nu_\alpha(f_\alpha(x_\alpha))\Big)$$
$$\geq \bigwedge_\Gamma \mu_\alpha(f_\alpha(x)) = \bigwedge_\Gamma \mu_\alpha \circ f_\alpha(x) = \mu(x).$$

Thus $\mu \leq N(\nu)$. So $(X, \mu, \nu) \in \mathbf{ISet}(H)$. By the definition of ν, $\nu \geq \nu_\alpha \circ f_\alpha$ for each $\alpha \in \Gamma$. Moreover, by the process of the proof of Theorem 2.1 in Hur [9], $f_\alpha : (X, \mu) \to (X_\alpha, \mu_\alpha)$ is an $\mathbf{Set}(H)$-mapping for each $\alpha \in \Gamma$. Hence $f_\alpha : (X, \mu, \nu) \to (X_\alpha, \mu_\alpha, \nu_\alpha)$ is an $\mathbf{ISet}(H)$-mapping for each $\alpha \in \Gamma$.

For any $(Y, \mu_Y, \nu_Y) \in \mathbf{ISet}(H)$, let $g : Y \to X$ be any mapping for which $f_\alpha \circ g : (Y, \mu_Y, \nu_Y) \to (X_\alpha, \mu_\alpha, \nu_\alpha)$ is an $\mathbf{ISet}(H)$-mapping for each $\alpha \in \Gamma$. We will show that $g : (Y, \mu_Y, \nu_Y) \to (X, \mu, \nu)$ is an $\mathbf{ISet}(H)$-mapping. By the process of the proof of Theorem 2.1 in Hur [9], $g : (Y, \mu_Y) \to (X, \mu)$ is a $\mathbf{Set}(H)$-mapping. Thus it is sufficient to show that $\nu_Y \geq \nu \circ g$. Since $f_\alpha \circ g : (Y, \mu_Y, \nu_Y) \to (X_\alpha, \mu_\alpha, \nu_\alpha)$ is an $\mathbf{ISet}(H)$-mapping for each $\alpha \in \Gamma$, $\nu_Y \geq \nu_\alpha \circ (f_\alpha \circ g) = (\nu_\alpha \circ f_\alpha) \circ g$ for each $\alpha \in \Gamma$. Let $y \in Y$. Then $\nu_Y(y) \geq (\nu_\alpha \circ f_\alpha) \circ g(y)$ for each $\alpha \in \Gamma$. Thus $\nu_Y(y) \geq \bigvee_\Gamma (\nu_\alpha \circ f_\alpha)(g(y)) = \nu(g(y)) = \nu \circ g(y)$. So $\nu_Y \geq \nu \circ g$.

Hence $\big(f_\alpha : (X, \mu, \nu) \to (X_\alpha, \mu_\alpha, \nu_\alpha)\big)_\Gamma$ is an initial source in $\mathbf{ISet}(H)$. This completes the proof. □

Example 2.5. (1) *Inverse image of an IHFS structure.* Let X be a set, let (Y, μ_Y, ν_Y) an IHFS and let $f : X \to Y$ a mapping. Then there exists the initial IHFS structure (μ_X, ν_X) on X for which $f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y)$ is an $\mathbf{ISet}(H)$-mapping. In this case, (μ_X, ν_X) is called the *inverse image* of (μ_Y, ν_Y) under f. In particular, let $X \subset Y$ and let $f : X \to Y$ be the canonical mapping. Then the inverse image (μ_X, ν_X) of (μ_Y, ν_Y) under f is called the *induced IHFS structure* and the triple (X, μ_X, ν_X) an *intuitionistic H-fuzzy subset* of (Y, μ_Y, ν_Y).

(2) *Product IHFS structure.* Let $\big((X_\alpha, \mu_\alpha, \nu_\alpha)\big)_\Gamma$ be a family of IHFSs. Then there exists the initial IHFS structure (μ, ν) on the product set $X = \Pi_{\alpha \in \Gamma} X_\alpha$ for which the projection $\pi_\alpha : (X, \mu, \nu) \to (X_\alpha, \mu_\alpha, \nu_\alpha)$ is an $\mathbf{ISet}(H)$-mapping for each $\alpha \in \Gamma$. In this case, (μ, ν) is called the *product* of $\big((\mu_\alpha, \nu_\alpha)\big)_\Gamma$, denoted by $(\mu, \nu) = (\prod \mu_\alpha, \prod \nu_\alpha)$, and the triple $(\prod X_\alpha, \prod \mu_\alpha, \prod \nu_\alpha)$ is called the *product IHFS* of $\big((X_\alpha, \mu_\alpha, \nu_\alpha)\big)_\Gamma$. In fact, $\Pi \mu_\alpha = \bigwedge_\Gamma \mu_\alpha \circ \pi_\alpha$ and $\Pi \nu_\alpha = \bigvee_\Gamma \nu_\alpha \circ \pi_\alpha$. In particular, if $\Gamma = \{1, 2\}$, then $\Pi \mu_\alpha = \mu_1 \times \mu_2 = (\mu_1 \circ \pi_1) \wedge (\mu_2 \circ \pi_2)$ and $\Pi \nu_\alpha = \nu_1 \times \nu_2 = (\nu_1 \circ \pi_1) \vee (\nu_2 \circ \pi_2)$.

The following is the immediate result of Theorem 2.4 and Result 1.B.

Corollary 2.6. $\mathbf{ISet}(H)$ *is complete and cocomplete.*

From Result 1.A, it is clear that $\mathbf{ISet}(H)$ *is cotopological. However, we will show that* **ISet (H)** *is cotopological.*

Theorem 2.7. $\mathbf{ISet}(H)$ *is cotopological over* **Set**.

Proof. Let X be any set and let $\big((X_\alpha, \mu_\alpha, \nu_\alpha)\big)_\Gamma$ any family of IHFSs indexed by a class Γ. Let $(f_\alpha : X_\alpha \to X)_\Gamma$ be any sink of mappings. We define two mappings $\mu, \nu : X \to H$ by for each $x \in X$

$$\mu(x) = \begin{cases} \bigvee_\Gamma \bigvee_{x_\alpha \in f_\alpha^{-1}(x)} \mu_\alpha(x_\alpha) & \text{if } \ f_\alpha^{-1} \neq \varnothing, \\ = 0 & \text{if } \ f_\alpha^{-1} = \varnothing \end{cases}$$

and

$$\nu(x) = \begin{cases} \bigwedge_\Gamma \bigwedge_{x_\alpha \in f_\alpha^{-1}(x)} \nu_\alpha(x_\alpha) & \text{if } f_\alpha^{-1} \neq \varnothing, \\ 0 & \text{if } f_\alpha^{-1} = \varnothing \end{cases}$$

Since $(X_\alpha, \mu_\alpha, \nu_\alpha) \in \mathbf{ISet}(H)$, $\mu_\alpha \leq N(\nu_\alpha)$ for each $\alpha \in \Gamma$. We may assume that $f^{-1}(x) \neq \varnothing$ without loss of generality. Let $x \in X$. Then

$$\begin{aligned} N(\nu(x)) &= N\Big(\bigwedge_\Gamma \bigwedge_{x_\alpha \in f_\alpha^{-1}(x)} \nu_\alpha(x_\alpha)\Big) \\ &= \bigvee_\Gamma \bigvee_{x_\alpha \in f_\alpha^{-1}(x)} N\big(\nu_\alpha(x_\alpha)\big) \\ &\geq \bigvee_\Gamma \bigvee_{x_\alpha \in f_\alpha^{-1}(x)} \mu_\alpha(x_\alpha) \\ &= \mu(x). \end{aligned}$$

Thus $\mu \leq N(\nu)$. So $(X, \mu, \nu) \in \mathbf{ISet}(H)$. By the process of the proof of Theorem 2.2 in Hur [9], $f_\alpha : (X_\alpha, \mu_\alpha) \to (X, \mu)$ is an $\mathbf{Set}(H)$-mapping. Moreover, by the definition of ν, $\nu_\alpha \geq \nu \circ f_\alpha$ for each $\alpha \in \Gamma$. Hence $f_\alpha : (X_\alpha, \mu_\alpha, \nu_\alpha) \to (X, \mu, \nu)$ is an $\mathbf{ISet}(H)$-mapping for each $\alpha \in \Gamma$.

For each $(Y, \mu_Y, \nu_Y) \in \mathbf{ISet}(H)$, let $g : X \to Y$ be any mapping for which $g \circ f_\alpha : (X_\alpha, \mu_\alpha, \nu_\alpha) \to (Y, \mu_Y, \nu_Y)$ is an $\mathbf{ISet}(H)$-mapping for each $\alpha \in \Gamma$. We will show that $g : (X, \mu, \nu) \to (Y, \mu_Y, \nu_Y)$ is an $\mathbf{ISet}(H)$-mapping. By the process of the proof of Theorem 2.2 in Hur [9], $g : (X, \mu) \to (Y, \mu_Y)$ is a $\mathbf{Set}(H)$-mapping. Since $g \circ f_\alpha : (X_\alpha, \mu_\alpha, \nu_\alpha) \to (Y, \mu_Y, \nu_Y)$ is an $\mathbf{ISet}(H)$-mapping, $\nu_\alpha \geq \nu_Y \circ (g \circ f_\alpha)$ for each $\alpha \in \Gamma$. Let $x \in X$ and let $x_\alpha \in f_\alpha^{-1}(x)$ for each $\alpha \in \Gamma$.

Then $\nu_\alpha(x_\alpha) \geq \nu_Y \circ (g \circ f_\alpha)(x_\alpha) = \nu_Y \circ g\big(f_\alpha(x_\alpha)\big) = \nu_Y \circ g(x)$ for each $\alpha \in \Gamma$. Thus $\bigwedge_\Gamma \bigwedge_{x_\alpha \in f_\alpha^{-1}(x)} \nu_\alpha(x_\alpha) \geq \nu_Y \circ g(x)$, *i. e.*, $\nu(x) \geq \nu_Y \circ g(x)$. So $\nu \geq \nu_Y \circ g$. Hence $g : (X, \mu, \nu) \to (Y, \mu_Y, \nu_Y)$ is an $\mathbf{ISet}(H)$-mapping. Therefore $\mathbf{ISet}(H)$ is cotopological over $\mathbf{Set}$. □

Example 2.8. (1) *Intuitionistic H-fuzzy quotient set structure.*

Let $(X, \mu, \nu) \in \mathbf{ISet}(H)$, let R be an equivalence relation on X and let $\varphi : X \to X/R$ the canonical mapping. Then there exists the final intuitionistic H-fuzzy set structure $(\mu_{X/R}, \nu_{X/R})$ on X/R for which $\varphi : (X, \mu, \nu) \to (X/R, \mu_{X/R}, \nu_{X/R})$ is an $\mathbf{ISet}(H)$-mapping. In this case, $(\mu_{X/R}, \nu_{X/R})$ is called the *intuitionistic H-fuzzy qoutient set structure* of X by R.

(2) *Sum of intuitionistic H-fuzzy set structures.* Let $\big((X_\alpha, \mu_\alpha, \nu_\alpha)\big)_\Gamma$ be a family of H-fuzzy sets, let X the sum of $(X_\alpha)_\Gamma$, *i. e.*, $X = \bigcup_{\alpha \in \Gamma}(X_\alpha \times \{\alpha\})$ and let j_α :

$X_\alpha \to X$ the canonical (injection) mapping for each $\alpha \in \Gamma$. Then there exists the final intuitionistic H-fuzzy set structure (μ, ν) on X. In fact, for each $(x_\alpha, \alpha) \in X$, $\mu(x_\alpha, \alpha) = \bigvee_\Gamma \mu_\alpha(x_\alpha)$ and $\nu(x_\alpha, \alpha) = \bigwedge_\Gamma \nu_\alpha(x_\alpha)$. In this case, (μ, ν) is called the *sum of* $\left((\mu_\alpha, \nu_\alpha)\right)_\Gamma$ and (X, μ, ν) called the *sum of* $\left((X_\alpha, \mu_\alpha, \nu_\alpha)\right)_\Gamma$.

Theorem 2.9. *Final episinks in* **ISet**(H) *are preserved by pullbacks.*

Proof. Let $\left(g_\alpha : (X_\alpha, \mu_\alpha, \nu_\alpha) \to (Y, \mu_Y, \nu_Y)\right)_\Gamma$ be any final episink in **ISet**(H) and let $f : (W, \mu_W, \nu_W) \to (Y, \mu_Y, \nu_Y)$ any **ISet**(H)-mapping. For each $\alpha \in \Gamma$, let $U_\alpha = \left\{(w, x_\alpha) \in W \times X_\alpha : f(w) = g_\alpha(x_\alpha)\right\}$ and let us define two mappings $\mu'_\alpha : U_\alpha \to H$ and $\nu'_\alpha : U_\alpha \to H$ by for each $(w, x_\alpha) \in U_\alpha$, $\mu'_\alpha(w, x_\alpha) = \mu_W(x) \wedge \mu_\alpha(x_\alpha)$ and $\nu'_\alpha(w, x_\alpha) = \nu_W(x) \vee \nu_\alpha(x_\alpha)$, where $e_\alpha : U_\alpha \to W$ and $p_\alpha : U_\alpha \to X_\alpha$ are the usual projections of U_α. Then clearly $(U_\alpha, \mu'_\alpha, \nu'_\alpha) \in$ **ISet**(H) for each $\alpha \in \Gamma$ and $e_\alpha : (U_\alpha, \mu'_\alpha, \nu'_\alpha) \to (W, \mu_W, \nu_W)$ and $p_\alpha : (U_\alpha, \mu'_\alpha, \nu'_\alpha) \to (X_\alpha, \mu_\alpha, \nu_\alpha)$ are **ISet**(H)-mappings for each $\alpha \in \Gamma$. Moreover the following diagram is a pullback square in **ISet**(H):

$$
\begin{array}{ccc}
(U_\alpha, \mu'_\alpha, \nu'_\alpha) & \xrightarrow{\quad p_\alpha \quad} & (X_\alpha, \mu_\alpha, \nu_\alpha) \\
{\scriptstyle e_\alpha}\downarrow & & \downarrow{\scriptstyle g_\alpha} \\
(W, \mu_W, \nu_W) & \xrightarrow[\quad f \quad]{} & (Y, \mu_Y, \nu_Y).
\end{array}
$$

By the process of the proof of Theorem 2.3 in Hur [9], $\left(e_\alpha : (U_\alpha, \mu'_\alpha, \nu'_\alpha) \to (W, \mu_W, \nu_W)\right)_\Gamma$ is an episink in **ISet**(H). Suppose (μ, ν) is another final intuitionistic H-fuzzy set structure on W with respect to $(e_\alpha)_\Gamma$. By the process of the proof of Theorem 2.3 in Hur [9], since $\mu = \mu_W$, it is sufficient to show that $\nu = \nu_W$. Let $w \in W$. Then:

$$
\begin{aligned}
\nu_W(w) &= \nu_W(w) \vee \nu_W(w) \\
&\geq \nu_W(w) \vee \nu_Y \circ f(w) \\
&\qquad\qquad \text{(Since } f : (W, \mu_W, \nu_W) \to (Y, \mu_Y, \nu_Y) \text{ is an } \mathbf{ISet}(H)\text{-mapping)} \\
&= \nu_W(w) \vee \nu_Y(f(w)) \\
&= \nu_W(w) \vee \Big[\bigwedge_\Gamma \bigwedge_{x_\alpha \in g_\alpha^{-1}(f(w))} \nu_\alpha(x_\alpha)\Big] \text{ (Since } (g_\alpha)_\Gamma \text{ is final)}
\end{aligned}
$$

$$= \bigwedge_{\Gamma} \bigwedge_{x_\alpha \in g_\alpha^{-1}(f(w))} [\nu_W(w) \vee \nu_\alpha(x_\alpha)]$$
$$= \bigwedge_{\Gamma} \bigwedge_{(w,x_\alpha) \in e_\alpha^{-1}(w)} \nu'_\alpha(w,(x_\alpha)).$$

Thus $\nu_W(w) \geq \nu(w)$ for each $w \in W$. So $\nu_W \geq \nu$. On the other hand, since $(e_\alpha : (U_\alpha, \mu'_\alpha, \nu'_\alpha) \to (W, \mu_W, \nu_W))_\Gamma$ is final, $1_W : (W, \mu, \nu) \to (W, \mu_W, \nu_W)$ is an **ISet**(H)-mapping and thus $\nu \geq \nu_W$. So $\nu = \nu_W$. Hence $(e_\alpha)_\Gamma$ is final. This completes the proof. □

For any singleton set $\{a\}$, since the H-fuzzy set structure (μ, ν) on $\{a\}$ is not unique, the category **ISet**(H) is not property fibred over **Set**. Hence, by Theorem 2.7 and Theorem 2.9, we obtain the following result.

Theorem 2.10. **ISet**(H) *satisfies all the conditions of a topological universe over* **Set** *except the terminal separator property.*

Theorem 2.11. **ISet**(H) *is cartesian closed over* **Set**.

Proof. It is clear that **ISet**(H) has products by Corollary 2.6. Thus it is sufficient to show that **ISet**(H) has exponential objects.

For any IHFSs $\mathbf{X} = (X, \mu, \nu)$ and $\mathbf{Y} = (Y, \mu_Y, \nu_Y)$, let Y^X be the set of all mappings from X to Y. We define two mappings $\mu : Y^X \to H$ and $\nu : Y^X \to H$ as follows: for each $f \in Y^X$,

$$\mu(f) = \bigwedge \{h \in H : \mu_X(x) \wedge h \leq \mu_Y(f(x)) \text{ for each } x \in X\}$$

and

$$\nu(f) = \bigvee \{h \in H : \nu_X(x) \vee h \geq \nu_Y(f(x)) \text{ for each } x \in X\}$$

Then clearly $(Y^X, \mu, \nu) \in$ **ISet**(H). Let $\mathbf{Y^X} = (Y^X, \mu, \nu)$ Then, by the definitions of μ and ν, for each $f \in Y^X$ and each $x \in X$,

$$\mu_X(x) \wedge \mu(f) \leq \mu_Y(f(x))$$

and

$$\nu_X(x) \vee \nu(f) \geq \nu_Y(f(x)).$$

Define $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(x, f) = f(x)$ for each $(x, f) \in X \times Y^X$. Let $(x, f) \in X \times Y^X$. Then:

$$(\mu_X \times \mu)(x, f) = \mu_X(x) \wedge \mu(f) \leq \mu_Y(f(x)) = \mu_Y \circ f(x) = \mu_Y \circ e_{X,Y}(x, f)$$

and

$$(\nu_X \times \nu)(x, f) = \nu_X(x) \vee \nu(f) \geq \nu_Y(f(x)) = \nu_Y \circ f(x) = \nu_Y \circ e_{X,Y}(x, f)$$

Thus $\mu_X \times \mu \leq \mu_Y \circ e_{X,Y}$ and $\nu_X \times \nu \geq \nu_Y \circ e_{X,Y}$. So $e_{X,Y} : \mathbf{X} \times \mathbf{Y}^{\mathbf{X}} \to \mathbf{Y}$ is an $\mathbf{ISet}(H)$-mapping.

For any $\mathbf{Z} = (Z, \mu_Z, \nu_Z) \in \mathbf{ISet}(H)$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be an $\mathbf{ISet}(H)$-mapping. We define $\overline{h} : Z \to Y^X$ by $[\overline{h}(z)](x) = h(x, z)$ for each $z \in Z$ and each $x \in X$. Let $z \in Z$ and let $x \in X$. Then:

$$\begin{aligned}(\mu_X \times \mu_Z)(x, z) &= \mu_X \wedge \mu_Z(z) \\ &\leq \mu_Y \circ h(x, z) \\ &\qquad \text{Since } h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y} \text{ is an } \mathbf{ISet}(H)\text{-mapping)} \\ &= \mu_Y([\overline{h}(z)](x))\end{aligned}$$

Thus, by the definition of μ, $\mu_Z(z) \leq \mu(\overline{h}(z)) = \mu \circ \overline{h}(z)$. So $\mu_Z \leq \mu \circ \overline{h}$.

On the other hand,

$$\begin{aligned}(\nu_X \times \nu_Z)(x, z) &= \nu_X \vee \nu_Z(z) \\ &\geq \nu_Y \circ h(x, z) \\ &\qquad \text{Since } h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y} \text{ is an } \mathbf{ISet}(H)\text{-mapping)} \\ &= \nu_Y([\overline{h}(z)](x))\end{aligned}$$

Thus, by the definition of ν, $\nu_Z(z) \leq \nu(\overline{h}(z)) = \nu \circ \overline{h}(z)$. So $\nu_z \geq \nu \circ \overline{h}$. Hence $\overline{h} : \mathbf{Z} \to \mathbf{Y}^{\mathbf{X}}$ is an $\mathbf{ISet}(H)$-mapping. Moreover, $\overline{h}$ is the unique $\mathbf{ISet}(H)$-mapping such that $e_{X,Y} \circ (1_X \times \overline{h}) = h$. This completes the proof. □

3. THE RELATIONS BETWEEN ISet(H)AND Set(H)

Lemma 3.1. *Define* $G_1, G_2 : \mathbf{ISet}(H) \to \mathbf{Set}(H)$ *by*

$$G_1(X, \mu, \nu) = (X, \mu), G_2(X, \mu, \nu) = \big(X, N(\nu)\big) \quad \text{and} \quad G_1(f) = G_2(f) = f.$$

Then G_1 *and* G_2 *are functors.*

Proof. Clearly $G_1(X, \mu, \nu) = (X, \mu) \in \mathbf{Set}(H)$ for each $(X, \mu, \nu) \in \mathbf{ISet}(H)$. Let (X, μ_X, ν_X), $(Y, \mu_Y, \nu_Y) \in \mathbf{ISet}(H)$ and let $f : (X, \mu_X, \nu_X) \to (Y, \mu_Y, \nu_Y)$ be an $\mathbf{ISet}(H)$-mapping. Then $\mu_X \leq \mu_Y \circ f$. Thus $G_1(f) = f : (X, \mu_X) \to (Y, \mu_Y)$ is a $\mathbf{Set}(H)$-mapping. Hence $G_1 : \mathbf{ISet}(H) \to \mathbf{Set}(H)$ is a functor.

Also $G_2(X,\mu,\nu) = \big(X, N(\nu)\big) \in \mathbf{Set}(H)$ for each $(X,\mu,\nu) \in \mathbf{ISet}(H)$. Now let (X,μ_X,ν_X), $(Y,\mu_Y,\nu_Y) \in \mathbf{ISet}(H)$ and let $f : (X,\mu_X,\nu_X) \to (Y,\mu_Y,\nu_Y)$ be an $\mathbf{ISet}(H)$-mapping. Then $\nu_X \geq \nu_Y \circ f$. Thus $N(\nu_X) \leq N(\nu_Y) \circ f$. So $G_2(f) = f : \big(X, N(\nu_X)\big) \to \big(Y, N(\nu_Y)\big)$ is a $\mathbf{Set}(H)$-mapping. Hence $G_2 : \mathbf{ISet}(H) \to \mathbf{Set}(H)$ is a functor. □

Lemma 3.2. *Define* $F_1 : \mathbf{Set}(H) \to \mathbf{ISet}(H)$ *by* $F_1(X,\mu) = (X,\mu,N(\mu))$ *and* $F_1(f) = f$. *Then* F_1 *is a functor.*

Proof. For each $(X,\mu) \in \mathbf{Set}(H)$, $\mu \leq NN(\mu)$. Thus $F_1(X,\mu) = \big(X,\mu,N(\mu)\big) \in \mathbf{ISet}(H)$. Let $(X,\mu_X),(Y,\mu_Y) \in \mathbf{Set}(H)$ and let $f : (X,\mu_X) \to (Y,\mu_Y)$ be an $\mathbf{Set}(H)$-mapping. Then $\mu_X \leq \mu_Y \circ f$.

Consider the mapping $F_1(f) = f : \big(X,\mu_X,N(\mu_X)\big) \to \big(Y,\mu_Y,N(\mu_Y)\big)$. Since $\mu_X \leq \mu_Y \circ f$, $N(\mu_X) \geq N(\mu_Y) \circ f$. So $f : \big(X,\mu_X,N(\mu_X)\big) \to \big(Y,\mu_Y,N(\mu_Y)\big)$ is an $\mathbf{ISet}(H)$-mapping. Hence F_1 is a functor. □

Lemma 3.3. *Define* $F_2 : \mathbf{Set}(H) \to \mathbf{ISet}(H)$ *by* $F_2(X,\mu) = \big(X, NN(\mu), N(\mu)\big)$ *and* $F_2(f) = f$. *Then* F_2 *is a functor.*

Proof. It is clear that $F_2(X,\mu) \in \mathbf{ISet}(H)$ for each $(X,\mu) \in \mathbf{Set}(H)$. Let (X,μ_X), $(Y,\mu_Y) \in \mathbf{Set}(H)$ and let $f : (X,\mu_X) \to (Y,\mu_Y)$ be an $\mathbf{Set}(H)$-mapping.

Consider the mapping $F_2(f) = f : F_2(X,\mu_X) \to F_2(Y,\mu_Y)$, where $F_2(X,\mu_X) = \big(X, NN(\mu_X), N(\mu_X)\big)$ and $F_2(Y,\mu_Y) = \big(Y, NN(\mu_Y), N(\mu_Y)\big)$. Since $f : (X,\mu_X) \to (Y,\mu_Y)$ is a $\mathbf{Set}(H)$-mapping, $\mu_X \leq \mu_Y \circ f$. Thus $NN(\mu_X) \leq NN(\mu_Y) \circ f$. Moreover $N(\mu_X) \geq N(\mu_Y) \circ f$. So $F_2(f) = f : F_2(X,\mu_X) \to F_2(Y,\mu_Y)$ is an $\mathbf{ISet}(H)$-mapping. Hence F_2 is a functor. □

Theorem 3.4. *The functor* $F_1 : \mathbf{Set}(H) \to \mathbf{ISet}(H)$ *is a left adjoint of the functor* $G_1 : \mathbf{ISet}(H) \to \mathbf{Set}(H)$.

Proof. For each $(X,\mu) \in \mathbf{Set}(H)$, $1_X : (X,\mu) \to G_1F_1(X,\mu) = (X,\mu)$ is a $\mathbf{Set}(H)$-mapping. Let $(Y,\mu_Y,\nu_Y) \in \mathbf{ISet}(H)$ and let $f : (X,\mu) \to G_1(Y,\mu_Y,\nu_Y)$ be an $\mathbf{ISet}(H)$-mapping. We will show that $f : F_1(X,\mu) = (X,\mu,N(\mu)) \to (Y,\mu_Y,\nu_Y)$ is an $\mathbf{ISet}(H)$-mapping. Since $f : (X,\mu) = G_1(Y,\mu_Y,\mu_Y) \to (Y,\mu_Y)$ is a $\mathbf{Set}(H)$-mapping, $\mu \leq \mu_Y \circ f$.

Then $N(\mu) \geq N(\mu_Y) \circ f$. Since $\mu_Y \leq N(\nu_Y)$, $\nu_Y \leq NN(\nu_Y) \leq N(\mu_Y)$. Thus $N(\mu) \geq \nu_Y \circ f$. So $f : F_1(X,\mu) \to (Y,\mu_Y,\nu_Y)$ is an $\mathbf{ISet}(H)$-mapping. Hence 1_X is a G_1-universal map for (X,μ) in $\mathbf{Set}(H)$. This completes the proof. □

For each $(X, \mu) \in \mathbf{Set}(H)$, $F_1(X, \mu) = (X, \mu, N(\mu))$ is called an *intuitionistic H-fuzzy set in X induced* by (X, μ). Let us denote the category of all induced intuitionistic H-fuzzy sets and $\mathbf{ISet}(H)$-mappings as $\mathbf{ISet}^*(H)$. Then it is clear that $\mathbf{ISet}^*(H)$ is a full subcategory of $\mathbf{ISet}(H)$.

Theorem 3.5. *Two categories* $\mathbf{Set}(H)$ *and* $\mathbf{ISet}^*(H)$ *are isomorphic.*

Proof. It is clear that $F_1 : \mathbf{Set}(H) \to \mathbf{ISet}^*(H)$ is a functor by Lemma 3.2. Consider the restriction $G_1 : \mathbf{ISet}^*(H) \to \mathbf{Set}(H)$ of the functor G_1 in Lemma 3.1. Let $(X, \mu) \in \mathbf{Set}(H)$. Then, by Lemma 3.2, $F_1(X, \mu) = \big(X, \mu, N(\mu)\big)$.

Thus $G_1 F_1(X, \mu) = G_1\big(X, \mu, N(\mu)\big) = (X, \mu)$. So $G_1 \circ F = 1_{\mathbf{Set}(H)}$. Now let $\big(X, \mu, N(\mu)\big) \in \mathbf{ISet}^*(H)$. Then, by Lemma 3.1, $G_1\big(X, \mu, N(\mu)\big) = (X, \mu)$. Thus $FG_1\big(X, \mu, N(\mu)\big) = F(X, \mu) = \big(X, \mu, N(\mu)\big)$. So $F \circ G_1 = 1_{\mathbf{ISet}^*(H)}$.

Hence $F : \mathbf{Set}(H) \to \mathbf{ISet}^*(H)$ is an isomorphism. This completes the proof. □

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