J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 12, Number 3 (August 2005), Pages 223–228

FURTHER SUMMATION FORMULAS FOR THE APPELL'S FUNCTION F_1

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ABSTRACT. In 2001, Choi, Harsh & Rathie [Some summation formulas for the Appell's function F_1. *East Asian Math. J.* **17** (2001), 233–237] have obtained 11 results for the Appell's function F_1 with the help of Gauss's summation theorem and generalized Kummer's summation theorem. We aim at presenting 22 more results for F_1 with the help of the generalized Gauss's second summation theorem and generalized Bailey's theorem obtained by Lavoie, Grondin & Rathie [Generalizations of Whipple's theorem on the sum of a ${}_3F_2$. *J. Comput. Appl. Math.* **72** (1996), 293–300]. Two interesting (presumably) new special cases of our results for F_1 are also explicitly pointed out.

1. INTRODUCTION AND PRELIMINARIES

We have obtained 11 explicit closed-form expressions for

$$F_1(a\,;\,b,\,b'\,;\,1+a+b-b'+i\,;\,1,\,-1)$$

for $i = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5$ in Choi, Harsh, & Rathie [2].

Here we aim at giving explicit closed-form expressions for

$$F_1\left(a\,;\,b,\,b'\,;\,\frac{1}{2}(1+a+b'+2b+i)\,;\,1,\,\frac{1}{2}\right)$$

and

$$F_1\left(a\,;\,b,\,1-a+i\,;\,c+b\,;\,1,\,\frac{1}{2}\right)$$

for $i = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5$. The results are derived with the help of Gauss's summation theorem, generalized Gauss's second summation theorem and generalized Bailey's summation theorem obtained by Lavoie, Grondin & Rathie [3].

Received by the editors July 2, 2005.
2000 *Mathematics Subject Classification.* Primary 33C20, 33C60; Secondary 33C70, 33C65.
Key words and phrases. generalized hypergeometric series ${}_pF_q$, summation theorems for ${}_pF_q$, Appell's function F_1.

The following results will be required in our present investigation: Gauss's summation theorem (see Bailey [1, p. 2] and Srivastava & Choi [4, p. 45])

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} \,\middle|\, 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (\Re(c-a-b)>0). \tag{1.1} $$

Generalized Gauss's Second Summation Theorem (*cf.* Choi, Harsh & Rathie [3]). *For* $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5,$

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+i+1) \end{matrix} \,\middle|\, \frac{1}{2}\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} \\ &\quad \times \left\{ \frac{\mathcal{A}_i}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)} + \frac{\mathcal{B}_i}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}-\left[\frac{i}{2}\right]\right)} \right\} \end{aligned} \tag{1.2} $$

where $[x]$ *denotes the greatest integer less than or equal to* x *and its modulus is denoted by* $|x|$. *The values of* $\mathcal{A}_i$ *and* $\mathcal{B}_i$ *are given in* Tables 1 and 2.

Table 1. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$ $(-4 \le i \le 5)$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
5	$-(b+a+6)^2+\frac{1}{2}(b-a+6)(b+a+6)$ $+\frac{1}{4}(b-a+6)^2+11(b+a+6)$ $-\frac{13}{2}(b-a+6)-20$	$(b+a+6)^2+\frac{1}{2}(b-a+6)(b+a+6)$ $-\frac{1}{4}(b-a+6)^2-17(b+a+6)$ $-\frac{1}{2}(b-a+6)+62$
4	$\frac{1}{2}(b+a+1)(b+a-3)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a-1)$
3	$-\frac{1}{2}(b+3a-2)$	$\frac{1}{2}(3b+a-2)$
2	$\frac{1}{2}(b+a-1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a-1)$	2
-3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$
-4	$\frac{1}{2}(b+a-3)(b+a+1)$ $-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a-1)$

Table 2. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$ $(i=-5)$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
-5	$(b+a-4)^2-\frac{1}{4}(b+a-4)(b-a-4)$ $-\frac{1}{4}(b-a-4)^2+4(b+a-4)$ $-\frac{7}{2}(b-a-4)$	$(b+a-4)^2+\frac{1}{2}(b+a-4)(b-a-4)$ $-\frac{1}{4}(b-a-4)^2+8(b+a-4)$ $-\frac{1}{2}(b-a-4)+12$

We see that the special case $i=0$ of (1.2) leads to the well-known Gauss's second summation theorem (see Bailey [1, p. 11], Srivastava & Choi [4, p. 250])

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ \frac{1}{2}(a+b+1)\end{matrix}\middle|\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}. \tag{1.3} $$

Generalized Summation Theorem (*cf.* Lavoie, Grondin & Rathie [3]).

$$ {}_2F_1\left[\begin{matrix} a, \quad 1-a+i \\ c\end{matrix}\middle|\frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(1-a)}{2^{c-i-1}\Gamma\left(1-a+\frac{1}{2}i+\frac{1}{2}|i|\right)} \times \left\{\frac{\mathcal{C}_i}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\left[\frac{i+1}{2}\right]\right)} + \frac{\mathcal{D}_i}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\}, \tag{1.4} $$

where $i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5$ *and the values of* $\mathcal{C}_i$ *and* $\mathcal{D}_i$ *are in* Table 3.

Table 3. Table for $\mathcal{C}_i$ and $\mathcal{D}_i$ $(-5\le i\le 5)$

i	$\mathcal{C}_i$	$\mathcal{D}_i$
5	$-4b^2+2ab+a^2+22b-13a-20$	$4b^2+2ab-a^2-34b-a+62$
4	$2(b-2)(b-4)-(a-1)(a-4)$	$12-4b$
3	$a-2b+3$	$a+2b-7$
2	$b-2$	-2
1	-1	1
0	1	0
-1	1	1
-2	b	2
-3	$2b-a$	$2b+a+2$
-4	$2b(b+2)-a(a+3)$	$4(b+1)$
-5	$4b^2-2ab-a^2+8b-7a$	$4b^2+2ab-a^2+16b-a+12$

The special case of (1.4) when $i = 0$ gives the well-known Bailey's summation theorem (see Bailey [1, p. 11] and Srivastava & Choi [4, p. 250])

$$ {}_2F_1\left[\begin{matrix} a, & 1-a \\ & c \end{matrix} \,\middle|\, \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}. \tag{1.5} $$

2. Main Summation Formulas

The following 22 (11 each) summation formulas for the Appell's function, given in the form of two general formulas, will be established.

Theorem 1.

$$ \begin{aligned} &F_1\left(a\,;\,b,\,b'\,;\,\frac{1}{2}(1+a+b'+2b+i)\,;\,1,\,\frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(1+a+b'+2b+i)\right)\Gamma\left(\frac{1}{2}(1-a+b'+i)\right)\Gamma\left(\frac{1}{2}(1+a-b'-i)\right)}{\Gamma\left(\frac{1}{2}(1-a+b'+2b+i)\right)\Gamma\left(\frac{1}{2}(1+a-b'+|i|)\right)} \\ &\times\left\{\frac{\mathcal{E}_i}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b'+\frac{1}{2}+\frac{1}{2}i-\left[\frac{i+1}{2}\right]\right)}+\frac{\mathcal{F}_i}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b'+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{2.1} $$

where $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$ and the values of $\mathcal{E}_i$ and $\mathcal{F}_i$ can be obtained by simply replacing b by b' in Table 1.

$$ \begin{aligned} &F_1\left(a\,;\,b,\,1-a+i\,;\,c+b\,;\,1,\,\frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c+b)\,\Gamma(c-a)\,\Gamma(1-a)}{2^{c-i-1}\,\Gamma(c+b-a)\,\Gamma\left(1-a+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\qquad\times\left\{\frac{\mathcal{C}_i}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\left[\frac{i+1}{2}\right]\right)}\right. \\ &\qquad\qquad\left.+\frac{\mathcal{D}_i}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\}, \end{aligned} \tag{2.2} $$

where $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$ and the values of $\mathcal{C}_i$ and $\mathcal{D}_i$ are the same as given in Table 3.

Proof. In order to prove (2.1), we begin by denoting, for convenience, the left-hand side of (2.1) by I and expressing the Appell's function F_1 in its series as follows:

$$I := F_1\left(a\,;\,b,\,b'\,;\,\frac{1}{2}(1+a+b'+2b+i)\,;\,1,\,\frac{1}{2}\right)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(b')_n}{\left(\frac{1}{2}(1+a+b'+2b+i)\right)_{m+n}\, m!\,n!}\,\frac{1}{2^n}.$$

Using the well-known identity $(\alpha)_{m+n} = (\alpha+n)_m\,(\alpha)_n$, we find that

$$I = \sum_{n=0}^{\infty} \frac{(a)_n\,(b')_n}{\left(\frac{1}{2}(1+a+b'+2b+i)\right)_n\, n!}\,\frac{1}{2^n} \sum_{m=0}^{\infty} \frac{(a+n)_m\,(b)_m}{\left(\frac{1}{2}(1+a+b'+2b+i)+n\right)_m\, m!}$$
$$= \sum_{n=0}^{\infty} \frac{(a)_n\,(b')_n}{\left(\frac{1}{2}(1+a+b'+2b+i)\right)_n\, n!\,2^n}\,{}_2F_1\left[\begin{matrix} a+n,\ b \\ \frac{1}{2}(1+a+b'+2b+i)+n\end{matrix}\,\middle|\,1\right].$$

If we use the result (1.1), we, after a little simplification, get

$$I = \frac{\Gamma\left(\frac{1}{2}(1+a+b'+2b+i)\right)\,\Gamma\left(\frac{1}{2}(1-a+b'+i)\right)}{\Gamma\left(\frac{1}{2}(1+a+b'+i)\right)\,\Gamma\left(\frac{1}{2}(1-a+b'+2b+i)\right)}\,{}_2F_1\left[\begin{matrix} a,\ b' \\ \frac{1}{2}(1+a+b'+i)\end{matrix}\,\middle|\,\frac{1}{2}\right].$$

Finally, if we apply the result (1.2) to ${}_2F_1(1/2)$ in the last equation, after a little simplification, we arrive at the right-hand side of (2.1). This completes the proof of (2.1).

In the exactly same manner as above, the result (2.2) can be established with the help of the results (1.1) and (1.3), so we omit the proof of (2.2).

Setting $i = 0$ in (2.1) and (2.2), we obtain the following two interesting (presumably) new identities for F_1:

$$F_1\left(a\,;\,b,\,b'\,;\,\frac{1}{2}(1+a+b'+2b)\,;\,1,\,\frac{1}{2}\right)$$
$$= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}b'\right)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b'+b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b'+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}b'+b\right)} \tag{2.3}$$

and

$$F_1\left(a\,;\,b,\,1-a\,;\,c+b\,;\,1,\,\frac{1}{2}\right)$$
$$= \frac{\Gamma(c+b)\,\Gamma(c-a)\,\Gamma\left(\frac{1}{2}\right)}{2^{c-1}\,\Gamma(c+b-a)\,\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}. \tag{2.4}$$

□

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