J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 12, Number 2 (May 2005), Pages 93–103

ASYMPTOTIC BEHAVIORS FOR LINEAR DIFFERENCE SYSTEMS

DONG MAN IM AND YOON HOE GOO

ABSTRACT. We study some stability properties and asymptotic behavior for linear difference systems by using the results in [W. F. Trench: Linear asymptotic equilibrium and uniform, exponential, and strict stability of linear difference systems. *Comput. Math. Appl.* **36** (1998), no. 10–12, pp. 261–267].

1. INTRODUCTION

We are concerned with the nonlinear difference system

$$\Delta x(n) = f(n, x(n)), \quad x(n_0) = x_0 \tag{1.1}$$

where $f : \mathbb{N}(n_0) \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots, n_0 + k, \ldots\}$, n_0 is a nonnegative integer, $\mathbb{R}^m$ is the m-dimensional Euclidean space. Here Δ is the forward difference operator, *i. e.*, $\Delta x(n) = x(n+1) - x(n)$.

Also, we consider the associated variational systems

$$\Delta v(n) = f_x(n, 0)v(n) \tag{1.2}$$

and

$$\Delta z(n) = f_x(n, x(n, n_0, x_0))z(n) \tag{1.3}$$

of the system (1.1).

We recall some stability notions in Agarwal [1], Choi & Koo [2] and Choi, Koo & Song [3].

The system (1.1) is said to be *strongly stable* if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any solution $x(n) = x(n, n_0, x_0)$, $n_0 \leq k \in \mathbb{N}(n_0)$ and $|x(k, n_0, x_0)| < \delta$ imply $|x(n, n_0, x_0)| < \varepsilon$ for all $n \in \mathbb{N}(n_0)$.

Received by the editors November 19, 2004 and, in revised form, January 12, 2005.
2000 *Mathematics Subject Classification.* 39A11.
Key words and phrases. summable similarity, asymptotic equivalence, asymptotic equilibrium.

For the linear difference system

$$\Delta x(n) = A(n)x(n), \tag{1.4}$$

where $A(n)$ is an $m \times m$ matrix function defined on $\mathbb{N}(n_0)$, the system (1.4) is said to be *restrictively stable* if it is stable and its adjoint system

$$\Delta y(n) = -A^T(n)y(n+1) \tag{1.5}$$

is stable.

Also, (1.4) is *reducible* (*reducible to zero*, respectively) if there exists an $m \times m$ matrix $L(n)$ which, together with its inverse $L^{-1}(n)$, is defined and bounded on $\mathbb{N}(n_0)$ such that

$$L^{-1}(n+1)A(n)L(n) + L^{-1}(n+1)L(n) - I$$

is a constant matrix (the zero matrix, respectively) on $\mathbb{N}(n_0)$, where I is the $m \times m$ identity matrix.

System (1.1) is called an *h-system* if there is a positive function $h : \mathbb{N}(n_0) \longrightarrow \mathbb{R}$ and a constant $c \geq 1$ and $\delta > 0$ such that

$$|x(n, n_0, x_0)| \leq |x_0|h(n), \quad n \geq n_0$$

if $|x_0| < \delta$. If h is bounded, then (1.1) is said to be *h-stable*.

To study asymptotic behavior for system (1.1), we need the following notions: system (1.1) has *asymptotic equilibrium* if there exists a single $\xi \in \mathbb{R}^m$ and $r > 0$ such that any solution $x(n, n_0, x_0)$ of (1.1) with $|x_0| < r$ satisfies

$$x(n, n_0, x_0) \longrightarrow \xi + o(1) \quad \text{as} \quad n \longrightarrow \infty, \tag{1.6}$$

and for every $\xi \in \mathbb{R}^m$, there exists a solution of (1.1) such that satisfies (1.6). A linear homogeneous system is said to have *linear asymptotic equilibrium* if every nontrivial solution approaches a nonzero limit as $n \longrightarrow \infty$.

Two difference systems (1.1) and

$$\Delta y(n) = g(n, y(n)) \tag{1.7}$$

are said to be *asymptotically equivalent* if, for every solution $x(n)$ of (1.1), there exists a solution $y(n)$ of (1.7) such that

$$x(n) = y(n) + o(1) \quad \text{as} \quad n \longrightarrow \infty \tag{1.8}$$

and conversely, for every solution $y(n)$ of (1.7), there exists a solution $x(n)$ of (1.1) such that the asymptotic relationship (1.8) holds.

Two $m \times m$ invertible matrices $A(n)$ and $B(n)$ are *summably similar* if there exists an $m \times m$ matrix $F(n)$ satisfying

$$\sum_{l=n_0}^{\infty} |F(l)| < \infty$$

such that

$$\Delta S(n) + S(n+1)B(n) - A(n)S(n) = F(n)$$

for some invertible bounded matrix $S(n)$ having bounded inverse.

The notion of summable similarity was introduced by Trench [6]. Trench's definition is a discrete analog of Conti's definition of t_∞-similarity of matrix functions Conti [4]. Trench [6] also weakened the definition of summable similarity as t_∞-quasisimilarity and obtained results under weakened conditions. The definition of n_∞-similarity in Choi & Koo [2] is quite different from Trench's definition.

In this paper, we investigate some stability properties for linear difference systems in Section 2, and study asymptotic behavior for linear difference system and its perturbed system by means of Trench's result Trench [6] in Section 3.

2. Stability for Linear System

Firstly, we study about h-system for the variational system (1.3). To do this we consider two linear homogeneous difference systems

$$\Delta x(n) = A(n)x(n), \quad x(n_0) = x_0 \tag{2.1}$$

and

$$\Delta y(n) = B(n)y(n), \quad y(n_0) = y_0, \tag{2.2}$$

where $A(n)$, $B(n)$ are $m \times m$ matrices, and $I + A(n)$, $I + B(n)$ are invertible $m \times m$ matrices on $\mathbb{N}(n_0)$. Their fundamental matrix solutions are denoted by $X(n)$ and $Y(n)$, respectively. If $A(n)$ and $B(n)$ are summably similar, then we have

$$X^{-1}(n)S(n)Y(n) = X^{-1}(n_0)S(n_0)Y(n_0) + \sum_{l=n_0}^{n-1} X^{-1}(l+1)F(l)Y(l) \tag{2.3}$$

by Lemma 1 in Trench [6]. In view of Medina & Pinto [5], the notion of h-system is characterized by means of the fundamental matrix solution Φ of (2.1):

$$|\Phi(n, n_0, 0)| \le ch(n)h^{-1}(n_0), \quad n \ge n_0 \tag{2.4}$$

for some constant $c \ge 1$ and positive function h defined on $\mathbb{N}(n_0)$.

Theorem 2.1. *Suppose that*

(i) $f_x(n,0)$ *and* $f_x(n,x(n,n_0,x_0))$ *are summably similar for* $n \geq n_0 \geq 0$,

(ii) $I+f_x(n,0)$ *and* $I+f_x(n,n_0,x_0)$, $n \geq n_0$, *are all invertible, the latter for all* $|x_0|$ *sufficiently small,*

(iii) $\sum_{n=n_0}^{\infty} \frac{h(n)}{h(n+1)}|F(n)| < \infty$, $h(n_0) \geq 1$.

Then the system (1.3) *is an* h*-system if* (1.2) *is an* h*-system.*

Proof. Let $\Phi(n,n_0,0)$ and $\Phi(n,n_0,x_0)$ be fundamental matrix solutions of the variational systems (1.2) and (1.3), respectively. In view of (2.3), we have

$$\Phi(n,n_0,x_0) = S^{-1}(n)[\Phi(n,n_0,0)S(n_0) + \sum_{l=n_0}^{n-1} \Phi(n,l+1,0)F(l)\Phi(l,n_0,x_0)].$$

From (2.4) and the boundedness of $S(n)$ and $S^{-1}(n)$, we obtain

$$|\Phi(n,n_0,x_0)| \leq c_1c_2h(n)h^{-1}(n_0) + c_1c_2 \sum_{l=n_0}^{n-1} h(n)h^{-1}(l+1)|F(l)||\Phi(l,n_0,x_0)|$$

for some positive constants c_1 and c_2. Thus

$$|\Phi(n,n_0,x_0)|h^{-1}(n) \leq c_1c_2h^{-1}(n_0)+c_1c_2 \sum_{l=n_0}^{n-1} \frac{h(l)}{h(l+1)}|F(l)||h^{-1}(l)||\Phi(l,n_0,x_0)|.$$

Applying the discrete Gronwall inequality (*cf.* Agarwal [1]), we have

$$\begin{aligned} |\Phi(n,n_0,x_0)| &\leq dh(n)h^{-1}(n_0) \prod_{l=n_0}^{n-1} \left(1+\frac{h(l)}{h(l+1)}|F(l)|\right) \\ &\leq dh(n)h^{-1}(n_0) \exp\left(\sum_{l=n_0}^{n-1} \frac{h(l)}{h(l+1)}|F(l)|\right) \\ &\leq ch(n)h^{-1}(n_0), \end{aligned}$$

where $c = d\exp\left(\sum_{l=n_0}^{\infty} \frac{h(l)}{h(l+1)}|F(l)|\right)$ and $d = c_1c_2$. It follows that

$$|\Phi(n,n_0,x_0)| \leq ch(n)h^{-1}(n_0),\ n \geq n_0 \geq 0.$$

This completes the proof. □

Remark 2.2. The condition (iii) in Theorem 2.1 can be replaced by $|F(n)| \in l_1(\mathbb{N}(n_0))$ if $\frac{h(n)}{h(n+1)}$ is bounded. But $\frac{h(n)}{h(n+1)}$ is not bounded in general even though $h(n)$ is bounded. For instance,

$$h(n) = \exp\left(-\sum_{s=n_0}^{n-1} s\right)$$

is bounded on $\mathbb{N}(n_0)$ but we have

$$\lim_{n\to\infty} \frac{h(n)}{h(n+1)} = \lim_{n\to\infty} \exp n = \infty.$$

The basic equivalence property about stability for the linear homogeneous system (2.1) is

$$|X(n, n_0)| \le c, \quad n \in \mathbb{N}(n_0), \tag{2.5}$$

for some constant $c > 0$, where $X(n, n_0)$ is the fundamental matrix solution of (2.1) (*cf.* Agarwal [1, Theorem 5.5.1]). Also, (2.1) is strongly stable if and only if

$$|X(n, n_0)| \le c \quad \text{and} \quad |X^{-1}(n, n_0)| \le c, \quad n \in \mathbb{N}(n_0) \tag{2.6}$$

for some constant $c > 0$. Using (2.5), we obtain the following theorem which appears in Agarwal [1, Theorem 5.5.2] without the proof.

Theorem 2.3. (2.1) *is restrictively stable if and only if it is strongly stable.*

Proof. Note that any solution $y(n, n_0, y_0)$ of the adjoint system

$$\Delta y(n) = -A^T(n) y(n+1) \tag{2.7}$$

is given by

$$y(n, n_0, y_0) = [X^T(n, n_0)]^{-1} y(n_0).$$

In fact, we have

$$\begin{aligned}
\Delta y(n) &= [Y^T(n+1, n_0)^{-1} - Y^T(n, n_0)^{-1}] y(n_0) \\
&= [Y^{-1}(n+1, n_0) - Y^{-1}(n, n_0)]^T y(n_0) \\
&= [-Y^{-1}(n+1, n_0) A(n)]^T y(n_0) \\
&= -A^T(n) Y^T(n+1)^{-1} y(n_0) \\
&= -A^T(n) y(n+1).
\end{aligned}$$

Then

$$\Delta Y(n) = -A^T(n) Y(n+1), \quad n \in \mathbb{N}(n_0),$$

where $Y(n)$ is the fundamental matrix solution of (2.7). Thus the result follows from (2.5). □

Note that the transformation $x(n) = L(n)y(n)$, where $L(n)$ is an invertible $m \times m$ matrix, converts (2.1) into

$$\Delta y(n) = [L^{-1}(n+1)A(n)L(n) + L^{-1}(n+1)L(n) - I\,]y(n).$$

Thus $x(n) = L(n)y(n)$ transforms (2.1) into a system with constant coefficients (into the system $\Delta y(n) = 0$). The following theorem is the result of Agarwal [1, Theorem 5.5.3].

Theorem 2.4. (2.1) *is restrictively stable if and only if it is reducible to zero.*

Proof. Suppose that (2.1) is restrictively stable. Then $X(n, n_0)$ and $X^{-1}(n, n_0)$ are bounded on $\mathbb{N}(n_0)$ by Theorem 2.3. Let $x(n) = X(n, n_0)y(n)$. Then we have

$$\begin{aligned}\Delta x(n) &= A(n)x(n)\\ &= X(n+1)[y(n+1) - y(n)] + X(n+1)y(n) - X(n)y(n)\\ &= X(n+1)\Delta y(n) + \Delta X(n)y(n)\\ &= X(n+1)\Delta y(n) + A(n)X(n)y(n)\\ &= X(n+1)\Delta y(n) + A(n)x(n).\end{aligned}$$

This implies that $\Delta y(n) = 0$ which means (2.1) is reducible to zero. Conversely, assume that (2.1) is reducible to zero. Then there exists an invertible $m \times m$ matrix $L(n)$ such that

$$L^{-1}(n+1)A(n)L(n) + L^{-1}(n+1)L(n) - I = 0. \tag{2.8}$$

Thus

$$\Delta L(n) = A(n)L(n),$$

i. e., $L(n)$ is the fundamental matrix solution of (2.1). From the boundedness of $L(n)$ and $L^{-1}(n)$, and (2.5), we conclude that (2.1) is restrictively stable. □

Now, we come to conclude that strong stability for (2.1) preserved under the notion of summable similarity.

Theorem 2.5. *Suppose that $A(n)$ and $B(n)$ are summably similar with $F(n) = 0$. Then* (2.2) *is strongly stable when* (2.1) *is strongly stable.*

Proof. Suppose that (2.1) is strongly stable. Then, from Theorems 2.3 and 2.4, (2.8) holds. It suffices to show that (2.2) is reducible to zero. Putting $T(n) = S^{-1}(n)L(n)$, we have

$$\begin{aligned}
&T^{-1}(n+1)B(n)T(n) + T^{-1}(n+1)T(n) - I \\
&= L^{-1}(n+1)S(n+1)B(n)S^{-1}(n)L(n) + L^{-1}(n+1)S(n+1)S^{-1}(n)L(n) - I \\
&= L^{-1}(n+1)[A(n) + I - S(n+1)S^{-1}(n)]L(n) \\
&\qquad\qquad\qquad\qquad + L^{-1}(n+1)S(n+1)S^{-1}(n)L(n) - I \\
&= 0
\end{aligned}$$

by the definition of summable similarity. This implies that (2.2) is reducible to zero. □

3. Asymptotic Behavior

Consider the linear difference system

$$\Delta x(n) = A(n)x(n) \tag{3.1}$$

and its perturbation

$$\Delta y(n) = A(n)y(n) + g(n), \tag{3.2}$$

where $g : \mathbb{N}(n_0) \longrightarrow \mathbb{R}^m$.

Theorem 3.1. *If* (3.1) *has linear asymptotic equilibrium and* $|g(n)| \in l(\mathbb{N}(n_0))$, *then* (3.2) *has also linear asymptotic equilibrium.*

Proof. Let $y(n, n_0, y_0)$ be any solution of (3.2). Using the fundamental matrix solution $\Phi(n, n_0)$ of (3.1), any solution $y(n)$ of (3.2) is given by

$$y(n) = \Phi(n, n_0)y_0 + \Phi(n, n_0) \sum_{s=n_0}^{n-1} \Phi^{-1}(s+1, n_0)g(s). \tag{3.3}$$

Set $p(n) = \sum_{s=n_0}^{n-1} \Phi^{-1}(s+1, n_0)g(s)$. Then $p(n)$ is a Cauchy sequence since $|g(n)| \in l(\mathbb{N}(n_0))$ and $\Phi^{-1}(n)$ is bounded. Thus $y(n)$ converges to a vector $\xi \in \mathbb{R}^m$.

For the converse, let ξ be any vector in $\mathbb{R}^m$. There exists a solution $y(n, n_0, y_0)$ of (3.2) with the initial point $y_0 = \Phi_\infty^{-1}\xi - p_\infty$ with the property that

$$\begin{aligned}\lim_{n\to\infty} y(n) &= \lim_{n\to\infty}\left[\Phi(n,n_0)y_0 + \Phi(n,n_0)\sum_{s=n_0}^{n-1}\Phi^{-1}(s+1,n_0)g(s)\right]\\ &= \Phi_\infty[y_0 + p_\infty]\\ &= \Phi_\infty[\Phi_\infty^{-1}\xi - p_\infty + p_\infty]\\ &= \xi,\end{aligned}$$

where $p_\infty = \lim_{n\to\infty} p(n)$ and $\Phi_\infty = \lim_{n\to\infty}\Phi(n)$. The proof is complete. □

Theorem 3.2. *The system* (3.1) *and* (3.2) *are asymptotically equivalent provided that* (3.1) *has linear asymptotic equilibrium and* $|g(n)| \in l(\mathbb{N}(n_0))$.

Proof. Let $x(n)$ be any solution of (3.1). Since (3.1) has linear asymptotic equilibrium, we have $\lim_{n\to\infty} x(n) = x_\infty$. If we put $y_0 = \Phi_\infty^{-1}x_\infty - p_\infty$ as in Theorem 3.1, there exists a solution $y(n, n_0, y_0)$ of (3.2) satisfying

$$\begin{aligned}\lim_{n\to\infty}\big[y(n) - x(n)\big] &= \Phi_\infty\big[y_0 + p_\infty - x_\infty\big]\\ &= \Phi_\infty\big[(\Phi_\infty^{-1}x_\infty - p_\infty) + p_\infty - x_\infty\big]\\ &= 0.\end{aligned}$$

The converse asymptotic relationship also holds if we put $x_0 = y_0 + p_\infty$. □

As an illustration of Theorem 3.2, we give the following example.

Example 3.3. We show that the following equations are asymptotically equivalent:

$$\Delta x(n) = a^n x(n) \tag{3.4}$$

and

$$\Delta y(n) = a^n y(n) + \alpha^n, \tag{3.5}$$

where $0 < a < 1$ and $0 < \alpha < 1$.

To show that (3.4) has linear asymptotic equilibrium we use Trench's result [6, Theorem 1]: for the fundamental matrix solution $\Phi(n)$ of (3.4), $\lim_{n\to\infty}\Phi(n)$ exists and is invertible.

Note that $\Phi(n)$ is given by $\prod_{s=n_0}^{n-1}(1 + a^s)$. Since $1 + a^n \le \exp a^n$ for $n \ge n_0 \ge 0$, $\Phi(n)$ is bounded, and nondecreasing on $\mathbb{N}(n_0)$. This implies $\lim_{n\to\infty}\Phi(n, n_0) = \Phi_\infty$

exists and it is nonzero. Also, we have $\lim_{n\to\infty} \Phi^{-1}(n, n_0) = \Phi_\infty^{-1}$. Thus (3.4) has linear asymptotic equilibrium. Since any solution $y(n, n_0, y_0)$ of (3.5) is given by

$$y(n) = \prod_{s=n_0}^{n-1} (1 + a^s) y_0 + \sum_{s=n_0}^{n-1} \left[\prod_{r=s+1}^{n-1} (1 + a^r) \alpha^s \right], \quad n \geq n_0 \geq 0,$$

and $\alpha^n \in l(\mathbb{N}(n_0))$, we conclude that (3.5) has linear asymptotic equilibrium by Theorem 3.1. In view of Theorem 3.2, equation (3.4) and equation (3.5) are asymptotically equivalent.

The following theorem states that the variational difference system

$$\Delta v(n) = f_x(n, 0) v(n) \tag{3.6}$$

inherits the property of having asymptotic equilibrium from the original nonlinear difference system

$$\Delta x(n) = f(n, x(n)). \tag{3.7}$$

Theorem 3.4. *If* (3.7) *has asymptotic equilibrium, then* (3.6) *has also asymptotic equilibrium.*

Proof. We claim that the fundamental matrix solution $\Phi(n, n_0, 0) = \frac{\partial}{\partial x} x(n, n_0, 0)$ of (3.6) converges as $n \to \infty$.

Let $x_0 \in \mathbb{R}^m$ be a vector of length h in the j-th coordinate direction for each $j = 1, 2, \dots, m$. Then $\lim_{n\to\infty} x(n, n_0, x_0) = x_\infty$ exists for fixed $h \neq 0$ since (3.7) has asymptotic equilibrium. Also, since $x(n, n_0, x_0)$ satisfies the Cauchy property for each $j = 1, 2, \dots, m$, we have

$$|x(n, n_0, x_0) - x(m, n_0, x_0)| < |h|^2, \quad n, m \geq N.$$

For each $j = 1, 2, \dots, m$, we obtain

$$\begin{aligned} &\left| \frac{\partial}{\partial x_{0j}} x(n, n_0, 0) - \frac{\partial}{\partial x_{0j}} x(m, n_0, 0) \right| \\ &= \left| \lim_{h\to 0} \frac{x(n, n_0, x_0) - x(n, n_0, 0)}{h} - \lim_{h\to 0} \frac{x(m, n_0, x_0) - x(m, n_0, 0)}{h} \right| \\ &= \left| \lim_{h\to 0} \frac{x(n, n_0, x_0) - x(m, n_0, x_0)}{h} \right| \\ &< \lim_{h\to 0} \frac{|h^2|}{|h|} \\ &< \varepsilon. \end{aligned}$$

for $n, m \geq N$.

It follows that $\lim_{n\to\infty} \Phi(n, n_0, 0) = \Phi_\infty$ exists.

Now, to show that the limit Φ_∞ is invertible, we consider linearly independent vectors $\hat{x}_{0j} \in \mathbb{R}^m$ in the j-th coordinate direction for each $j = 1, 2, \ldots, m$. Then, for the solutions $x_j(n, n_0, x_{0j})$ of (3.7),

$$\lim_{n\to\infty} x_j(n, n_0, x_{0j}) = h\hat{x}_{0j}, \quad j = 1, 2, \ldots, m, \quad h \neq 0.$$

Hence we have

$$\begin{aligned}
&\lim_{n\to\infty} \Phi(n, n_0, 0) \\
&= \lim_{n\to\infty} \left[\frac{\partial}{\partial x_{01}} x_1(n, n_0, x_{01}), \ldots, \frac{\partial}{\partial x_{0m}} x_m(n, n_0, x_{0m}) \right] \\
&= \lim_{n\to\infty} \left[\lim_{h\to 0} \frac{x_1(n, n_0, x_{01}) - x(n, n_0, 0)}{h}, \ldots, \lim_{h\to 0} \frac{x_m(n, n_0, x_{0m}) - x(n, n_0, 0)}{h} \right] \\
&= \left[\lim_{h\to 0} \frac{\lim_{n\to\infty} x_1(n, n_0, x_{01})}{h}, \ldots, \lim_{h\to 0} \frac{\lim_{n\to\infty} x_m(n, n_0, x_{0m})}{h} \right] \\
&= [\hat{x}_{01}, \ldots, \hat{x}_{0m}] \\
&= \Phi_\infty.
\end{aligned}$$

Therefore Φ_∞ is invertible since $\hat{x}_{01}, \cdots, \hat{x}_{0m}$ are linearly independent. This completes the proof. □

The following example shows that the converse of Theorem 3.4 need not be true.

Example 3.5. Consider the nonlinear difference equation

$$\Delta x(n) = x^2(n), \quad x(n_0) = x_0 = 1, \tag{3.8}$$

and its variational difference equation

$$\Delta v(n) = 0, \quad v(n_0) = v_0 \neq 0. \tag{3.9}$$

Here, $f(n, x(n)) = x^2(n)$ and $f_x(n, x) = 2x$. It is clear that (3.8) has asymptotic equilibrium since the fundamental solution of (3.9) is $\phi(n) = 1 \neq 0$. However, (3.8) does not have any asymptotic equilibrium since there exists a solution $x(n, 0, 1)$ of (3.8) such that $x(n, 0, 1) = x(n) > n$ for each $n \geq 1$.

4. ACKNOWLEDGEMENTS

We express our appreciation to anonymous referees who made some very helpful suggestions and pointed out some inaccuracies in an earlier version of this paper.

References

1. R. P. Agarwal: *Difference equations and inequalities.* Theory, methods, and applications. 2nd ed., Marcel Dekker, Inc., New York, 2000. MR **2001f**:39001
2. S. K. Choi & N. J. Koo: Variationally stable difference systems by n_∞-similarity. *J. Math. Anal. Appl.* **249** (2000), no. 2, 553–568. MR **2002m**:39013
3. S. K. Choi, N. J. Koo & S. M. Song: h-stability for nonlinear perturbed difference systems. *Bull. Korean Math. Soc.* **41** (2004), no. 3, 435–450. MR **2005f**:39029
4. R. Conti: Sulla t-similitudine tra matrici e la stabilità dei sistemi differenziali lineari. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8)* **19** (1955), 247–250. MR **18**,483e
5. R. Medina & M. Pinto: Stability of nonlinear difference equations. In: G. S. Ladde and M. Sambandham (Eds.), *Proceedings of Dynamic Systems and Applications. Vol. 2. (Second International Conference held at Morehouse College, Atlanta, Georgia, May 24–27, 1995)* (pp. 397–404). Publishers, Inc., Atlanta, GA, 1996. MR **98f**:39012
6. W. F. Trench: Linear asymptotic equilibrium and uniform, exponential, and strict stability of linear difference systems. *Comput. Math. Appl.* **36** (1998), no. 10–12, 261–267. MR **99j**:39016

(D. M. Im) Department of Mathematics Education, Cheongju University, 36 Naedeok-dong, Sangdang-gu, Cheongju, Chungbuk 360-764, Korea
Email address: dmim@chongju.ac.kr

(Y. H. Goo) Department of Mathematics, Hanseo University, 360 Daegok-ri, Haemi-myeon, Seosan, Chungnam 356-706, Korea
Email address: yhgoo@hanseo.ac.kr