NATURAL FRENET EQUATIONS OF NULL CURVES

Dae Ho Jin

ABSTRACT. The purpose of this paper is to study the geometry of null curves in a
Lorentzian manifold (M, g). We show that it is possible to construct new type of
Frenet equations of null curves in M, supported by two examples.

1. INTRODUCTION

The theory of space curves of a Riemannian manifold is fully developed and its
local and global geometry is well-known. Spivak [11] published his work on *curves
in a Riemannian manifold*. He showed briefly how the Frenet-Serret formulas in $\mathbb{R}^3$
generalize and derived several results which are needed to discuss higher dimension.
His study was restricted to Riemannian manifold. In case of curves in a Lorentzian
manifold, there are three categories of curves, namely, spacelike, timelike and null,
depending on their causal character. We know from O'Neill [10] that the study of
timelike curves has many similarities with the spacelike curves.

Duggal & Bejancu [3] published their work on *general theory of null curves in
Lorentz manifolds*. They constructed a Frenet frame and proved the fundamental
existence and uniqueness theorem for this class of null curves.

The objective of the present paper is also to study on null curves in a Lorentzian
manifolds (M, g). We show that it is possible to construct new type of Frenet
frames suitable for (M, g) (called *natural Frenet equations*) which is more simple
type than Duggal and Bejancu [3]. In particular, we study some invariant properties
of curves. Much of the work will be immediately generalized in a formal way to semi-
Riemannian manifolds.

2. TRANSVERSAL VECTOR BUNDLES

Let (M, g) be a real $(m+2)$-dimensional Lorentzian manifold and C be a smooth null curve in M locally given by

$$x^i = x^i(t), \quad t \in I \subset \mathbb{R}, \quad i \in \{0, 1, \ldots, (m+1)\}$$

for a coordinate neighborhood $\mathcal{U}$ on C. Then the tangent vector field $\lambda = \frac{d}{dt}$ on $\mathcal{U}$ satisfies

$$g(\lambda, \lambda) = 0.$$

Denote by TC the tangent bundle of C and $TC^{\perp}$ the TC perpendicular. Clearly, $TC^{\perp}$ is a vector bundle over C of rank $(m+1)$. Since λ is null, the tangent bundle TC of C is a vector subbundle of $TC^{\perp}$, of rank 1. This implies that $TC^{\perp}$ is not complementary to TC in $TM|_C$. Thus we must find complementary vector bundle to TC in TM which will play the role of the normal bundle $TC^{\perp}$ consistent with the classical non-degenerate theory.

Suppose $S(TC^{\perp})$ denotes the complementary vector subbundle to TC in $TC^{\perp}$, i.e., we have

$$TC^{\perp} = TC \perp S(TC^{\perp})$$

where $\perp$ means the orthogonal direct sum. It follows that $S(TC^{\perp})$ is a non-degenerate vector subbundle of TM, of rank m. We call $S(TC^{\perp})$ a *screen vector bundle* of C, which being non-degenerate, we have

$$TM|_C = S(TC^{\perp}) \perp S(TC^{\perp})^{\perp}, \tag{1}$$

where $S(TC^{\perp})^{\perp}$ is a complementary orthogonal vector subbundle to $S(TC^{\perp})$ in $TM|_C$ of rank 2.

We denote by $F(C)$ the algebra of smooth functions on C and by $\Gamma(E)$ the $F(C)$ module of smooth sections of a vector bundle E over C. We use the same notation for any other vector bundle.

Theorem 1 (Duggal & Bejancu [3]). *Let C be a null curve of a Lorentzian manifold (M, g) and $S(TC^{\perp})$ be a screen vector bundle of C. Then there exists a unique vector bundle $\mathrm{ntr}(C)$ over C, of rank 1, such that on each coordinate neighborhood $\mathcal{U} \subset C$ there is a unique section $N \in \Gamma(\mathrm{ntr}(C)|_{\mathcal{U}})$ satisfying*

$$g\big(\lambda, N\big) = 1, \quad g(N, N) = g(N, X) = 0, \quad {}^{\forall}X \in \Gamma(S(TC^{\perp})|_{\mathcal{U}}). \tag{2}$$

We call the vector bundle $\mathrm{ntr}(C)$ the *null transversal bundle* of C with respect to $S(TC^\perp)$. Next consider the vector bundle

$$\mathrm{tr}(C) = \mathrm{ntr}(C) \perp S(TC^\perp),$$

which according to (1) and (2) is complementary but not orthogonal to TC in $TM|_C$. More precisely, we have

$$TM|_C = TC \oplus \mathrm{tr}(C) = (TC \oplus \mathrm{ntr}(C)) \perp S(TC^\perp). \tag{3}$$

We call $\mathrm{tr}(C)$ the *transversal vector bundle* of C with respect to $S(TC^\perp)$. The vector field N in Theorem 1 is called the *null transversal vector field* of C with respect to λ. As $\{\lambda, N\}$ is a null basis of $\Gamma((TC \oplus \mathrm{ntr}(C))|_\mathcal{U})$ satisfying (2), we obtain

Proposition 2 (Duggal & Bejancu [3]). *Let C be a null curve of a Lorentzian manifold (M, g). Then any screen vector bundle $S(TC^\perp)$ of C is Riemannian.*

3. GENERAL FRENET FRAMES

Let C be a null curve of an $(m + 2)$-dimensional Lorentzian manifold (M, g). Since any screen vector bundle $S(TC^\perp)$ of C will be Riemannian and $g(\nabla_\lambda \lambda, \lambda) = 0$; $g(\nabla_\lambda \lambda, N) = h$, from the decomposition (3), we have

$$\nabla_\lambda \lambda = h\lambda + R_1,$$

where $R_1 \in \Gamma(S(TC^\perp))$ is a spacelike vector field perpendicular to λ and N. Define the first curvature function κ_1 by $\kappa_1 = \|R_1\|$ and set $W_1 = \frac{R_1}{\kappa_1}$, then W_1 is a unit spacelike vector field along C. Thus the above equation becomes

$$\nabla_\lambda \lambda = h\lambda + \kappa_1 W_1.$$

Also, from $g(\nabla_\lambda N, \lambda) = -h$, $g(\nabla_\lambda N, N) = 0$ and $g(\nabla_\lambda N, W_1) = \kappa_2$, where κ_2 denotes the second curvature function, we also have

$$\nabla_\lambda N = -hN + \kappa_2 W_1 + R_2$$

where $R_2 \in \Gamma(S(TC^\perp))$ is also spacelike vector field perpendicular to λ, N and W_1. Define the third curvature function κ_3 by $\kappa_3 = \|R_2\|$ and set $W_2 = \frac{R_2}{\kappa_3}$, then W_2 is also a unit spacelike vector field along C. Thus we have

$$\nabla_\lambda N = -hN + \kappa_2 W_1 + \kappa_3 W_2.$$

Repeating above process we obtain the following equations

$$\nabla_\lambda \lambda = h\,\lambda + \kappa_1\,W_1$$

$$\nabla_\lambda N = -\,h\,N + \kappa_2\,W_1 + \kappa_3\,W_2$$

$$\nabla_\lambda W_1 = -\,k_2\,\lambda - \kappa_1\,N + \kappa_4\,W_2 + \kappa_5\,W_3$$

$$\nabla_\lambda W_2 = -\,\kappa_3\,\lambda - \kappa_4\,W_1 + \kappa_6\,W_3 + \kappa_7\,W_4$$

$$\nabla_\lambda W_3 = -\,\kappa_5\,W_1 - \kappa_6\,W_2 + \kappa_8\,W_4 + \kappa_9\,W_5 \qquad (4)$$

$$\vdots$$

$$\nabla_\lambda W_{m-1} = -\,\kappa_{2m-3}\,W_{m-3} - \kappa_{2m-2}\,W_{m-2} + \kappa_{2m}\,W_m$$

$$\nabla_\lambda W_m = -\,\kappa_{2m-1}\,W_{m-2} - \kappa_{2m}\,W_{m-1},$$

where h and $\{\kappa_1, \ldots, \kappa_{2m}\}$ are smooth functions on $\mathcal{U}$, $\{W_1, \ldots, W_m\}$ is a certain orthonormal basis of $\Gamma(S(TC^\perp)|_\mathcal{U})$. In general, for any $m > 0$, we call $F = \{\lambda, N, W_1, \ldots, W_m\}$ a *general Frenet frame* on M along C with respect to the screen vector bundle $S(TC^\perp)$ and the equations (4) are called its *general Frenet equations* of C. Finally, the functions $\{\kappa_1, \ldots, \kappa_{2m}\}$ are called *curvature functions* of C with respect to the General Frenet frame F.

4. NATURAL FRENET FRAMES

First, we consider two Frenet frames F and F^* along neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ with respect to a given screen vector bundle $S(TC^\perp)$ respectively. Then the general transformations that relate elements of F and F^* on $\mathcal{U} \cap \mathcal{U}^* \neq \varnothing$ are

$$\lambda^* = \frac{dt}{dt^*}\,\lambda, \quad N^* = \frac{dt^*}{dt}\,N, \quad W_\alpha^* = \sum_{\beta=1}^{m} A_\alpha^\beta\,W_\beta,\ 1 \leq \alpha \leq m, \qquad (5)$$

where A_α^β are smooth functions on $\mathcal{U} \cap \mathcal{U}^*$ and the matrix $(\,A_\alpha^\beta(x)\,)$ is an element of the orthogonal group $O(m)$ for any $x \in \mathcal{U} \cap \mathcal{U}^*$. We call (5) the *transformation of coordinate neighborhood* of C with respect to $S(TC^\perp)$. Using (5) and the first equation of the Frenet equations (4) for both F and F^*, we obtain

$$\frac{d^2 t}{dt^{*\,2}} - h^*\,\frac{dt}{dt^*} = -h\Big(\frac{dt}{dt^*}\Big)^2.$$

Next, we also consider two Frenet frames F and $\overline{F}$ *(resp.)* with respect to $(t,\,S(TC^\perp),\,\mathcal{U})$ and $(\overline{t},\,\overline{S}(TC^\perp),\,\overline{\mathcal{U}})$ *(resp.)*. Then the general transformations

that relate elements of F and $\overline{F}$ on $\mathcal{U} \cap \overline{\mathcal{U}} \neq \varnothing$ are

$$\overline{\lambda} = \frac{dt}{d\overline{t}}\,\lambda,$$

$$\overline{N} = -\frac{1}{2}\frac{dt}{d\overline{t}}\sum_{\alpha=1}^{m}(c_{\alpha})^2\,\lambda + \frac{d\overline{t}}{dt}\,N + \sum_{\alpha=1}^{m} c_{\alpha}\,W_{\alpha}, \qquad (6)$$

$$\overline{W}_{\alpha} = \sum_{\beta=1}^{m} B_{\alpha}^{\beta}\left(W_{\beta} - \frac{dt}{d\overline{t}}\,c_{\beta}\,\lambda\right), \qquad 1 \le \alpha \le m,$$

where c_{α} and B_{α}^{β} are smooth funtions on $\mathcal{U} \cap \overline{\mathcal{U}}$ and the $m \times m$ matrix $[\,B_{\alpha}^{\beta}(x)\,]$ is an element of the orthogonal group $O(m)$ for each $x \in \mathcal{U} \cap \overline{\mathcal{U}}$. We call (6) the *transformation of screen vector bundle* of C. Also, using (6) and the first equation of the Frenet equations (4) for both F and $\overline{F}$ we obtain

$$\frac{d^2 t}{d\overline{t}^2} - \overline{h}\,\frac{dt}{d\overline{t}} = -h\left(\frac{dt}{d\overline{t}}\right)^2 - \kappa_1\,c_1\left(\frac{dt}{d\overline{t}}\right)^3.$$

Now we consider the differential equation of the form

$$\frac{d^2 t}{d\overline{t}^2} - \tilde{h}\,\frac{dt}{d\overline{t}} = 0$$

whose general solution comes from

$$t = a\int_{\overline{t}_0}^{\overline{t}} \exp\left(\int_{s_0}^{s} \tilde{h}(\tilde{t})d\tilde{t}\right)ds + b, \quad a,\,b \in R.$$

It follows that any of these solutions, with $a \neq 0$, might be taken as special parameter on C, such that $h = 0$. Denote one such solution by $p = \frac{t-b}{a}$, where t is the general parameter as defined in above equation. We call p a *distinguished parameter* of C, in terms for which $h = 0$. It is important to note that when t is replaced by p and $\xi = \frac{d}{dp}$ in the Frenet equations (4), the first two equations become

$$\nabla_{\xi}\,\xi = \kappa_1\,W_1,$$

$$\nabla_{\xi}\,N = \kappa_2\,W_1 + \kappa_3\,W_2$$

and the other equations remain unchanged.

In case $\kappa_1 = 0$, the first equation of the last relations takes the following familiar form

$$\frac{d^2 x^h}{dp^2} + \sum_{i,\,j=0}^{m} \Gamma_{ij}^{h}\,\frac{dx^i}{dp}\,\frac{dx^j}{dp} = 0, \qquad h \in \{0,\,\ldots,\,m\}$$

where Γ_{ij}^{h} are the Christoffel symbols of the second type induced by ∇. Hence C is a null geodesic of M. The converse follows easily. Thus we have

Theorem 3 (Duggal & Bejancu [3]). *Let C be a null curve of a Loremtzian manifold (M, g). Then C is a null geodesic of M if and only if the first curvature κ_1 vanishes identically on C.*

Theorem 4. *Let C be a non-geodesic null curve of a Lorentzian manifold (M, g). Then there exists a screen vector bundle $\overline{S}(TC)$ which induces Frenet frame $\overline{F}$ on $\overline{\mathcal{U}}$ such that $\overline{\kappa}_4 = \overline{\kappa}_5 = 0$.*

Proof. From (6) and the first equation of the Frenet equations (4), we have

$$\overline{\kappa}_1 B_1^1 = \kappa_1 \left(\frac{dt}{\overline{dt}}\right)^2; \qquad \overline{\kappa}_1 B_1^\alpha = 0, \ \alpha \in \{2, \ldots, m\}.$$

Since $k_1 \neq 0$ on $\mathcal{U} \cap \overline{\mathcal{U}}$, we have $\overline{\kappa}_1 \neq 0$ on $\mathcal{U} \cap \overline{\mathcal{U}}$ and $B_1^2 = \cdots = B_1^m = 0$. Also $[B_\alpha^\beta(x)]$ is an orthogonal matrix, we infer that $B_1^1 = B_1 = \pm 1$ and $B_2^1 = \cdots = B_m^1 = 0$. Also, from the third equation of the Frenet equations (4), we have

$$\overline{\kappa}_4 B_2^2 + \overline{\kappa}_5 B_3^2 = B_1^1 \left(\kappa_4 + \kappa_1 c_2 \frac{dt}{\overline{dt}}\right)\frac{dt}{\overline{dt}},$$

$$\overline{\kappa}_4 B_2^3 + \overline{\kappa}_5 B_3^3 = B_1^1 \left(\kappa_5 + \kappa_1 c_3 \frac{dt}{\overline{dt}}\right)\frac{dt}{\overline{dt}},$$

$$\overline{\kappa}_4 B_2^\alpha + \overline{\kappa}_5 B_3^\alpha = B_1^1 \kappa_1 c_\alpha \left(\frac{dt}{\overline{dt}}\right)^2, \qquad \alpha \in \{4, \ldots, m\}.$$

Taking into account that

$$c_2 = -\frac{\kappa_4}{\kappa_1}\frac{d\overline{t}}{dt}; \quad c_3 = -\frac{\kappa_5}{\kappa_1}\frac{d\overline{t}}{dt}; \quad c_\alpha = 0, \ \alpha \in \{4, \ldots, m\}$$

in the last equations and after some computations, we have $\overline{\kappa}_4 = \overline{\kappa}_5 = 0$.

Remark. If we take $t = \overline{t}$ in Theorem 4, then $c_2 = -\frac{\kappa_4}{\kappa_1}$; $c_3 = -\frac{\kappa_5}{\kappa_1}$ and

$$\overline{N} = -\frac{1}{2}\left(\frac{\kappa_4^2 + \kappa_5^2}{\kappa_1^2}\right)\lambda + N - \frac{\kappa_4}{\kappa_1}W_2 - \frac{\kappa_5}{\kappa_1}W_3,$$

$$\overline{W}_2 = W_2 + \frac{\kappa_4}{\kappa_1}\lambda,$$

$$\overline{W}_3 = W_3 + \frac{\kappa_5}{\kappa_1}\lambda,$$

$$\overline{W}_i = W_i, \quad i \in \{1, 4, \ldots, m\}.$$

Relabeling $N = \overline{N}$, $W_1 = \overline{W}_1, W_2 = \overline{W}_2$, $\kappa_i = \overline{\kappa}_i$, $i \in \{1, 2, 3\}$ and $S(TC^\perp) = \overline{S}(TC^\perp)$ in the process of the above theorem and we take only the first four equations

in (4) as follows:

$$\nabla_\lambda \lambda = h\,\lambda + \kappa_1\,W_1,$$

$$\nabla_\lambda N = -\,h\,N + \kappa_2\,W_1 + \kappa_3\,W_2,$$

$$\nabla_\lambda W_1 = -\,\kappa_2\,\lambda - \kappa_1\,N,$$

$$\nabla_\lambda W_2 = -\,\kappa_3\,\lambda + R_3,$$

where R_3 is a spacelike vector field in $\Gamma(S(TC^\perp)$ perpendicular to λ, N, W_1 and W_2. Define the new fourth curvature function κ_4 by $\kappa_4 = \|\,R_3\,\|$ and let $W_3 = \frac{R_3}{\kappa_4}$, then W_3 is also a unit spacelike vector field along C. Thus we have

$$\nabla_\lambda W_2 = -\,\kappa_3\,\lambda + \kappa_4\,W_3.$$

Repeating above process we obtain the following;

Theorem 5. *Let C be a non-geodesic null curve of a Lorentzian manifold (M, g). Then there exists a Frenet frame $\mathcal{F} = \{\lambda,\, N,\, W_1,\, \ldots,\, W_m\}$ satisfying the following equations*

$$\nabla_\lambda \lambda = h\lambda + \kappa_1 W_1,$$

$$\nabla_\lambda N = -\,hN + \kappa_2 W_1 + \kappa_3 W_2,$$

$$\nabla_\lambda W_1 = -\,\kappa_2\lambda - \kappa_1 N,$$

$$\nabla_\lambda W_2 = -\,\kappa_3\lambda + \kappa_4 W_3,$$

$$\nabla_\lambda W_3 = -\,\kappa_4 W_2 + \kappa_5 W_4, \tag{7}$$

$$\vdots$$

$$\nabla_\lambda W_i = -\,\kappa_{i+1} W_{i-1} + \kappa_{i+2} W_{i+1}, \quad i \in \{\,2,\, \ldots,\, m-1\,\},$$

$$\vdots$$

$$\nabla_\lambda W_m = -\,\kappa_{m+1} W_{m-1},$$

where $\{\kappa_1,\, \ldots,\, \kappa_{m+1}\}$ are smooth functions on $\mathcal{U}$, $\{W_1,\, \ldots,\, W_m\}$ is a certain orthonormal basis of $\Gamma(S(TC^\perp)|_{\mathcal{U}})$.

Corollary. *Let $C(p)$ be a non-geodesic null curve of a Lorentzian manifold (M, g), where p is a distinguished parameter on C. Then there exists a Frenet frame $\{\lambda,\, N,\, W_1,\, \ldots,\, W_m\}$ satisfying the equations (7) such that $h = 0$.*

Definition. We call the frame $\mathcal{F} = \{\lambda,\, N,\, W_1,\, \ldots,\, W_m\}$ in Theorem 5 a *natural Frenet frame* on M along C with respect to the given screen vector bundle $S(TC^\perp)$

and the equations (7) are called its *natural Frenet equations* of C. Finally, the functions $\{\kappa_1, \ldots, \kappa_{m+1}\}$ are called the *curvature functions* of C with respect to the Natural Frenet frame $\mathcal{F}$.

Example 1. Consider a null curve C and its Frenet frames in $\mathbf{R}_1^4$ given by

$$C(p) = \frac{1}{\sqrt{2}}\left(\sinh p,\ \cosh p,\ \sin p,\ \cos p\right),$$

$$\xi = \frac{1}{\sqrt{2}}\left(\cosh p,\ \sinh p,\ \cos p,\ -\sin p\right),$$

$$N = \frac{1}{\sqrt{2}}\left(-\cosh p,\ -\sinh p,\ \cos p,\ -\sin p\right),$$

$$W_1 = \frac{1}{\sqrt{2}}\left(\sinh p,\ \cosh p,\ -\sin p,\ -\cos p\right),$$

$$W_2 = \frac{1}{\sqrt{2}}\left(\sinh p,\ \cosh p,\ \sin p,\ \cos p\right),$$

where $p \in \mathbb{R}$, then $\nabla_\xi \xi = W_1,\ \nabla_\xi N = W_2,\ \nabla_\xi W_1 = -N,\ \nabla_\xi W_2 = -\xi$.

Example 2. Consider a null curve C and its Frenet frames in $\mathbf{R}_1^5$ given by

$$C(p) = \frac{1}{2\sqrt{2}}\left(\frac{1}{3}p^3 + 2p,\ p^2,\ \frac{1}{3}p^3,\ 2\cos p,\ 2\sin p\right),$$

$$\xi = \frac{1}{2\sqrt{2}}\left(p^2 + 2,\ 2p,\ p^2,\ -2\sin p,\ 2\cos p\right),$$

$$N = -\frac{1}{8\sqrt{2}}\left(p^2 + 10,\ 2p,\ p^2 + 8,\ 6\sin p,\ -6\cos p\right),$$

$$W_1 = \frac{1}{\sqrt{2}}\left(p,\ 1,\ p,\ -\cos p,\ -\sin p\right),$$

$$W_2 = -\frac{1}{\sqrt{2}}\left(p,\ 1,\ p,\ \cos p,\ \sin p\right),$$

$$W_3 = \frac{1}{4}\left(p^2 - 2,\ 2p,\ p^2 - 4,\ 2\sin p,\ -2\cos p\right).$$

where $p \in \mathbb{R}$. Then we have the following Frenet equations;

$$\nabla_\xi \xi = W_1, \qquad \nabla_\xi N = \frac{1}{4}W_1 + \frac{1}{2}W_2,$$

$$\nabla_\xi W_1 = -\frac{1}{4}\xi - N, \quad \nabla_\xi W_2 = -\frac{1}{2}\xi + \frac{1}{\sqrt{2}}W_3, \quad \nabla_\xi W_3 = -\frac{1}{\sqrt{2}}W_2.$$

5. INVARIANCE OF FRENET FRAMES

In this section we investigate the invariant properties of the Natural Frenet equations with respect to the transformations of the coordinate neighborhood and the screen vector bundle of C.

First, with respect to a given screen vector bundle $S(TC^\perp)$, we consider two Frenet frames $\mathcal{F}$ and $\mathcal{F}^*$ along two neighborhoods $\mathcal{U}$ and $\mathcal{U}^*$ respectively with non-null intersection. Then using (5) and the first equation of (7), we have

$$\kappa_1^* A_1^1 = \kappa_1 \left(\frac{dt}{dt^*} \right)^2 ; \qquad \kappa_1^* A_1^\alpha = 0, \quad \alpha \in \{2, \dots, m\}.$$

Proposition 6. *Let C be a non-geodesic null curve of a Lorentzian manifold (M, g) and $\mathcal{F}$ and $\mathcal{F}^*$ be two Natural Frenet frames on $\mathcal{U}$ and $\mathcal{U}^*$ induced by the same screen vector bundle $S(TC^\perp)$. Suppose $\prod_{i=3}^{m+1} \kappa_i \neq 0$ on $\mathcal{U} \cap \mathcal{U}^* \neq \phi$. Then we have*

$$\begin{aligned}
\kappa_1^* &= \kappa_1 A_1 \left(\frac{dt}{dt^*} \right)^2, \\
\kappa_2^* &= \kappa_2 A_1, \\
\kappa_3^* &= \kappa_3 A_2, \\
\kappa_\alpha^* &= \kappa_\alpha A_{\alpha-1} \frac{dt}{dt^*}, \quad 4 \leq \alpha \leq m+1
\end{aligned} \tag{8}$$

where $A_i = \pm 1, \ 1 \leq i \leq m - 1$.

Proof. From the last relations, we have $\kappa_1^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ and $A_1^2 = \cdots = A_1^m = 0$. Since $[A_\alpha^\beta(x)]$ is an orthogonal matrix, we infer that $A_1^1 = A_1 = \pm 1$ and $A_2^1 = \cdots = A_m^1 = 0$. Then from the second equation of (7) with respect to $\mathcal{F}$ and $\mathcal{F}^*$, and taking into account that $\kappa_3 \neq 0$, we obtain $\kappa_3^* \neq 0$ on $\mathcal{U} \cap \mathcal{U}^*$ which implies $A_2^3 = A_3^2 = \cdots = A_m^2 = A_m^2 = 0$ and $A_2^2 = A_2 = \pm 1$. Repeating this process for all other equations of (7) and set $A_\alpha = A_{\alpha-1}^{\alpha-1} A_\alpha^\alpha \ (\alpha \geq 3)$, we obtain all the relations in (8), which completes the proof. $\quad\square$

Proposition 7. *Let C be a non-geodesic null curve of a Lorentzian manifold (M, g) and $\mathcal{F}$ and $\mathcal{F}^*$ be two Natural Frenet frames on $\mathcal{U}$ and $\mathcal{U}^*$ induced by the same screen vector bundle $S(TC^\perp)$. Then the second curvature κ_2 and the third curvature κ_3 are invariant to the transformations of coordinate neighborhood of C.*

Let $\mathcal{F}$ and $\overline{\mathcal{F}}$ be two Natural Frenet frames with respect to $(t, S(TC^\perp), \mathcal{U})$ and $(\overline{t}, \overline{S}(TC^\perp), \overline{\mathcal{U}})$ respectively. Then the general transformations that relate elements

of $\mathcal{F}$ and $\overline{\mathcal{F}}$ on $\mathcal{U} \cap \overline{\mathcal{U}}$ are given by

$$\overline{\lambda} = \frac{dt}{d\overline{t}}\,\lambda,$$

$$\overline{N} = -\frac{1}{2}\frac{dt}{d\overline{t}}\sum_{i=1}^{m}(c_i)^2\,\lambda + \frac{d\overline{t}}{dt}\,N + \sum_{i=1}^{m} c_i\,W_i, \qquad (9)$$

$$\overline{W}_1 = B_1\left(W_1 - \frac{dt}{d\overline{t}}\,c_1\,\lambda\right),$$

$$\overline{W}_\alpha = \sum_{\beta=2}^{m} B_\alpha^\beta\left(W_\beta - \frac{dt}{d\overline{t}}\,c_\beta\,\lambda\right), \quad \alpha \in \{2,\dots,m\},$$

and $B_1^i = B_i^1 = 0\,(i \neq 1)\,;\ B_1^1 = B_1 = \pm 1$.

Proposition 8. *Let C be a non-geodesic null curve of a Lorentzian manifold (M, g) and $\mathcal{F}$ and $\overline{\mathcal{F}}$ be two Natural Frenet frames on $\mathcal{U}$ and $\overline{\mathcal{U}}$ respectively. Suppose $\prod_{i=1}^{m+1} \kappa_i \neq 0$ on $\mathcal{U} \cap \overline{\mathcal{U}} \neq \phi$. Then, their curvature functions are related by*

$$\overline{\kappa}_1 = \kappa_1\,B_1\left(\frac{dt}{d\overline{t}}\right)^2;$$

$$\overline{\kappa}_2 = \left\{\kappa_2 + \overline{h}\,c_1 + \frac{dc_1}{d\overline{t}} - \frac{1}{2}\kappa_1\,c_1^2\left(\frac{dt}{d\overline{t}}\right)^2\right\}B_1; \qquad (10)$$

$$\overline{\kappa}_3 = \kappa_3\,B_2\,;$$

$$\overline{\kappa}_\alpha = \kappa_\alpha\,B_{\alpha-1}\frac{dt}{d\overline{t}}, \qquad \alpha \in \{4,\dots,m\},$$

where $B_i = \pm 1,\ 1 \leq i \leq m-1$ and $c_1 \neq 0\,;\ c_2 = \dots = c_m = 0$.

Proof. From the third equation of (7) and (9), we have the first equation of (10) and $\overline{\kappa}_1 c_\alpha = 0\,(\alpha \neq 1)$ on $\mathcal{U} \cap \overline{\mathcal{U}}$. Thus we have $c_\alpha = 0\,(\alpha \neq 1)$. Also, from the second equation of (7) and (9), we have the second equation of (10) and $\overline{\kappa}_3 B_2^2 = \kappa_3\,;\ \overline{\kappa}_3 B_2^\alpha = 0\ (\alpha \geq 3)$. Thus we have $B_2 = B_2^2 = \pm 1$ and $B_2^\alpha = B_\alpha^2 = 0\ (\alpha \geq 3)$. Repeating this process for all other equations of (7) and set $B_\alpha = B_{\alpha-1}^{\alpha-1} B_\alpha^\alpha\ (\alpha \geq 3)$, we obtain all the relations in (10), which completes the proof. $\qquad\square$

Proposition 9. *Let C be a non-geodesic null curve of a Lorentzian manifold (M, g) and $\mathcal{F}$ and $\overline{\mathcal{F}}$ be two Natural Frenet frames on $\mathcal{U}$ and $\overline{\mathcal{U}}$ respectively. Then the third curvature κ_3 is invariant to the transformations of the screen vector bundle of C.*

REFERENCES

1. A. Bejancu: Lightlike curves in Lorentz manifolds. *Publ. Math. Debrecen* **44** (1994), no. 1-2, 145–155. MR **95h**:53090

2. W. B. Bonnor: Null curves in a Minkowski space-time. *Tensor (N. S.)* **20** (1969), 229–242. MR **40#1909**

3. K. L. Duggal & A. Bejancu: *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications.* Kluwer Acad. Publishers, Dordrecht, 1996. MR **97e:53121**

4. K. L. Duggal & D. H. Jin: Geometry of null curves. *Math. J. Toyama Univ.* **22** (1999), 95–120. MR **2001h:53096**

5. L. K. Graves: Codimension one isometric immersions between Lorentz spaces. *Trans. Amer. Math. Soc.* **252** (1979), 367–392. MR **80j:53060**

6. T. Ikawa: On curves and submanifolds in an indefinite Riemannian manifold. *Tsukuba J. Math.* **9** (1985), 353–371. MR **87d:53033**

7. J. Inoguchi: Biharmonic curves in Minkowski 3-space. *Int. J. Math. Math. Sci.* (2003), no. 21, 1365–1368. MR **2004e:53108**

8. J. Inoguchi & S. Lee: *Null curves in Minkowski 3-space.* (Private communication), pp. 1–27, March, 2005.

9. D. H. Jin: Frenet Equations of Null Curves. *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **10** (2003), no. 2, 71–102. MR **2004c:53111**

10. B. O'Neill: *Semi-Riemannian Geometry. With Applications to Relativity.* Pure and Applied Mathematics, 103. Academic Press, Inc., New York, 1983. MR **85f:53002**

11. M. Spivak: *A Comprehensive Introduction to Differential Geometry, Vol. 3.* Publish or Perish, Boston, Mass., 1975. MR **51#8962**

DEPT. OF MATHEMATICS, DONGGUK UNIVERSITY, 707 SEOKJANG-DONG, GYEONGJU, GYEONGBUK 780-714, KOREA
Email address: jindh@dongguk.ac.kr