J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 12, Number 3 (August 2005), Pages 229–235

ON SPHERICALLY CONCAVE FUNCTIONS

SEONG-A KIM

ABSTRACT. The notions of spherically concave functions defined on a subregion of the Riemann sphere $\mathbb{P}$ are introduced in different ways in Kim & Minda [The hyperbolic metric and spherically convex regions. *J. Math. Kyoto Univ.* **41** (2001), 297–314] and Kim & Sugawa [Charaterizations of hyperbolically convex regions. *J. Math. Anal. Appl.* **309** (2005), 37–51]. We show continuity of the concave function defined in the latter and show that the two notions of the concavity are equivalent for a function of class C^2. Moreover, we find more characterizations for spherically concave functions.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane. We denote $\mathbb{P}$ for the Riemann sphere and consider the spherical geometry for $\mathbb{P}$. Let

$$\lambda_X(w)|dw| = \frac{|dw|}{1+\varepsilon|w|^2} \text{ for } w \in X. \tag{$*$}$$

The metric given by $(*)$ is the *spherical metric* on $\mathbb{P}$ when $X = \mathbb{P}$ and $\varepsilon = 1$, and the *hyperbolic metric* on the unit disk $\mathbb{D}$ when $X = \mathbb{D}$ and $\varepsilon = -1$. We call $\Omega \subset \mathbb{P}$ is hyperbolic if its complement with respect to $\mathbb{P}$ has at least 3 points. For a hyperbolic region $\Omega \subset \mathbb{P}$ the hyperbolic metric $\lambda_\Omega(w)|dw|$ is uniquely determined from $\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z)$ for all $z \in \mathbb{D}$, where f is any meromorphic covering projection of the unit disk onto Ω. Note that the density λ_Ω is real analytic and hence it is smooth. Also, the hyperbolic metric has constant Gaussian curvature -4 (*cf.* Minda [4]) that is, $-\Delta \log \lambda_\Omega = -4\lambda_\Omega^2$.

Let X be either a hyperbolic region in $\mathbb{P}$ or $\mathbb{P}$ itself. The distance $d_X(A,B)$ between A and B in X measured by the metric $\lambda_X(w)|dw|$ is given by

$$d_X(A,B) = \inf \int_\gamma \lambda_X(w)|dw|,$$

Received by the editors July 26, 2005.
2000 *Mathematics Subject Classification.* 30F45.
Key words and phrases. spherically concave function, spherical metric, hyperbolic metric.

where the infimum is taken over all paths γ in X joining A and B. There exists an arc δ joining the two points A and B such that $d_X(A,B) = \int_\delta \lambda_X(w)|dw|$. We call such an arc δ a geodesic arc joining A and B in X. A geodesic arc is the shorter arc of the great circle joining two distinct points in $\mathbb{P}$ in the case of $X = \mathbb{P}$ and the part of the circular arc joining two distinct points in $\mathbb{D}$ which is perpendicular to the boundary of $\mathbb{D}$ in the case of $X = \mathbb{D}$. For example, the spherical distance between two points A and B is

$$d_{\mathbb{P}}(A,B) = \arctan\left|\frac{A-B}{1+\overline{A}B}\right|.$$

Note that $d_{\mathbb{P}}(A,B) \le \pi/2$ with equality if and only if A and B are antipodal.

We know that a real-valued function r defined on a plane region $\Omega \subset \mathbb{C}$ is called (*Euclidean*) *concave* in Ω if the inequality

$$r((1-t)w_0 + tw_1) \ge (1-t)r(w_0) + tr(w_1)$$

holds for every $t \in [0,1]$ whenever the line segment $[w_0, w_1]$ joining two points w_0 and w_1 in Ω is contained in Ω. Note that Ω need not be convex. In Kim & Minda [1], it is shown that the reciprocal of the hyperbolic density of the hyperbolic metric on $\Omega \subset \mathbb{C}$ is (Euclidean) concave when Ω is (Euclidean) convex.

The spherical concavity of a real-valued function of class C^2 was first defined by Kim & Minda [2]. They give some characterizations for the spherical concavity and show that the reciprocal of the spherical density of the hyperbolic metric on Ω is spherically concave on Ω if and only if a hyperbolic region $\Omega \subset \mathbb{P}$ is convex relative to the spherical metric on Ω. Here, the spherical density of the hyperbolic metric on Ω is $(1+|w|^2)\lambda_\Omega(w)$ which is the ratio of the hyperbolic metric on Ω to the spherical metric.

In this paper, we consider a geometric definition of spherical concavity which is analogus to the Euclidean concavity. This definition is briefly mentioned as a remark when characterizations for hyperbolically concave functions are established by Kim & Sugawa [3]. The geometric definition is as follows; Let Ω be a subregion of $\mathbb{P}$. A real-valued function r on Ω is said to be *spherically concave* if the inequality

$$r(w_t) \ge \frac{\sin(2(1-t)d)r(w_0) + \sin(2td)r(w_1)}{\sin(2d)} \tag{1}$$

holds for each $t \in [0,1]$, where $d = d_{\mathbb{P}}(w_0, w_1)$ and w_t is the unique point in Ω such that $d_{\mathbb{P}}(w_0, w_t) = td$ and that $d_{\mathbb{P}}(w_t, w_1) = (1-t)d$, for $w_0,\ w_1 \in \Omega$, whenever the geodesic arc γ (relative to the spherical geometry) joining w_0 and w_1 with $d_{\mathbb{P}}(w_0, w_1) < \pi/2$ lies entirely in Ω. We remark that the point w_t lies in γ necessarily.

For brief notation, we will write $w_t = P_t(w_0, w_1)$. Note that the above definition does not require a real-valued function to be of class C^2 while the other definition in Kim & Minda [2] does require.

In this paper, we show continuity of the spherically concave function defined geometrically (see Theorem 1). In Theorem 2, we show that the composition of spherically concave function with a certain function can be spherically concave. Also, we show that the two definitions of spherical concavity mentioned above are actually equivalent for a real valued function of class C^2 in Theorem 3. Moreover, we find more equivalent conditions for spherically concave function. Our methods of proofs of the Theorems 2 and 3 are analogous to the methods used for the hyperbolic concavity in Kim & Sugawa [3].

2. The invariant differential operators

There are differential operators relative to spherical geometry which are called spherical differential operators. We consider these differential operators for the region $\Omega \subset \mathbb{P}$ (*cf.* Kim & Minda [2]): For a C^2 function $r : \Omega \to \mathbb{R}$, we consider

$$\begin{aligned} \partial_s r &= \frac{1}{\lambda_{\mathbb{P}}} \partial r, \\ \partial_s^2 r &= \frac{1}{\lambda_{\mathbb{P}}^2} \left[\partial^2 r - 2(\partial_s \lambda_{\mathbb{P}})\partial r\right] = \frac{1}{\lambda_{\mathbb{P}}^2} \left[\partial^2 r - 2(\partial \log \lambda_{\mathbb{P}})\partial r\right], \\ \Delta_s r &= \frac{1}{\lambda_{\mathbb{P}}^2} \Delta r. \end{aligned}$$

Here, ∂ and $\bar\partial$ are the partial differential operators and the Laplacian $\Delta = 4\partial\bar\partial$. We recall that for $w = u + iv$ in the complex plane,

$$\begin{aligned} \partial &= \frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \\ \bar\partial &= \frac{\partial}{\partial \bar w} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \end{aligned}$$

We note that ∂_s^2 is not equal to $\partial_s \partial_s$ unlike the Euclidean case: $\partial^2 = \partial\partial$. In detail,

$$\partial_s^2 r(w) = \partial_s(\partial_s r(w)) + \overline{w}\partial_s r(w)$$

A behavior of these operators was observed under an isometry of the spherical metric or equivalently, a rotation of $\mathbb{P}$ in Kim & Minda [2]. In particular, the quantities $|\partial_s r|$, $|\partial_s^2 r|$ and $\Delta_s r$ are invariant under spherical isometries; For an isometry of the spherical metric, $T(z) = e^{i\theta}(z-a)/(1+\bar{a}z)$, we have

$$\begin{aligned}\partial_s(r\circ T) &= \frac{T'}{|T'|}[(\partial_s r)\circ T],\\ \partial_s^2(r\circ T) &= \left(\frac{T'}{|T'|}\right)^2[(\partial_s^2 r)\circ T],\\ \Delta_s(r\circ T) &= (\Delta_s r)\circ T.\end{aligned}$$

3. MAIN RESULTS

First, we show a continuity of spherically concave function. In the proof of the next theorem, set $D_{\mathbb{P}}(w_0, d) = \{w \in \mathbb{P} : d_{\mathbb{P}}(w_0, w) \le d\}$ for a point $w_0 \in \Omega$.

Theorem 1. *Suppose that Ω is a subregion of $\mathbb{P}$. A spherically concave function $r : \Omega \to \mathbb{R}$ is continuous.*

Proof. Let w_0 be an arbitrary point in Ω. First we assume that $r \ge -M$ in the neighborhood $V = D_{\mathbb{P}}(w_0, d_0) \subset \Omega$, where M and d_0 are positive constants. Note that for $t \in [0,1]$, $D_{\mathbb{P}}(w_0, td_0) = \{P_s(w_0, w) : 0 \le s \le t, w \in \partial V\}$. We take an arbitrary point w_1 on ∂V and set $w_t = P_t(w_0, w_1)$. Here, we may assume that $0 < d_{\mathbb{P}}(w_0, w_1) < \pi/4$. By (1), for each t where $0 \le t \le 1$, we get the inequality

$$r(w_t) - r(w_0) \ge \left(\frac{\sin(2(1-t)d_0)}{\sin(2d_0)} - 1\right) r(w_0) - \frac{\sin(2td_0)}{\sin(2d_0)} M \tag{2}$$

In order to get an upper bound of $[r(w_t) - r(w_0)]$, we choose a point $w_{-1} \in \partial V$ so that w_0 is the spherical midpoint of w_1 and w_{-1} in Ω. Since $w_0 = P_{1/(1+t)}(w_{-1}, w_t)$, we obtain the inequality

$$r(w_0) \ge \frac{\sin(2td_0)r(w_{-1}) + \sin(2d_0)r(w_t)}{\sin[2(1+t)d_0]}$$

by (1), and hence,

$$r(w_t) - r(w_0) \le \left(\frac{\sin(2(1+t)d_0)}{\sin(2d_0)} - 1\right) r(w_0) + \frac{\sin(2td_0)}{\sin(2d_0)} M. \tag{3}$$

Both of the right-hand sides in (2) and (3) tend to 0 as $t \to 0$. Hence we have shown that r is continuous at w_0 if it is locally bounded below.

On the other hand, the local lower boundedness of r can be easily shown. Indeed, consider a compact spherical triangle T in Ω. Then, by (1), the function r is bounded below on each side of T. Applying (1) again, we can deduce that r is bounded below on T. Since Ω is covered by such triangles, the local lower boundedness follows. □

Remark. For a convex region Ω relative to spherical geometry, $r(w) = 1/[(1+|w|^2)\lambda_\Omega(w)]$ is spherically concave as mentioned in Introduction. This is the ratio of the spherical metric to the hyperbolic metric on Ω.

Next, we show that the composition of spherically concave function with a certain function can be spherically concave.

Theorem 2. *Suppose that a concave and non-decreasing function $g : (0,b) \to \mathbb{R}$ satisfies that $g(x)/x$ is non-decreasing on $(0,b)$ where $b > 0$. If a spherically concave function $r : \Omega \subset \mathbb{P} \to \mathbb{R}$ takes its values in $(0,b)$, then the composed function $g \circ r$ is also spherically concave on Ω.*

Proof. We need to show the inequality (1) for $g \circ r$. First, we put

$$c = \frac{\sin(2(1-t)d) + \sin(2td)}{\sin(2d)} \quad \text{and} \quad s = \frac{\sin(2td)}{\sin(2(1-t)d) + \sin(2td)}.$$

Note that $c \geq 1$. Now the spherical concavity of r gives

$$r(w_t) \geq c[(1-s)r(w_0) + sr(w_1)].$$

By the condition that $g(x)/x$ is non-decreasing on $(0,b)$, $g(cx) \geq cg(x)$ for $x \in (0,b)$. This together with other conditions on g implies

$$\begin{aligned} g(r(w_t)) &\geq g(c[(1-s)r(w_0) + sr(w_1)]) \\ &\geq cg([(1-s)r(w_0) + sr(w_1)]) \\ &\geq c[(1-s)g(r(w_0)) + sg(r(w_1))], \end{aligned}$$

which is the desired inequality. □

Now, we give equivalent conditions for spherical concavity when r is of class C^2. First, we introduce a kind of minimum principle for solutions to a boundary value problem for an ordinary differential equation. See Walter [5] for a proof of the following result.

Lemma 1. *Let u and v be real-valued functions of class C^2 on the interval $[a,b]$ and suppose that $v'' \leq -4v$ and $u'' = -4u$ there. If $u(a) = v(a)$ and $u(b) = v(b)$, then either $v = u$ on $[a,b]$ or $v > u$ on (a,b).*

We establish characterizations of spherical concavity in the next theorem. Remember that we use geometric definition of spherical concavity here. In Kim & Minda [2], r is called *spherically concave* if r satisfies the condition (iii) of the next theorem.

Theorem 3. *Let Ω be a subregion of $\mathbb{P}$ and r be a real-valued function of class C^2 on Ω. Then the following are equivalent:*

(i) *r is spherically concave on Ω.*

(ii) *Whenever the geodesic arc joining w_0 and w_1 in Ω is contained in Ω, the midpoint m of it satisfies the inequality,*

$$r(m) \geq \frac{r(w_0) + r(w_1)}{2\cos d_{\mathbb{P}}(w_0, w_1)}. \tag{4}$$

(iii) *Whenever the geodesic arc $w(s)$ parametrized by spherical arclength is contained in Ω, the function $v(s) = r(w(s))$ satisfies the differential inequality $v''(s) + 4v(s) \leq 0$.*

(iv) *The inequality*

$$|\partial_s^2 r(w)| + \frac{1}{4}\Delta_s r(w) + 2r(w) \leq 0 \tag{5}$$

holds on Ω.

Proof. (i)⇒(ii): Just put $t = 1/2$ in the inequality (1).

(ii)⇒(iii): Let $w(s)$ be a geodesic arc in Ω parametrized by spherical arclength and set $s = s_0$. For $w_0 = w(s_0 - \delta)$ and $w_1 = w(s_0 + \delta)$, we obtain the inequality

$$v(s_0) \geq \frac{v(s_0 - \delta) + v(s_0 + \delta)}{2\cos(2\delta)}$$

by (4) or, equivalently,

$$\frac{v(s_0 - \delta) + v(s_0 + \delta) - 2v(s_0)}{\delta^2} \leq 2\,\frac{\cos(2\delta) - 1}{\delta^2}\,v(s_0).$$

Letting $\delta \to 0$, we obtain the inequality $v''(s_0) \leq -4v(s_0)$.

(iii)⇒(i): Let $w_0, w_1 \in \Omega$ and suppose that the geodesic arc joining w_0 and w_1 is contained in Ω. Set $w(s) = w_{s/d} = P_{s/d}(w_0, w_1)$ for $s \in [0, d]$, where $d = d_{\mathbb{P}}(w_0, w_1)$ and $u(s) = [\sin(2(d - s))\, r(w_0) + \sin(2s)\, r(w_1)] / \sin(2d)$. Then u satisfies the differential equation $u'' + 4u = 0$ and the boundary conditions $u(0) = r(w_0), u(d) = r(w_1)$. Applying Lemma 1 to the function $v(s) = r(w(s))$ yields the inequality $r(w_{s/d}) \geq u(s)$ for $s \in [0, d]$, which is the same as (1).

(iii)⇔(iv): Kim & Minda [2] give a proof for this relation. □

REFERENCES

1. S. Kim & D. Minda: The hyperbolic and quasihyperbolic metrics in convex regions. *J. Analysis* **1** (1993), 109–118. MR **94h**:30005
2. ______: The hyperbolic metric and spherically convex regions. *J. Math. Kyoto Univ.* **41** (2001), 285–302. MR **2002j**:30070
3. S. Kim & T. Śugawa: Charaterizations of hyperbolically convex regions. *J. Math. Anal. Appl.* **309** (2005), no. 1, 37–51. CMP 2154025
4. D. Minda: Hyperbolic curvature on Riemann surfaces. *Complex Variables Theory Appl.* **12** (1989), no. 1-4, 1–8. MR **91b**:30135
5. W. Walter: *Ordinary Differential Equations.* Translated from the sixth German edition (1996) by Russell Thompson. Graduate Texts in Mathematics, 182. Springer-Verlag, New York, 1998. MR **2001k**:34002

DEPARTMENT OF MATHEMATICS, WOOSUK UNIVERSITY, 490 HUJEONG-RI, SAMNYE-EUP, WANJU-GUN, JEONBUK 565-701, KOREA
Email address: `sakim@woosuk.ac.kr`