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EXISTENCE AND ASYMPTOTICS FOR THE TOPOLOGICAL CHERN-SIMONS VORTICES OF THE $CP(1)$ MODEL

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ABSTRACT. In this paper we study the existence and local asymptotic limit of the topological Chern-Simons vortices of the $CP(1)$ model in $\mathbb{R}^2$. After reducing to semilinear elliptic partial differential equations, we show the existence of topological solutions using iteration and variational arguments & prove that there is a sequence of topological solutions which converges locally uniformly to a constant as the Chern-Simons coupling constant goes to zero and the convergence is exponentially fast.

1. INTRODUCTION

The classical $CP(1)$ model which is equivalent to the $O(3)$ sigma model is a basic model in field theory (see Rajaraman [10] and references therein). The model is useful as a toy model for the instantons in non-abelian Yang-Mills theories, but it is scale invariant and yield instantons of arbitrary size. This makes the model unsuitable as a model for real particles and there have been many attempts to break the scale invariance. Among them, one of the most elegant way is to introduce a gauge field which incorporates the kinetic term.

While the generalization of $O(3)$ model to $O(N)$ model does not yield instantons when $N > 3$, the $CP(N)$ model yield instanton solutions for arbitrarily large N. For this reason, we consider the $2+1$ dimensional self-dual Chern-Simons $CP(1)$ model (*cf.* Kimm, K. Lee & T. Lee [8]) where the gauge field dynamics is solely governed by the Chern-Simons term in this paper.

The addition of Chern-Simons gauge terms to classical $CP(1)$ model and the particular choice of potential terms give the finite energy solitons and a Bogomol'nyi

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limit or self-dual equations which are easy to analyze mathematically compared to full second order Euler-Lagrange equation.

On the other hand, like many other low dimensional systems, this model provides an insight for the study of phenomena expected to occur in $3+1$ dimensional gauge theories & can be interpreted as an effective theory for the description of strongly correlated electrons such as superconductors and quantum Hall effect (*cf.* Wilczek [14]). We also note that the Bogomol'nyi limit in superconductivity plays an important role as it permits to distinguish between the type of superconductors (*cf.* Bogomol'nyi [2]). For an overview of self-dual Chern-Simons theories, see Dunne [4] and references therein.

The $2+1$ dimensional $CP(1)$ model consists of two complex scalar fields z_1, z_2 in $\mathbb{R}^2$. Denoting $\mathbf{z} = (z_1, z_2)$, the model requires that $|\mathbf{z}|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ and $\mathbf{z}$ is equivalent to the overall phase rotations. Thus if we can find $\phi = z_2/z_1$, then we have that $\mathbf{z} \sim (1,\phi)/\sqrt{1+|\phi|^2}$.

The Lagrangian for the self-dual Chern-Simons $CP(1)$ model is

$$\mathcal{L} = \frac{\kappa}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + |\nabla_\mu \mathbf{z}|^2 - V(\mathbf{z})$$

where $\varepsilon^{\mu\nu\rho}$ is the totally skew-symmetric tensor with $\varepsilon^{012} = 1$, $R = \mathrm{diag}(1/2, -1/2)$, $A_\mu R$ is the matrix valued gauge field, $V(\mathbf{z})$ is a potential term which will be fixed later & the "covariant derivatives" ∇_μ and D_μ are defined as follows:

$$\nabla_\mu \mathbf{z} = D_\mu \mathbf{z} - (\bar{\mathbf{z}} D_\mu \mathbf{z})\mathbf{z}, \quad D_\mu \mathbf{z} = \partial_\mu \mathbf{z} - iA_\mu R\,\mathbf{z}.$$

The Gauss law constraint obtained from the variation of A_0 is given by

$$\kappa F_{12} = i\big\{\nabla_0\bar{\mathbf{z}}\big[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}\big] - \text{h.c.}\big\}$$

where $F_{12} = \partial_1 A_2 - \partial_2 A_1$. The theory possesses the following conserved topological current K^μ and global $U(1)$ current J^μ for the generator R,

$$K^\mu = -i\varepsilon^{\mu\nu\rho}\partial_\nu(\bar{\mathbf{z}}D_\rho\mathbf{z}), \quad J^\mu = i\{\nabla^\mu\bar{\mathbf{z}}[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}] - \text{h.c.}\}.$$

For the static configuration, we choose the potential as given by

$$V(\mathbf{z}) = \frac{1}{\kappa^2}\Big|\big[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}\big](\bar{\mathbf{z}}R\mathbf{z} - s)\Big|^2$$

for a free real parameter s & we rewrite the energy density as

$$\begin{aligned}\mathcal{E} &= |\nabla_0\mathbf{z}|^2 + |\nabla_1\mathbf{z}|^2 + |\nabla_2\mathbf{z}|^2 + V(\mathbf{z}) \\ &= |(\nabla_1 \pm i\nabla_2)\mathbf{z}|^2 + \Big|\nabla_0\mathbf{z} \mp \frac{i}{\kappa}\{\big[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}\big](\bar{\mathbf{z}}R\mathbf{z} - s)\}\Big|^2 \pm (K_0 - \frac{s}{\kappa}J_0).\end{aligned}$$

Thus the field configurations saturating the energy bound

$$E = \pm \int (K_0 - (s/\kappa)J_0) \geq 0$$

satisfy the Gauss law constraint and the following self-dual equations,

$$\begin{cases} (\nabla_1 \pm i\nabla_2)\mathbf{z} = 0 \\ \nabla_0 \mathbf{z} \mp \dfrac{i}{\kappa}\Big\{ \big[R\mathbf{z} - (\bar{\mathbf{z}}R\mathbf{z})\mathbf{z}\big](\bar{\mathbf{z}}R\mathbf{z} - s)\Big\} = 0. \end{cases} \tag{1}$$

2. Theorem

In this paper, we will prove the following theorem:

Theorem 1. *For any $\kappa > 0$, $-1 < 2s < 1$ and disjoint sets of points $P = \{p_1, \ldots, p_n\}$, $Q = \{q_1, \ldots, q_m\}$, there exists a topological finite action solution $(\mathbf{z}, A)$ to the self-dual equation (1) with the following properties.*

(i) *$(\mathbf{z}, A)$ is globally smooth.*

(ii) *The sets of zeroes of z_1 and z_2 are P and Q respectively, and*

$$z_1(x) \sim c_{1,l}(x - p_\ell)^{n_\ell} \quad \text{near } p_\ell \text{ and } c_{1,l} \neq 0$$
$$z_2(x) \sim c_{2,k}(x - q_k)^{m_k} \quad \text{near } q_k \text{ and } c_{2,k} \neq 0$$

where n_ℓ and m_k are the multiplicities of p_ℓ and q_k in the sets P and Q respectively.

(iii) *$|z_1|^2 \to (1 + 2s)/2$ and $|z_2|^2 \to (1 - 2s)/2$ as $|x| \to \infty$.*

Moreover, for any decreasing sequence $\{\kappa_n\}$ which converges to 0, and

$$K \subset\subset \mathbb{R}^2 \setminus \{\cup_k \{q_k\} \cup_\ell \{p_\ell\}\},$$

there is a sequence of topological solutions $\{(\mathbf{z}^{\kappa_n}, A^{\kappa_n})\}$ such that

$$\left| |z_1^\kappa|^2 - \frac{1+2s}{2} \right| \leq Ce^{-c_0/\kappa}, \quad \left| |z_2^\kappa|^2 - \frac{1-2s}{2} \right| \leq Ce^{-c_0/\kappa} \quad \text{in } K.$$

We remark that multiple existence of multivortex solutions under the doubly periodic boundary condition and its asymptotic limits are studied in Chae & Nam [3], Han & Nam [5], and Nam [9]. The studies on the Chern-Simons-Higgs theory (*cf.* Spruck & Yang [11], Tarantello [12], Wang [13]) motivate our work here.

3. Proof of Theorem 1

From the self-dual equations (1), without loss of generality, we can choose the upper signs. Introducing complex differentiation

$$\bar{\partial} = \frac{(\partial_1 + i\partial_2)}{2} \quad \text{and} \quad \bar{\alpha} = \frac{(A_1 + iA_2)}{2},$$

we obtain that

$$z_2\bar{\partial}z_1 - z_1\bar{\partial}z_2 = 2iz_1z_2\bar{\alpha}.$$

Thus, away from the zeroes of z_1 and z_2, $\phi = z_2/z_1$ satisfies

$$\bar{\partial}\ln\phi = -2i\bar{\alpha}. \tag{2}$$

Following the argument of Jaffe-Taubes [7], the zeroes of z_1 and z_2 are discrete and we can set

$$z_1 = \psi_1\prod_\ell (z - p_\ell)^{n_\ell}, \quad z_2 = \psi_2\prod_j (z - q_j)^{m_j}, \quad \sum_\ell n_\ell = n, \quad \sum_j m_j = m$$

where p_ℓ's and q_j's are distinct and ψ_1, ψ_2 are nonvanishing smooth functions. Thus if the field configuration $(\mathbf{z}, A)$ exists, it is globally smooth.

Noting that the action $\int\mathcal{L}$ is invariant under the following gauge transformations

$$\mathbf{z} \to \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\mathbf{z}, \quad A_0 \to A_0, \quad A_j \to A_j - \partial_j\theta, \tag{3}$$

where θ is a real-valued function and $j = 1, 2$, we introduce a real valued function $\tilde{u}$ defined by

$$\phi = \frac{z_2}{z_1} = \exp\Big(\frac{\tilde{u}}{2} + i\big(\sum_j m_j \arg(z - q_j) - \sum_\ell n_\ell \arg(z - p_\ell)\big)\Big), \tag{4}$$

so that the self-dual equations (1) reduce to

$$\Delta\tilde{u} = \frac{4(1+2s)}{\kappa^2}\frac{e^{\tilde{u}}}{(1+e^{\tilde{u}})^3}\Big(e^{\tilde{u}} - \frac{1-2s}{1+2s}\Big) + 4\pi\sum_j m_j\,\delta_{q_j} - 4\pi\sum_\ell n_\ell\,\delta_{p_\ell}$$

in $\mathbb{R}^2$. Here we impose the topological boundary condition

$$\tilde{u}(x) \to \ln\frac{1-2s}{1+2s} \quad \text{as } |x| \to \infty.$$

If $\tilde{u}$ is once found, then we can recover $(\mathbf{z}, A)$ by equations (2), (4) and the equivalence relation

$$\mathbf{z} \sim \frac{(1, \phi)}{\sqrt{1 + |\phi|^2}}.$$

For simplicity we change the variable $\tilde{u} - \ln(1-2s)/(1+2s)$ by u to obtain the following boundary value problem:

$$(5)\qquad \begin{cases} \Delta u = \dfrac{4(1+2s)^2}{\kappa^2(1-2s)}\dfrac{e^u(e^u-1)}{(e^u+\frac{1+2s}{1-2s})^3} + 4\pi\sum_j m_j\delta_{q_j} - 4\pi\sum_\ell n_\ell\delta_{p_\ell} \quad \text{in } \mathbb{R}^2, \\ u \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

To show the existence of solution, we consider the following associated problem:

$$(6)\qquad \begin{cases} \Delta u_- = \dfrac{4(1+2s)^2}{\kappa^2(1-2s)}\dfrac{e^{u_-}(e^{u_-}-1)}{(e^{u_-}+\frac{1+2s}{1-2s})^3} + 4\pi\sum_j m_j\delta_{q_j} \quad \text{in } \mathbb{R}^2, \\ u_- \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

If u_- exists, then $u_- < 0$ by the maximum principle and is a subsolution to the original problem(5). The existence of u_- follows from iteration and variational arguments and we will briefly sketch the proof.

Introducing background functions

$$u_0 = \sum_j m_j \ln \frac{|x-q_j|^2}{1+|x-q_j|^2}, \quad g = \sum_j \frac{4m_j}{(1+|x-q_j|^2)^2},$$

$v = u_- - u_0$ satisfies

$$(7)\qquad \begin{cases} \Delta v = \dfrac{4(1+2s)^2}{\kappa^2(1-2s)}\dfrac{e^{u_0+v}(e^{u_0+v}-1)}{(e^{u_0+v}+\frac{1+2s}{1-2s})^3} + g \quad \text{in } \mathbb{R}^2, \\ v \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

For $K \geq 1/\kappa^2$, we define an iterative scheme by

$$(8)\qquad \begin{cases} (\Delta - K)v_{n+1} = \dfrac{4(1+2s)^2}{\kappa^2(1-2s)}\dfrac{e^{u_0+v_n}(e^{u_0+v_n}-1)}{(e^{u_0+v_n}+\frac{1+2s}{1-2s})^3} - Kv_n + gv_n, \\ v_{n+1} \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

with the starting point $v_0 = -u_0$. Then the sequence of functions $\{v_n\}$ is well defined and $v_n \geq v_{n+1}$ for all $n \geq 0$ by the maximum principle. To guarantee the convergence to a solution, we use variational arguments and consider the associated functional given by

$$F(v) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla v|^2 + \frac{(1+2s)^2}{\kappa^2}\Big(\frac{e^{u_0+v}-1}{e^{u_0+v}+\frac{1+2s}{1-2s}}\Big)^2 + gv, \quad v \in H^1(\mathbb{R}^2).$$

Since $u_0 + v_n \leq 0$ and the function

$$\phi(v) \equiv \frac{(1+2s)^2}{\kappa^2}\Big(\frac{e^{u_0+v}-1}{e^{u_0+v}+\frac{1+2s}{1-2s}}\Big)^2 - \frac{K}{2}v^2$$

is concave, we obtain that

$$F(v_{n+1}) \leq F(v_n) \quad \text{for all } n \geq 1. \tag{9}$$

Indeed, multiplying (8) by $v_{n+1} - v_n$ and integrating by parts, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \phi(v_{n+1}) - \phi(v_n) + g(v_{n+1} - v_n) \\
&\quad \leq \int_{\mathbb{R}^2} \phi'(v_n)(v_{n+1} - v_n) + g(v_{n+1} - v_n) \\
&\quad = \int_{\mathbb{R}^2} (v_{n+1} - v_n)(\Delta - K)v_{n+1} \\
&\quad = \int_{\mathbb{R}^2} -\nabla(v_{n+1} - v_n)\nabla v_{n+1} - K(v_{n+1} - v_n)v_{n+1} \\
&\quad = \int_{\mathbb{R}^2} -\frac{1}{2}|\nabla v_{n+1}|^2 + \frac{1}{2}|\nabla v_n|^2 - \frac{1}{2}|\nabla(v_{n+1} - v_n)|^2 - \frac{K}{2}v_{n+1}^2 + \frac{K}{2}v_n^2 \\
&\qquad\qquad - \frac{K}{2}(v_{n+1} - v_n)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
F(v_{n+1}) &= \int_{\mathbb{R}^2} \frac{1}{2}|\nabla v_{n+1}|^2 + \phi(v_{n+1}) + \frac{K}{2}v_{n+1}^2 + gv_{n+1} \\
&\leq \int_{\mathbb{R}^2} \frac{1}{2}|\nabla v_n|^2 + \phi(v_n) + \frac{K}{2}v_n^2 + gv_n - \frac{1}{2}|\nabla(v_{n+1} - v_n)|^2 - \frac{K}{2}(v_{n+1} - v_n)^2 \\
&= F(v_n) - \frac{1}{2}\|\nabla(v_{n+1} - v_n)\|_{L^2}^2 - \frac{K}{2}\|v_{n+1} - v_n\|_{L^2}^2,
\end{aligned}$$

and (9) is proved.

On the other hand, variational argument in Wang [13] implies that $F(v_n)$ is coercive in $H^1(\mathbb{R}^2)$ and

$$\|v_n\|_{H^1} \leq C(F(v_n) + 1) \quad \text{for all } n \geq 1. \tag{10}$$

Indeed, from the following estimates

$$\begin{aligned}
\frac{2(1+2s)^2}{\kappa^2}\left(\frac{e^{u_0+v} - 1}{e^{u_0+v} + \frac{1+2s}{1-2s}}\right)^2 &\geq \frac{(1+2s)^2(1-2s)}{2\kappa^2}\frac{2|u_0 + v|^2}{(1 + |u_0 + v|)^2} \\
&\geq \frac{(1+2s)^2(1-2s)}{2\kappa^2}\frac{v^2 - 2u_0^2}{(1 + |u_0| + |v|)^2},
\end{aligned}$$

and

$$\left|\int_{\mathbb{R}^2} gv\right| \leq c\|v\|_{L^4} \leq C\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2} \leq \eta\|v\|_{L^2} + \eta\|\nabla v\|_{L^2}^2 + \frac{C}{\eta^2},$$

we have that

$$
\begin{aligned}
F(v) \geq & \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{(1+2s)^2(1-2s)}{2\kappa^2}\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2} \\
& - \frac{(1+2s)^2(1-2s)}{\kappa^2}\int_{\mathbb{R}^2}\frac{u_0^2}{(1+|u_0|)^2} - \eta\|v\|_{L^2} - \eta\|\nabla v\|_{L^2}^2 - \frac{C}{\eta^2} \\
\geq & \frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{(1+2s)^2(1-2s)}{2\kappa^2}\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2} - \eta\|v\|_{L^2} - \frac{C}{\eta^2} - C.
\end{aligned}
$$

By Hölder inequality and Young's inequality, we have that

$$
\begin{aligned}
\|v\|_{L^2}^2 &= \int_{\mathbb{R}^2} v^2\frac{1+|v|+|u_0|}{1+|v|+|u_0|} \\
&\leq \Big(\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2}\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2}(|v|+|v|^2+|vu_0|)^2\Big)^{\frac{1}{2}} \\
&\leq C\Big(\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2}\Big)^{\frac{1}{2}}(\|v\|_{L^2}+\|v\|_{L^2}\|\nabla v\|_{L^2}+1) \\
&\leq \frac{1}{2}\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^4 + C\Big(\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2}\Big)^2 + C,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|v\|_{L^2}^2 &\leq C\Big(\|\nabla v\|_{L^2}^2 + \int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2} + 1\Big) \\
&\leq C\Big(F(v) - \frac{(1+2s)^2(1-2s)}{2\kappa^2}\int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2} + \eta\|v\|_{L^2} + \frac{C}{\eta^2} \\
&\qquad + C + \int_{\mathbb{R}^2}\frac{v^2}{(1+|v|+|u_0|)^2} + 1\Big) \\
&\leq C(F(v) + \eta\|v\|_{L^2} + 1).
\end{aligned}
$$

Since C is independent of η, we can choose η sufficiently small to obtain that

$$\|v\|_{L^2} \leq C(F(v)+1)$$

and hence (10) holds.

Moreover, the standard elliptic regularity estimates imply that $\{v_n\}$ is bounded in $H^k(\mathbb{R}^2)$ for $k \geq 1$ and converges to a solution of (7) in C^k_{loc} (a subsequence if necessary).

Similarly, we can show the existence of $u_+ > 0$ as a solution of

$$
\begin{cases}
\Delta u_+ = \dfrac{4(1+2s)^2}{\kappa^2(1-2s)}\dfrac{e^{u_+}(e^{u_+}-1)}{(e^{u_+}+\frac{1+2s}{1-2s})^3} - 4\pi\sum_\ell n_\ell\delta_{p_\ell} \quad \text{in } \mathbb{R}^2, \\
u_+ \to 0 \quad \text{as } |x| \to \infty,
\end{cases}
$$

which is a supersolution of the original problem (5). Again by the iterative scheme, we obtain the existence of a topological solution u^κ for the original problem for any κ.

Now, we give a proof for asymptotic limits. For any given decreasing sequence $\{\kappa_n\}$ $(n = 0, 1, \dots)$, we observe that the solutions $u_-^{\kappa_0} < 0$ of (6) is a subsolution of (6) for κ_n. By similarity, we can construct sequences of solutions $\{u_-^{\kappa_n}\}$, $\{u_+^{\kappa_n}\}$, $\{u^{\kappa_n}\}$ which satisfy

$$u_-^{\kappa_0} < u_-^{\kappa_n} < u^{\kappa_n} < u_+^{\kappa_n} < u_+^{\kappa_0}, \quad u_-^{\kappa_n} < 0 < u_+^{\kappa_n},$$

and it suffices to show the exponential decay of $u_-^{\kappa_n}$ and $u_+^{\kappa_n}$.

For any compact subsets $K \subset\subset K' \subset\subset \mathbb{R}^2 \setminus \cup_k\{q_k\} \cup_\ell \{p_\ell\}$, choose any $x_0 \in K$ and $d > 0$ such that $B_{x_0}(2d) \subset K'$. Since $u_-^{\kappa_0} < u_-^{\kappa_n} < 0$,

$$\Delta u_-^{\kappa_n} = \frac{4(1+2s)^2}{\kappa^2(1-2s)} \frac{e^{u_-^{\kappa_n}}(e^{u_-^{\kappa_n}} - 1)}{(e^{u_-^{\kappa_n}} + \frac{1+2s}{1-2s})^3} \le a u_-^{\kappa_n} \quad \text{in } K'$$

where

$$a = \frac{4(1+2s)^2}{\kappa_n^2(1-2s)} \frac{e^{a_0}(e^{a_0}-1)}{a_0(e^{a_0} + \frac{1+2s}{1-2s})^3} > 0, \quad -\infty < a_0 = \min_{K'} u_-^{\kappa_0} < 0.$$

For $w(x) = C\exp((|x-x_0|^2 - d^2)/\kappa)$, if d is sufficiently small and n is sufficiently large, we have that

$$(\Delta - a)(u_-^{\kappa_n} + w) \le \Big(\frac{2|x-x_0|^2}{\kappa_n^2} + \frac{4}{\kappa_n} - a\Big) w \le 0 \quad \text{for } |x-x_0| < d.$$

Since $u_-^{\kappa_n}$ is bounded from below by $u_-^{\kappa_0}$, we can choose C sufficiently large such that $u_-^{\kappa_n} + w > 0$ on $|x - x_0| = d$. Hence $u_-^{\kappa_n} > -w$ for $|x - x_0| < d$. Since K is compact and $x_0 \in K$ is arbitrary, we can conclude that the sequence $\{u_-^{\kappa_n}\}$ locally uniformly converges to 0 as κ goes to 0 and the convergence is exponentially fast. Similarly, $\{u_+^{\kappa_n}\}$ locally uniformly converges to 0 exponentially fast and this completes the proof of Theorem 1.

4. CONCLUSIONS

In this paper we showed the existence and asymptotic limits of the topological Chern-Simons vortices of the $CP(1)$ model in $\mathbb{R}^2$. While the range of the parameter κ for which the existence is guaranteed is limited in doubly periodic domain (*cf.* Chae & Nam [3]), the topological boundary condition .plays a role in the whole plane such that the variational argument can work for any κ. We expect that the boundary condition for the case of whole plane is closely related to the asymptotic limits in the doubly periodic domain. We will pursuit this direction in the next paper.

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