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MULTIPLE EXISTENCE AND UNIQUENESS OF AN ELLIPTIC EQUATION WITH EXPONENTIAL NONLINEARITY

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ABSTRACT. In this paper we consider a Dirichlet problem in the unit disk. We show that the equation has a unique or multiple solutions according to the range of the parameter. Moreover, we prove that the equation admits a nonradial bifurcation at each branch of radial solutions.

1. INTRODUCTION

We are interested in the following equation

$$(1)\qquad \begin{aligned} -\Delta u &= \lambda\frac{|x|^{2\alpha}e^u}{\int_{B_1}|x|^{2\alpha}e^u}, &&\text{in } B_1,\\ u &= 0, &&\text{on } \partial B_1 \end{aligned}$$

where λ and α are positive constants. B_1 will denote the unit disk $\{x \in \mathbb{R}^2 \mid |x| < 1\}$.

When $\alpha = 0$ and the domain B_1 is replaced with a smooth bounded domain in $\mathbb{R}^2$ we have

$$(2)\qquad \begin{aligned} -\Delta u &= \lambda\frac{e^u}{\int_\Omega e^u}, &&\text{in } \Omega,\\ u &= 0, &&\text{on } \partial\Omega. \end{aligned}$$

Equation (2) arises from Onsager's vortex model for turbulent Euler flows. In this case, u can be interpreted as the stream function in the infinite vortex limit. See Marchioro & Pulvirenti [14]. Equation (2) is the Euler-Lagrange equation of the following functional

$$(3)\qquad I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \ln \int_\Omega e^u dx, \quad u \in H_0^1(\Omega).$$

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Caglioti, Lions, Marchioro & Pulvirenti [8] and Kiessling [12] proved that I_λ has a global minimizer for $\lambda < 8\pi$, which is based on the Moser-Trudinger inequality.

If $\lambda > 8\pi$, then the functional I_λ is not bounded below for $\lambda \geq 8\pi$, and it does not have a minimizer. In this case, the solution structure depends on the geometry of Ω. For example if Ω is a unit disk, then the Pohozaev identity implies that (2) admits a solution if and only if $\lambda < 8\pi$. It also follows from the maximum principle Gidas, Ni & Nirenberg [11] that if $\Omega = B_1$ and $0 < \lambda < 8\pi$ then every solution of (2) is radially symmetric, and the solution is unique. Moreover, the solution blows up as $\lambda \to 8\pi^-$.

When Ω is an annulus, Ding, Jost, Li, & Wang [10] proved that equation has a solution for $8\pi < \lambda < 16\pi$ by using the minimax method. Later, Chen & Lin [7] established existence results for $\lambda \in (0,\infty) \setminus \{8\pi m \mid m \in \mathbb{N}\}$ by computing the degree of the associated operator

$$T(u) = u + \lambda\Delta^{-1}\Big(\frac{e^u}{\int_\Omega e^u dx}\Big).$$

If $\alpha > 0$ we see that the weight function $|x|^{2\alpha}$ appears in equation (1). This kind of equation is related to the self-dual equations arising in the relativistic Chern-Simons-Higgs model (see Chae & Imanuvilov [4] and references therein), and the electroweak theory. For the electroweak theory we refer the readers to Bartolucci & Tarantello [3] and references therein. Equation (1) is also closely related to the conjecture of Wolansky. See Wolansky [17].

It follows from the Pohozaev identity that equation (1) admits a solution only if $\lambda < 8\pi(1+\alpha)$. Moreover, equation (1) has a unique radial solution

$$u_\lambda(r) = \ln\frac{(c_\lambda+1)^2}{(c_\lambda + r^{2\alpha+2})^2}, \quad r = |x|, \tag{4}$$

with $c_\lambda = 8\pi(1+\alpha)/\lambda - 1$.

In order to establish multiple existence result, we adopt the variational formulation for (1). It is easily checked that every solution of equation (1) is a critical point of the following functional

$$J_\lambda(u) = \frac{1}{2}\|\nabla u\|_2^2 - \lambda\ln\int_{B_1} r^{2\alpha}e^u\,dx \tag{5}$$

for $u \in H_0^1(B_1)$ and $\lambda < 8\pi(1+\alpha)$. We note that J_λ is not bounded from below for $8\pi < \lambda < 8\pi(1+\alpha)$. Moreover, it follows from the Green's representation formula and the argument of Struwe & Tarantello [15] that if $\lambda > 0$ is sufficiently small then u_λ is a unique critical point of J_λ, and hence a strict local minimizer of J_λ. We will find a λ_1 which is an infimum of λ_* such that u_λ is a strict local minimizer of J_λ for $0 < \lambda < \lambda_*$. If $\lambda_1 > 8\pi$ then we will be able to find a saddle point of J_λ, which is a second solution of (1).

Concerning the multiple existence, we have

Theorem 1. *If $8\pi < \lambda < 4\pi(\alpha+2)$ then equation (1) admits a nonradial solution.*

In order to prove Theorem 1, we will study an associated eigenvalue problem for the operator $J''_\lambda(u_\lambda)$ which is the second derivative of J_λ evaluated at u_λ. We will prove that the first eigenvalue λ_1 is equal to $4\pi(\alpha+2)$. Then we will be able to find a critical point of J_λ by using the mountain pass argument employed by Struwe (see Bartolucci & Tarantello [3], Ding, Jost, Li & Wang [10], Struwe & Tarantello [15] and references therein).

Moreover, it will turn out that $\lambda_n = 4\pi(\alpha+n+1)$ $(1 \le n < \alpha+1)$ is also an eigenvalue for the associated eigenvalue problem. If we restrict ourselves to the suitable subspace X of $H_0^1(B_1)$ we will find that λ_n is a simple eigenvalue of $J''_\lambda(u_\lambda)$ in X. In this case, the argument of Crandall & Rabinowitz [9] works, and we will have a continuous branch bifurcating from u_λ. See Theorem 5 below.

Finally, we will also establish uniqueness result for (1). For this purpose, we will use the isoperimetric inequality as well as the Schwartz symmetrization which was used in Bandle [1], Chang, Chen & Lin [5] and Suzuki [16].

We have the following uniqueness result.

Theorem 2. *If $0 < \lambda \le 8\pi$ then equation (1) admits a unique solution.*

Remark. Theorem 2 also holds true when B_1 is replaced with a smooth bounded simply connected domain Ω containing the origin. Moreover, every nonradial solution of (1) always blows up as $\lambda \to 8\pi^+$.

In the rest of this paper, we will prove Theorem 1 and Theorem 2.

2. Existence and Uniqueness results

We begin this section by proving Theorem 2.

Proof of Theorem 2. We will prove the following linearized equation

$$\Delta\varphi + \lambda\frac{|x|^{2\alpha}e^v\varphi}{\int_\Omega |x|^{2\alpha}e^v dx} - \lambda\frac{(\int_\Omega |x|^{2\alpha}e^v\varphi)|x|^{2\alpha}e^v}{(\int_\Omega |x|^{2\alpha}e^v)^2} = 0$$

admits only a trivial solution in $H_0^1(\Omega)$ when $0 \leq \lambda \leq 8\pi$. The proof follows from the argument of Suzuki [16] and Chang, Chen & Lin [5], and it is based on the symmetrization procedure with a variant of Alexandrov-Bol's inequality: If $p \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $-\Delta \ln p \leq |x|^{2\alpha} p$ in $\Omega \subset \mathbb{R}^2$, then we have

$$[l(\partial\Omega)]^2 \geq \frac{1}{2} m(\Omega)(8\pi - m(\Omega)),$$

where $l(\partial\Omega) = \int_{\partial\Omega} |x|^\alpha p\, ds$ and $m(\Omega) = \int_\Omega |x|^{2\alpha} p\, dx$.

Note that the function $U(x) = -2\ln(1 + \frac{1}{8}|x|^2)$ satisfies $-\Delta U = e^U$ in $\mathbb{R}^2$. We adopt a spherical symmetrization procedure with respect to e^U and

$$p_\lambda = \frac{\lambda e^v}{\int_\Omega |x|^{2\alpha} e^v\, dx}.$$

For instance, we may adopt the procedure of spherically decreasing rearrangement. We define an open ball Ω_t^* by

$$\int_{\Omega_t^*} e^U\, dx = \int_{\{v>t\}} |x|^{2\alpha} p_\lambda\, dx$$

and define $v^*(x) = \sup\{t \mid x \in \Omega_t^*\}$. It is well known that this rearrangement is equi-measurable and it decreases the L^2-norm of the first derivative.

Then we can repeat the argument in Suzuki [16] and Chang, Chen & Lin [5] to complete the proof. We skip the details. □

In the rest of this paper, we will establish the multiple existence result for (1).

2.1. Proof of Theorem 1

The proof will be given in several steps. Clearly, $v = u_\lambda$ is a critical point of J_λ. For convenience, we let

$$g_\lambda = \frac{\lambda r^{2\alpha} e^{u_\lambda}}{\int_{B_1} r^{2\alpha} e^{u_\lambda}}.$$

In order to determine the values of λ where $v = u_\lambda$ is a strict local minimum of J_λ, we consider the second derivative of J_λ. For $\varphi \in H_0^1(B_1)$, we have

$$J_\lambda''(u_\lambda)(\varphi, \varphi) = \|\nabla\varphi\|_2^2 - \int_{B_1} g_\lambda \varphi^2\, dx + \frac{1}{\lambda}\Big(\int_{B_1} g_\lambda \varphi\, dx\Big)^2. \tag{6}$$

In order to proceed, we consider an eigenvalue problem following the argument of Suzuki [16]. We introduce $H_c^1(B_1) = \{v \in H^1(B_1) \mid v = \text{const. on } \partial B_1\}$ and define an eigenvalue $K = K(\lambda)$ by

$$K = \inf\Big\{\|\nabla v\|_2^2 \mid v \in H_c^1(B_1),\ \int_{B_1} g_\lambda v^2\, dx = 1,\ \int_{B_1} g_\lambda v\, dx = 0\Big\} \tag{7}$$

Then it suffices to find the values of λ at which $K > 1$. Indeed, given $\varphi \in H_0^1(B_1)$, let $\tilde{\varphi} = \varphi - \frac{1}{\lambda}\int_{B_1} g_\lambda \varphi\, dx \in H_c^1(B_1)$. Since $\int_{B_1} g_\lambda \tilde{\varphi}\, dx = 0$, we obtain

$$J''_\lambda(u_\lambda)(\varphi,\varphi) = \|\nabla\tilde{\varphi}\|_2^2 - \int_{B_1} g_\lambda \tilde{\varphi}^2\, dx \geq \Big(1 - \frac{1}{K}\Big)\|\nabla\varphi\|_2^2.$$

The value K in (7) is the second eigenvalue for the eigenvalue problem: To find a $\psi \in H_c^1(B_1) \setminus \{0\}$ and $K \in \mathbb{R}$ such that

$$\int_{B_1} \nabla\psi \cdot \nabla v dx = K \int_{B_1} g_\lambda \psi v\, dx \quad \text{for any } v \in H_c^1(B_1).$$

Then the second eigenvalue for (7) is attained by a function $\psi \in H_c^1(B_1)$ satisfying

$$\begin{aligned} -\Delta\psi &= K g_\lambda \psi, \quad B_1, \\ \psi &= \text{const.}, \quad \partial B_1, \\ \int_{\partial B_1} \frac{\partial\psi}{\partial\nu}\, d\sigma &= 0. \end{aligned} \tag{8}$$

Since g_λ is radially symmetric, it is more convenient to consider the Fourier expansion of $\psi \in C^2(B_1) \cap C^0(\overline{B_1})$:

$$\psi(r,\theta) = \psi_0(r) + \sum_{n=1}^{\infty}(\psi_{1n}(r)\cos n\theta + \psi_{2n}(r)\sin n\theta). \tag{9}$$

Then equation (8) is reduced to a series of ODE's. Substituting (9) into (8), we obtain

$$\begin{aligned} -\partial_r^2\psi_0 - \frac{1}{r}\partial_r\psi_0 &= K g_\lambda \psi_0, \quad 0 < r < 1 \\ \partial_r\psi_0(1) &= 0, \end{aligned} \tag{10}$$

and

$$\begin{aligned} -\partial_r^2\psi_{in} - \frac{1}{r}\partial_r\psi_{in} + \frac{n^2}{r}\psi_{in} &= K g_\lambda \psi_{in}, \quad 0 < r < 1, \\ \psi_{in}(1) &= 0, \end{aligned} \tag{11}$$

where $n \geq 1,\ i = 1,2$. Concerning the eigenvalue problem (10) and (11), we define the eigenvalues $K^{(0)}$ and $K^{(n)}$, $n \in \mathbb{N}$, $n < 1 + \alpha$ by

$$K^{(0)} = \inf\Big\{\|\nabla v\|_{L^2(B_1)}^2 \mid v \in H_c^1(B_1) \text{ is radial },\ \int_{B_1} g_\lambda v^2\, dx = 1, \int_{B_1} g_\lambda v\, dx = 0\Big\} \tag{12}$$

and

$$K^{(n)} = \inf\Big\{\int_0^1 (r[\partial_r v]^2 + \frac{n^2}{r}v^2)dr \mid v(1) = 0, \int_0^1 r g_\lambda v^2\, dr = 1\Big\}. \tag{13}$$

It is easy to check that there is a minimizer v_0 (*resp.* v_n) which corresponds to $K^{(0)}$ (*resp.* $K^{(n)}$) such that v_0 changes sign once and only once in B_1, while v_n is definite. It is clear that K defined in (7) is given by $K = \min_{\nu\geq 0}\{K^{(\nu)}\}$. The following lemma shows that $K > 1$ if and only if $\lambda < 4\pi(\alpha+2)$.

Lemma 1. *$K^{(0)} > 1$ if $\lambda < 8\pi(1+\alpha)$ and $K^{(n)} > 1$ if and only if $\lambda < 4\pi(\alpha+n+1)$. Moreover, $K^{(n)} < K^{(n+1)}$ for $n \geq 1$.*

Proof. Note that the following ODE

$$-\partial_r^2 v - \frac{1}{r}\partial_r v = g_\lambda v, \quad r > 0$$

has two independent solutions ξ_1, ξ_2 given by

$$\xi_1(r) = \frac{c_\lambda - r^{2\alpha+2}}{c_\lambda + r^{2\alpha+2}}, \qquad \xi_2(r) = -1 + \frac{c_\lambda - r^{2\alpha+2}}{2(c_\lambda + r^{2\alpha+2})}\ln\frac{c_\lambda}{r^{2\alpha+2}}.$$

Let $\Omega_\pm = \{\pm v_0 > 0\}$ be the two connected nodal domains. Without loss of generality, we may assume $\Omega_+ = B_{R_0}$. It follows from maximum principle that $d = \psi(\partial\Omega) \neq 0$, and we assume $d < 0$. Since $\lambda < 8\pi(1+\alpha)$, we have two cases; either $\int_{\Omega_+} g_\lambda\, dx < 4\pi(1+\alpha)$ or $\int_{\Omega_+} g_\lambda\, dx \geq 4\pi(1+\alpha)$.

The condition $\int_{\Omega_+} g_\lambda dx < 4\pi(1+\alpha)$ is equivalent to $R_0^{2\alpha+2} < c_\lambda$. Consequently, ξ_1 is positive in $\overline{\Omega_+}$, which in turn implies $K^{(0)} > 1$.

If $\int_{\Omega_+} g_\lambda dx \geq 4\pi(1+\alpha)$, the solution ξ of the following equation

$$-\partial_r^2\psi - \frac{1}{r}\partial_r\psi = g_\lambda\psi, \quad R_0 < r < 1,$$
$$\psi(1) = -1,$$
$$\partial_r\psi(1) = 0$$

is given by

$$\xi(r) = (1-B^2)\Big[-1 + \frac{c_\lambda - r^{2\alpha+2}}{2(c_\lambda + r^{2\alpha+2})}\ln\frac{(1-B)c_\lambda}{(1+B)r^{2\alpha+2}}\Big] - B\frac{c_\lambda - r^{2\alpha+2}}{c_\lambda + r^{2\alpha+2}},$$

where $B = (c_\lambda - 1)/(c_\lambda + 1)$. Since $R_0^{2\alpha+2} \geq c_\lambda$ and ξ is strictly decreasing in $(0,1)$, we have

$$\xi(r) \leq \xi(c_\lambda^{\frac{1}{2\alpha+2}}) = -1 + B^2 < 0 \quad \text{for } r \in (R_0, 1),$$

which in turn implies that $K^{(0)} > 1$. Consequently, $K^{(0)} > 1$ for $\lambda \in (0, 8\pi(1+\alpha))$.

The minimizer v_n for $K^{(n)}$ satisfies

$$-\partial_r^2 v_n - \frac{1}{r}\partial_r v_n + \frac{n^2}{r^2}v_n = K^{(n)} g_\lambda v_n, \quad 0 < r < 1$$
$$v_n(1) = 0.$$

We note that for each $n = 1, 2, \dots,$

$$w_n(r) = \frac{(\alpha+n+1)c_\lambda - (\alpha-n+1)r^{2\alpha+2}}{c_\lambda + r^{2\alpha+2}} r^n \tag{14}$$

satisfies the following ODE (see Baraket & Pacard [2]).

$$-\partial_r^2 v - \frac{1}{r}\partial_r v + \frac{n^2}{r^2} v = g_\lambda v, \quad r > 0.$$

Consequently $K^{(n)} > 1$ if and only if $w_n > 0$ on $(0, 1]$, which is equivalent to the condition $\lambda < 4\pi(\alpha + n + 1)$. The last inequality is obvious. □

Remark. $v = u_\lambda$ is not a local minimum of J_λ if $\lambda = 4\pi(\alpha + 2)$. Indeed, for any nonzero $b \in \operatorname{span}\{w_1 \cos\theta, w_1 \sin\theta\}$, we have

$$J_\lambda(tb) - J_\lambda(0) = -\frac{t^4}{24}\int_{B_1} g_\lambda b^4\, dx + O(t^6) < 0$$

for sufficiently small $|t| > 0$.

Therefore we have proved the following proposition.

Proposition 1. *$v = u_\lambda$ is a strict local minimum of the functional J_λ if and only if $\lambda \in (0, 4\pi(\alpha + 2))$.*

On the other hand, it can be shown (*cf.* Wang & Wei [18]) that given $\lambda \in (8\pi, 4\pi(\alpha + 2))$, one can choose a small interval $I = (\lambda_0 - \delta, \lambda_0 + \delta)$ such that $\lambda \in I \subset\subset (8\pi, 4\pi(\alpha + 2))$ and construct a test function $\zeta_I \in H_0^1(B_1)$ such that $J_\lambda(\zeta_I) < J_\lambda(u_\lambda)$ for all $\lambda \in I$. Moreover, there is a constant $\rho = \rho(I) > 0$ such that $J_\lambda(u_{\lambda_0}) < \inf_{\|\varphi - u_{\lambda_0}\|_{H_0^1(B_1)} = \rho} J_\lambda(\varphi)$ for all $\lambda \in I$ if $\delta > 0$ is sufficiently small.

Let $S = \{\gamma \in C^0([0,1], H_0^1(B_1)) \mid \gamma(0) = u_{\lambda_0} \text{ and } \gamma(1) = \zeta_I\}$ and define a *minimax* value $c_\lambda = \inf_{\gamma \in S} \sup_{t \in [0,1]} J_\lambda(\gamma(t))$. Then it is easily checked that the map $\lambda \to \frac{c_\lambda}{\lambda}$ is monotonically decreasing for $\lambda \in I$. Applying Struwe's argument (*cf.* Struwe [15]), we obtain the following lemma.

Lemma 2. *There exists a dense subset $\Lambda \subset (8\pi, 4\pi(\alpha + 2))$ such that equation* (1) *admits a nonradial solution v_λ such that $J_\lambda(v_\lambda) = c_\lambda$ for any $\lambda \in \Lambda$.*

Let v_{λ_n} be a sequence of nonradial solutions with $\lambda_n \in \Lambda$ and $\lambda_n \to \lambda$. It can be shown from the method of moving planes that there is a constant $\delta = \delta(\alpha) > 0$ such that ∇v_{λ_n} does not vanish in $\{\delta < |x| < 1\}$. Indeed, given $x_0 \in \partial B_1$, let us consider the disk $B_R(p)$ with $p = (R + 1)x_0$. By rotating, we may assume $x_0 = (-1, 0)$. We perform a reflection $x \in B_1 \mapsto y = Tx = p + \frac{R(x-p)}{|x-p|^2}$ with respect to $B_R(p)$, and define $w_n(y) = v_{\lambda_n}(x)$. Then equation (1) transforms to

$$\Delta w_n + \frac{\lambda_n}{\int_{B_1} |x|^{2\alpha} e^{v_{\lambda_n}}\, dx} f(y) e^{w_n} = 0, \quad f(y) = |y-p|^{-4} \Big| p + \frac{R^2(y-p)}{|y-p|^2} \Big|^{2\alpha}$$

Moreover we have

$$\frac{\partial f}{\partial y_1} = -2|y-p|^{-8} \Big| p + \frac{y-p}{|y-p|^2} \Big|^{2\alpha-2} \Big(2(y_1-p_1)|y-p|^2 \Big| p + \frac{y-p}{|y-p|^2} \Big|^2 - \alpha\big(R^2 p_1 |y-p|^2 - 2R^2 p_1 (y_1-p_1)^2 - R^4 (y_1 - p_1)\big) \Big).$$

It is easily checked that $\partial f / \partial y_1 < 0$ when $R = 1/(\alpha+1)$. An application of the method of moving plane shows that w_n cannot have a stationary point in a small neighborhood of $x_0 \in \partial B_1$ in $T(B_1)$ (see Gidas, Ni & Nirenberg [11] for details and see Chen & Li [6] as well).

Let us fix a compact domain Σ in B_1 such that v_{λ_n} does not have any stationary point in Σ. We are in position to apply the following version of Brezis-Merle's result.

Theorem 3 (Bartolucci & Tarantell [3]). *Let u_{λ_n} be a sequence of solutions of the equation $-\Delta u_{\lambda_n} = \lambda_n |x|^{2\alpha} e^{u_{\lambda_n}}$ in B_1 with $\alpha > 0$ and $\lambda_n \to \lambda \geq 0$. Then (passing to a subsequence) one of the following alternative holds:*

(i) $\sup_\Sigma |v_{\lambda_n}| \leq C_\Sigma$,

(ii) $\sup_\Sigma v_{\lambda_n} \to -\infty$,

(iii) *There exists a finite and nonempty set $S = \{q_1, \ldots, q_l\} \subset \Sigma$, $l \in \mathbb{N}$, and a sequence of points $\{x_n^1, x_n^2, \ldots, x_n^l\} \subset B_1$ such that $x_n^i \to q_i$ and*

$$u_{\lambda_n}(x_n^i) + 2\alpha \ln |x_n^i| \to \infty \quad \textit{for} \quad i = 1, \ldots, l.$$

Moreover $\sup_\Sigma\{u_{\lambda_n}\} \to -\infty$ *on any compact set $K \subset \Sigma \setminus S$ and $|x|^{2\alpha} e^{u_{\lambda_n}} \to \sum_{i=1}^{l} \beta_i \delta_{q_i}$ in a measure sense, furthermore $\beta_i \in 8\pi\mathbb{N}$ if $q_i \neq 0$ and $\beta_i \geq 8\pi$ if $q_i = 0$ for some $i = 1, \ldots, l$.*

Suppose that $q_1 = 0$ and fix a domain D such that $D \cap S = \{0\}$. If u_{λ_n} satisfies a "mild" boundary condition on ∂D, in other words, if there is a constant C_0 such that $\max_{\partial D} u_{\lambda_n} - \min_{\partial D} u_{\lambda_n} \leq C_0$ *then $\beta_1 = 8\pi(1+\alpha)$.*

We are now in a position to prove Theorem 1.

Proof of Theorem 1. v_{λ_n} be a sequence of nonradial solutions with $\lambda_n \in \Lambda$ and $\lambda_n \to \lambda$. The Dirichlet boundary condition for v_{λ_n} implies that v_{λ_n} satisfies the above mild boundary condition in a suitable neighborhood of the origin. Consequently, Theorem 3 and the maximum principle imply that $\{v_{\lambda_n}\}$ is uniformly bounded in Σ if $\lambda_n \to \lambda \in (8\pi, 4\pi(\alpha+2)) \setminus 8\pi\mathbb{N}$. Then Theorem 1 follows from the standard elliptic estimates. $\square$

2.2. Bifurcation phenomena In the previous section we obtained the first eigenvalues for the eigenvalue problems (10) and (11). We will prove that equation (1) admits a continuum of non-trivial solution bifurcating from these eigenvalues.

For this purpose, we introduce a compact operator $G = \Delta^{-1} : C_0^{1,\gamma}(\overline{B_1}) \to C_0^{1,\gamma}(\overline{B_1})$ and a nonlinear operator $F : (0, 8\pi(\alpha+1)) \times C_0^{1,\gamma}(\overline{B_1}) \to C_0^{1,\gamma}(\overline{B_1})$ defined by

$$F(\lambda, v) = v + u_\lambda + \lambda G\Big(\frac{|x|^{2\alpha} e^{u_\lambda + v}}{\int_{B_1} |x|^{2\alpha} e^{u_\lambda + v}}\Big) \tag{15}$$

for a fixed exponent $\gamma \in (0,1)$. Then we have $F(\lambda, 0) = 0$ for all λ and

$$F_v(\lambda, 0)\omega = \omega + G(g_\lambda \omega) + \frac{1}{\lambda} u_\lambda \int_{B_1} g_\lambda \omega \, dx.$$

Recall that w_n is given in (14) and let us define $\varphi_{n+}, \varphi_{n-} \in C_0^{1,\gamma}(\overline{B_1})$ by

$$\varphi_{n+} = w_n \cos n\theta, \quad \varphi_{n-} = w_n \sin n\theta.$$

The proof of Lemma 1 shows that $\operatorname{Ker} F_v(\mu_n, 0) = \operatorname{span}\{\varphi_{n+}, \varphi_{n-}\}$ with $\mu_n = 4\pi(\alpha + n + 1)$.

The next lemma shows that the set $\{\mu_n\}_{1 \le n < \alpha+1}$ exhausts all the values λ where $\operatorname{Ker} F_v(\lambda, 0)$ is nontrivial.

Lemma 3. *For each $n = 1, 2, \cdots, [\alpha]+1$, define the second eigenvalue by*

$$K_*^{(n)} = \inf\Big\{ \int_0^1 \big(r v_r^2 + \frac{n^2}{r} v^2\big) dr \mid v(1) = 0, \int_0^1 r g_\lambda v^2 \, dr = 1, \int_0^1 r g_\lambda v_n v \, dr = 0 \Big\},$$

where v_n is the minimizer of (13). Then $K_*^{(n)} > 1$ for each $n = 1, \dots, [\alpha]+1$.

Proof. The eigenvalue $K_*^{(n)}$ is attained by a function ψ_n satisfying

$$-\partial_r^2 \psi_n - \frac{1}{r}\partial_r \psi_n + \frac{n^2}{r^2}\psi_n = K_*^{(n)} g_\lambda \psi_n, \quad 0 < r < 1,$$
$$\psi_n(1) = 0,$$
$$\int_0^1 r g_\lambda v_n \psi_n \, dr = 0.$$

We introduce two independent solutions $w_n(r)$ and $\tilde{w}_n(r) = -w_n(r) \int_r^1 \frac{ds}{s w_n^2(s)}$ of the ODE $-\partial_r^2 v - \frac{1}{r}\partial_r v + \frac{n^2}{r^2} v = g_\lambda v$, $0 < r < 1$. Let $R_1 \in (0,1)$ be the unique zero of ψ_n and we consider the two cases: either $\int_0^{R_1} rg_\lambda\, dr < 4\pi(\alpha+n+1)$ or $\int_0^{R_1} rg_\lambda\, dr \geq 4\pi(\alpha+n+1)$. Let us study only the latter part. In this case, we have $w_n < 0$ on $(R_1, 1]$. The solution of the following ODE

$$-\partial_r^2\phi - \frac{1}{r}\partial_r\phi + \frac{n^2}{r^2}\phi = g_\lambda\phi, \quad R_1 < r < 1$$
$$\phi(1) = 0, \qquad \partial_r\phi(1) = 1$$

is given by $\phi(r) = w_n(1)\tilde{w}_n(r)$, which is negative on $(R_1, 1)$. Moreover, $\lim_{r\to R_1}\phi(r)$ exists and is negative. This implies $K_*^{(n)} > 1$. □

Consequently, the above arguments show that $F_v(\lambda, 0) : C_0^{1,\gamma}(\overline{B_1}) \to C_0^{1,\gamma}(\overline{B_1})$ is an isomorphism if and only if $\lambda \in (0, 8\pi(\alpha+1)) \setminus \{\mu_n\}_{1\leq n<\alpha+1}$, and $\operatorname{Ker} F_v(\mu_n, 0) = \operatorname{span}\{\varphi_{n+}, \varphi_{n-}\}$.

We will apply the theorem of Crandall & Rabinowitz [9].

Theorem 4. *Let X, Y be Banach spaces, V a neighborhood of 0 in X and $F : (-1,1)\times V \to Y$ have the properties*

(a) $F(t, 0) = 0$ *for* $|t| < 1$,

(b) *The partial derivatives* F_t, F_x *and* F_{tx} *exist and are continuous,*

(c) $N(F_x(0,0))$ *and* $Y/R(F_x(0,0))$ *are one-dimensional, and*

(d) $F_{tx}(0,0)x_0 \notin R(F_x(0,0))$, *where* $N(F_x(0,0)) = \operatorname{span}\{x_0\}$.

If Z is any complement of $N(F_x(0,0))$ in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R}\times X$, an interval $(-a, a)$, and continuous functions $\phi : (-a,a) \to \mathbb{R}$, $\psi : (-a,a) \to Z$ such that $\phi(0) = 0$, $\psi(0) = 0$ and

$$F^{-1}(0)\cap U = \{(\phi(s), sx_0 + s\psi(s)) \mid |s| < a\} \cup \{(t,0) \mid (t,0)\in U\}.$$

Motivated by the results in Lin [13], we define the subspace $\tilde{C}_0^{1,\gamma}(\overline{B_1}) \subset H_0^1(B_1)$ by

$$\tilde{C}_0^{1,\gamma}(\overline{B_1}) = \{v \in C_0^{1,\gamma}(\overline{B_1}) \mid v(x_1, x_2) = v(x_1, -x_2)\}.$$

Then the mapping $F : (0, 8\pi(\alpha+1)) \times \tilde{C}_0^{1,\gamma}(\overline{B_1}) \to \tilde{C}_0^{1,\gamma}(\overline{B_1})$ is well-defined and (a), (b), (c) of Theorem 4 are all satisfied with $\operatorname{Ker} F_v(\mu_n, 0) = \operatorname{span}\{\varphi_{n+}\}$. We also note that

$$F_{\lambda v}(\mu_n, 0)\varphi_{n+} = G\Big(\varphi_{n+}\frac{d}{d\lambda}\Big|_{\lambda=\mu_n} g_\lambda\Big).$$

It is easy to check

$$\frac{d}{d\lambda}g_\lambda = \frac{64\pi(\alpha+1)^3}{\lambda^2}\frac{c_\lambda - r^{2\alpha+2}}{(c_\lambda + r^{2\alpha+2})^3}r^{2\alpha}.$$

The next lemma shows that the condition (d) of Theorem 4 also holds.

Lemma 4. *For $1 \le n < \alpha + 1$ and $\mu_n = 4\pi(\alpha + n + 1)$, we have*

$$\int_{B_1} \varphi_{n\pm}^2 \Big(\frac{d}{d\lambda}\Big|_{\lambda=\mu_n} g_\lambda \Big)\, dx \neq 0.$$

Proof. For convenience, we set $c_n = c_{\mu_n}$. It suffices to show that

$$A := \int_0^1 \frac{(c_n - r^{2\alpha+2})(1 - r^{2\alpha+2})^2}{(c_n + r^{2\alpha+2})^5} r^{2n+2\alpha+1}\, dr > 0.$$

Notice that integration by parts gives

$$\begin{aligned}
&\int_0^1 \frac{1}{(c_n + t)^4}\, t^{\frac{n}{\alpha+1}}\, dt \\
&= \frac{\alpha+1}{\alpha+n+1} \int_0^1 \frac{1}{(c_n+t)^4} \frac{d}{dt}\, t^{\frac{\alpha+n+1}{\alpha+1}}\, dt \\
&= \frac{4(\alpha+1)}{\alpha+n+1} \int_0^1 \Big(\frac{1}{(c_n+t)^4} - \frac{c_n}{(c_n+t)^5} \Big) t^{\frac{n}{\alpha+1}}\, dt + \frac{\alpha+1}{(\alpha+n+1)(1+c_n)^4},
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^1 \frac{c_n - t}{(c_n+t)^5}\, t^{\frac{n}{\alpha+1}}\, dt &= \int_0^1 \Big(\frac{2c_n}{(c_n+t)^5} - \frac{1}{(c_n+t)^4} \Big) t^{\frac{n}{\alpha+1}}\, dt \\
&= \frac{\alpha+1}{(3\alpha+3-n)(1+c_n)^4} + \frac{2c_n(\alpha+1-n)}{3\alpha+3-n} \int_0^1 \frac{t^{\frac{n}{\alpha+1}}}{(c_n+t)^5}\, dt \\
&> 0.
\end{aligned}$$

Since the map $t \mapsto (t-1)^2$ is strictly decreasing in $(0,1)$, the quantity A is positive. Indeed, we obtain, with substitution $t = r^{2\alpha+2}$,

$$\begin{aligned}
(2\alpha+2)A &= \int_0^{c_n} \frac{c_n - t}{(c_n+t)^5} (t-1)^2 t^{\frac{n}{\alpha+1}}\, dt - \int_{c_n}^1 \frac{t - c_n}{(c_n+t)^5} (t-1)^2 t^{\frac{n}{\alpha+1}}\, dt \\
&> (1-c_n)^2 \int_0^{c_n} \frac{c_n - t}{(c_n+t)^5} t^{\frac{n}{\alpha+1}}\, dt - (1-c_n)^2 \int_{c_n}^1 \frac{t-c_n}{(c_n+t)^5} t^{\frac{n}{\alpha+1}}\, dt \\
&> 0.
\end{aligned}$$
□

Therefore, we have proved the following theorem.

Theorem 5. *The branch u_λ of radial solutions of (1) has a nonradial bifurcation at each $\lambda = \mu_n = 4\pi(\alpha + n + 1)$ for $1 \le n < \alpha + 1$, $n \in \mathbb{N}$. Furthermore, the set of bifurcating nonradial solutions is two dimensional in a neighborhood of (μ_n, u_{μ_n}).*

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