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THE ORDER OF CONVERGENCE IN THE FINITE ELEMENT METHOD

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ABSTRACT. We investigate the error estimates of the h and p versions of the finite element method for an elliptic problems. We present theoretical results showing the p version gives results which are not worse than those obtained by the h version in the finite element method.

1. INTRODUCTION

There are several types of partial differential equations, but specially we are interested in the following model problem:

$$Lu = -\text{div } (a \nabla u) = f \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a is a bounded polygonal domain. When we approximate the above solution $u(x)$, there are three ways in the finite element method (FEM). The first one is h version of the FEM. This is the classical FEM, where piecewise polynomials of fixed, usually low degree p are used and the mesh size h is reduced for accuracy. The next one is p version. This is the name given to the FEM where the mesh is fixed (h constant) and accuracy is achieved by increasing the polynomial degree. The last one is a combination of the h and p versions. This is considered in Abramowitz & Stegun [1] where it is demonstrated that particular couplings of refined meshes and increasing polynomial degree distributions yield arbitrarily high orders of convergence in the energy norm with respect to the number of degrees of freedom. In this paper, some direct energy norm estimates are obtained. These show that, when

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both the h and p version estimates are expressed in terms of the number of degree of freedom, the order of convergence for the p version can be no worse than that of the h version with a quasi-uniform sequence of mesh refinements.

2. PRELIMINARIES

We will consider $\mathbb{R}^2$ as the two-dimensional Euclidean space and $\Omega \subset \mathbb{R}^2$ as a bounded domain with a piecewise smooth boundary $\partial\Omega$. In particular, we will deal with polygonal domains. $\xi(\overline{\Omega})$ shall be the space of all real C^∞ functions on Ω with that allows continuously extension of all derivatives to $\overline{\Omega}$. All functions in $\xi(\overline{\Omega})$ that have compact support in Ω form a subspace $D(\Omega) \subset \xi(\overline{\Omega})$. $L_2(\Omega) = H^0(\Omega)$ will be the space of all square-integrable functions on Ω.

In addition, for any integer $k \geq 1$, the Sobolev spaces $H^k(\Omega)$ (respectively $H_0^k(\Omega)$) will be the completions of $\xi(\overline{\Omega})$ (respectively $D(\Omega)$) under the norm

$$\|u\|_{k,\Omega}^2 = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{0,\Omega}^2 \tag{3}$$

for each multi-integer (α_1, α_2). Here $|\alpha| = \alpha_1 + \alpha_2$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}}. \tag{4}$$

The standard inner product in $H^k(\Omega)$ will be denoted by $(\cdot,\cdot)_{k,\Omega}$. Finally, we introduce the space $P_\rho(\Omega) \subset \Xi(\overline{\Omega})$ of all algebraic polynomials of degree not higher than p. Let $P_\rho^{[\varsigma]}(\Omega)$ consists of all functions in $H^1(\Omega)$ which are piecewise polynomials of degree at most p. Furthermore, let $P_{p,0}^{[\varsigma]}(\Omega) = P_p^{[\varsigma]}(\Omega) \cap H_0^1(\Omega)$.

Theorem 2.1. *Let $u \in H^k(\Omega)$. Then there exists a sequence $z_p \in P_\rho^{[\varsigma]}(\Omega)$, $p = 1, 2, \ldots$ such that for any $0 \leq l \leq k$ (k, l not necessarily integer),*

$$\|u - z_p\|_{1,\Omega} \leq Cp^{-(k-l)}\|u\|_{k,\Omega}, \tag{5}$$

where C is independent of u and p. (C depends on l and k).

Proof. The theorem has been proved in Babuška, Guo & Suri [3] and Babusuka, Szabo & Katz [?]. □

Theorem 2.2. *Let $u \in H^k(\Omega) \cap H_0^1(\Omega)$. Then there exists a sequence $z_p \in P_{p,0}^{[\varsigma]}(\Omega)$, $p = 1, 2, \ldots$ such that for any $k > 1$ (not necessarily integer) and any $\varepsilon > 0$, we have*

$$\|u - z_p\|_{1,\Omega} \leq Cp^{-(k-l)}\|u\|_{k,\Omega}, \tag{6}$$

where C is independent of p and u (C depends on l and k).

Proof. The theorem has been proved in Babusuka & Dorr [4]. □

3. Rate of Convergence of the h and p Version

Using Theorem 2.1 and Theorem 2.2, we get the following theorems

Theorem 3.1. *Let $u_0 \in H^k(\Omega)$, where $k > 1$, be the solution of the problem (1) and (2) and let u_p be the finite element approximation. Then*

$$\|u_0 - u_p\|_{1,\Omega} \le c(k, \varepsilon) p^{-(k-1)+\varepsilon} \|u_0\|_{k,\Omega}, \quad \forall \varepsilon > 0. \tag{7}$$

If the Neumann boundary conditions are under consideration, then ε can be taken to be zero.

Proof. See Babusuka & Dorr [4] and Babusuka, Szabo & Katz [5]. □

Since a polynomial of degree p has degrees of freedom $N = (p+1)(p+2)/2$, equation (7) can be rewritten in the following form:

$$\|u_0 - u_p\|_{1,\Omega} \le C(k, \varepsilon) N^{-(k-l)/2+\varepsilon} \|u_0\|_{k,\Omega}. \tag{8}$$

On the other hand, for the h version case, we have

$$\|u_0 - u_p\|_{1,\Omega} \le C h^{\mu} \|u_0\|_{k,\Omega}, \tag{9}$$

where $\mu = \min(k-1, p)$ and is the degree of the complete polynomial used in the elements (see Dorr [6]). Since the number of degrees of freedom N satisfies the relation $N \approx h^{-2}$ (9) becomes

$$\|u_0 - u_h\|_{1,\Omega} \le C N^{-\mu/2} \|u_0\|_{k,\Omega}. \tag{10}$$

The relations (8) and (10) give the results that the p version is not worse than the h version on a quasi-uniform mesh (as $\varepsilon \to 0$) if we compare the number of degrees of freedom that are required to obtain a certain accuracy. Assume

$$\|u_0 - u_h\|_{1,\Omega} \le K p^{-r} \text{ for some } r > 0. \tag{11}$$

Theorem 3.2. *Let $u_0 \in H^1(\Omega)$ and suppose that (11) holds. Then*

(i) *$u_0 \in H^{1+r-\varepsilon}(\Omega^*)$ where (Ω^*) is any domain such that $\Omega^* \subset K_i$ for some $i = 1, 2, \ldots, m$ where K_i are the triangles of the triangulation T and*

$$\|u_0\|_{1+r-\varepsilon,\Omega^*} \le C_1(\|u_0\|_{1,\Omega} + K). \tag{12}$$

(ii) $u_0 \in H^{1+r/2-\varepsilon}(K_i)$ *for each* $i = 1, 2, \ldots, m$ *and*

$$\|u_0\|_{1+r/2-\varepsilon,K_i} \leq C_2(\|u_0\|_{1,\Omega} + K). \tag{13}$$

Proof. The theorem has been proved in Babusuka, Szabo & Katz [5] and Dorr [6]. □

4. Practical Experiments

Consider the following problem:

$$u''(x) = -q(x), \quad x \in \Omega = (-1, 1), \tag{14}$$

where the loading function $q(x)$ and the Dirichlet boundary conditons will be specified later. The energy inner product is given by

$$B(u, v) = (u, v)_E = u'(x)v'(x)dx. \tag{15}$$

Hence we must find a solution $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$(u, v)_E = \int_{-1}^{1} u'(x)v'(x)dx = \int_{-1}^{1} q(x)v(x)dx. \tag{16}$$

We choose as basis functions

$$\Phi_i(x) = \int_{-1}^{x} \varphi_i(t)dt \quad \text{for} \quad i \geq 1, \tag{17}$$

where $\varphi_i(t)$ is the Legendre polynomial of degree i. Observe that $\Phi_i(x)(i = 1, 2, \ldots)$ forms an orthogonal family with respect to the energy inner product, because

$$(\Phi_i, \Phi_j)_E = \int_{-1}^{1} \Phi_i'\Phi_j'dx = \int_{-1}^{1} \varphi_i\varphi_j dx = \Big(\frac{2}{2i+1}\Big)\delta_{ij}. \tag{18}$$

First we consider the convergence when Ω is not divided; i.e., we use only one interval.

Clearly, the finite element solution u_p satisfies, for $i = 1, 2, \ldots, p$

$$(u_p, \Phi_i)_E = \int_{-1}^{1} q(x)\Phi_i dx. \tag{19}$$

And we can let

$$u_p(x) = \frac{1-x}{2}u(-1) + \frac{1+x}{2}u(1) + \sum_{i=1}^{p} a_i\Phi_i(x),$$

$$\int_{-1}^{1} q(x)\Phi_i(x)dx = (u_p(x), \Phi_i(x))_E = \Big(\sum_{j=1}^{p} a_j\Phi_j(x), \Phi_i(x)\Big)_E$$
$$= (a_i\Phi_i(x), \Phi_i(x))_E = a_i(\Phi_i(x), \Phi_i(x))_E = a_i\frac{2}{2i+1}.$$

Hence we get, for $i = 1, 2, \ldots, p$

$$a_i = \frac{2i+1}{2}\int_{-1}^{1} q(x)\Phi_i(x)dx. \tag{20}$$

If we denote the error by $e_p(x) = u(x) - u_p(x)$, then

$$\|e_p\|_E^2 = \int_{-1}^{1} q(x)(u-u_p)dx = \|u\|_E^2 - \|u_p\|_E^2 = \Big\|\sum_{i=p+1} a_i\Phi_i(x)\Big\|_E^2.$$

Since

$$\Big\|\sum_{i=p+1} a_i\Phi_i(x)\Big\|_E^2 = \sum_{i=p+1} a_i^2\frac{2}{2i+1}, \quad \|e_p\|_E^2 = \sum_{i=p+1} a_i^2\frac{2}{2i+1}.$$

Consider following problem

$$u(x) = |x|^{3/2}, \quad q(x) = -\frac{d^2}{dx^2}|x|^{3/2}, \tag{21}$$

where boundary conditions are $u(-1) = u(1) = 1$.

Equation (20) becomes

$$a_i = \frac{2i+1}{2}\int_{-1}^{1} q(x)\Phi_i(x)dx = \frac{2i+1}{2}\int_{-1}^{1} -\frac{d^2}{dx^2}|x|^{3/2}\Phi_i(x)dx.$$

By integration by parts,

$$\int_{-1}^{1}\frac{d^2}{dx^2}|x|^{3/2}\Phi_i(x)dx = \frac{d}{dx}|x|^{3/2}\Phi_i(x)\Big|_{-1}^{1} - \int_{-1}^{1}\left(\frac{d}{dx}|x|^{3/2}\right)\varphi_i(x)dx$$
$$= \int_{-1}^{1}|x|^{3/2}\frac{d}{dx}\varphi_{2m}(x)dx$$
$$= \varphi_{2m}(x)|x|^{3/2}\Big|_{-1}^{1} - \int_{-1}^{1}\frac{3}{2}|x|^{\frac{1}{2}}\varphi_{2m}(x)dx$$
$$= -\frac{3}{2}\int_{-1}^{1}|x|^{\frac{1}{2}}\varphi_{2m}(x)dx.$$

Hence

$$a_{2m} = \frac{3(4m+1)}{2}\int_{0}^{1}|x|^{\frac{1}{2}}\varphi_{2m}(x)dx.$$

From Abramowitz & Stegun [1, formula (22.13.8), p. 786],

$$\int_0^1 x^\lambda\varphi_{2m}(x)dx = \frac{(-1)^m\Gamma(m-\frac{\lambda}{2})\Gamma(\frac{1}{2}+\frac{\lambda}{2})}{2\Gamma(-\frac{\lambda}{2})\Gamma(m+\frac{3}{2}+\frac{\lambda}{2})} \quad (\lambda > -1).$$

Then we obtain

$$\int_0^1 |x|^{\frac{1}{2}}\varphi_{2m}(x)dx = \frac{(-1)^m\Gamma(m-\frac{1}{4})\Gamma(\frac{3}{4})}{2\Gamma(-\frac{1}{4})\Gamma(m+\frac{7}{4})}.$$

Therefore

$$a_{2m} = \frac{3(4m+1)}{2}\frac{(-1)^m\Gamma(\frac{1}{4})}{2\Gamma(-\frac{1}{4})(m+\frac{3}{4})(m-\frac{1}{4})}.$$

It follows that for i odd, $a_i = O(\frac{1}{i})$ as $i \to \infty$. Hence

$$\|e_p\|_E^2 = \sum_{i=p+1} a_i^2 \frac{2}{2i+1} = O\Big(\sum_{i=p+1}\frac{1}{i^3}\Big) = O\Big(\frac{1}{p^2}\Big) = O\Big(\frac{1}{N^2}\Big). \tag{22}$$

Thus, we obtain the same rate of convergence for the square of the error $\|e_h\|_E^2$ as obtained for the h version. This illustrates the importance of the statement that in order to get the full power of the p version, singularities must be located at vertices of the finite element mesh.

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