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NOVEL METHOD FOR CONSTRUCTING NEW WAVELET ANALYSIS

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ABSTRACT. In this paper, a new wavelet analysis of differential operator spline is generated, and it is of the symmetry and $(3-\varepsilon)$-order regularity $(0 < \varepsilon < 3)$. Finally, using this wavelet basis, we expand Lebesgue square integrable functions efficiently and quickly.

1. INTRODUCTION

Wavelets are studied by several authors for a long time (*cf.* Chui [1]; Chui & Wang [2, 3]; Cui [4]; Cui, Lee & Lee [5]; Daubechies [6]; Kontorovich & Krylov [7, 8]).

Chui & Wang [2, 3] constructed a multiresolution analysis based on polynomial spline function, from the view of operator spline, it is constructed by the solution of the fourth order differential equation

$$L(D)u \equiv D^4u(x) = \delta(x) \text{ and } g(x) = x_+^3,$$

where $x_+ = x$ for $x \geq 0$, $x_+ = 0$ for $x < 0$, through

$$V_k = \{V_k, g\} = \left\{u \,\middle|\, u(x) = \sum_{j\in\mathbb{Z}} C_{jk}\, g(2^jx - k),\ (C_{jk})_{j\in\mathbb{Z}} \in \ell^2\right\} \tag{1.1}$$

for $k \in \mathbb{Z}$.

When $L(D) \neq D^m$, $m \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, we may prove that, by using the solution of differential equation

$$L(D)u = \sum_{k\in\mathbb{Z}} C_k\delta(x-k),\ (C_k)_{k\in\mathbb{Z}} \in \ell^2$$

and the method of (1.1), it is not possible to obtain a multiresolution analysis.

In this paper, we get the following results:

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a. Based on a solution $g(x)$ of generalized differential equation, we obtain a wavelet analysis $\{V_j, g\}_{j\in\mathbb{Z}}$ defined by (1.1). Here, we directly defined a wavelet analysis not using multiresolution analysis. Then we prove that W_j=$\{W_j, g\}$ defined in this paper satisfies the definition of usual wavelet analysis.
b. This wavelet analysis is symmetry and has regular order which can be close to 3 arbitrarily.
c. A new method is given to expand a Lebesgue square integrable function according to this wavelet basis. It is worth to emphasize that when function is expanded using this method, it is enough to calculate some function-values, better than inner-product.

2. A Differential Operator Spline Function

Let D be the first differential operator, I be the unit operator, then it is easy to verify that for every $t \in \mathbb{R}$, $g(x-t) = \frac{1}{6}e^{-|x-t|} - \frac{1}{12}e^{-2|x-t|}$ is a solution of generalized differential equation

$$L(D) \equiv (D^4 - 5D^2 + 4I)u = \delta(x-t). \tag{2.1}$$

For a partition Π: $\{k\}_{k\in\mathbb{Z}}$, let us denote the spline function space of differential operator by $S(\Pi, L(D))$

$$S(\Pi, L(D)) = \left\{u \,\middle|\, u(x) = \sum_{k\in\mathbb{Z}} C_{0k}g(x-k),\ (C_{jk})_{j,\,k\in\mathbb{Z}} \in A\right\}, \tag{2.2}$$

where set $A = \{(C_{jk})_{j,\,k\in\mathbb{Z}} \mid C_{jk} \in \mathbb{R},\ \sum_{j,\,k\in\mathbb{Z}} C_{jk}^2 < +\infty\}$. In the following, we will write

$$W_0 = S(\Pi, L(D)).$$

It is obvious that $\sum_{k\in\mathbb{Z}} C_{0k}g(x-k)$ satisfies the following equation:

$$(D^4 - 5D^2 + 4I)u = \sum_{k\in\mathbb{Z}} C_{0k}\delta(x-t). \tag{2.3}$$

Taking Fourier transform for both sides of (4), we have

$$(1+\omega^2)(4+\omega^2)\hat{u}(\omega) = \sum_{k\in\mathbb{Z}} C_{0k}e^{-i\omega k}.$$

Therefore, W_0 can again be defined as

$$W_0 = \left\{u \,\middle|\, \hat{u}(\omega) = \frac{1}{(1+\omega^2)(4+\omega^2)}\mu_0(\omega),\ \mu_0 \in P(2\pi)\right\}, \tag{2.4}$$

where $P(2\pi)$ denotes the set of all functions with period equaling to 2π, $\{C_{0k}\}_{k\in\mathbb{Z}}$ denotes the Fourier coefficients of $\mu_0(\omega)$.

For every $j\in\mathbb{Z}$, let

$$W_j=\{W_j,g\}=\Big\{u\,\Big|\,u(x)=2^{\frac{j}{2}}\sum_{k\in\mathbb{Z}}C_{jk}\,g(2^jx-k),\ (C_{jk})_{j,k\in\mathbb{Z}}\in A\Big\}. \tag{2.5}$$

From (2.1), it follows that $g(2^jx-k)$ satisfies the following equation.

$$(2^{-4j}D^4-2^{-2j}5D^2+4I)u=\delta(2^jx-k). \tag{2.6}$$

Namely,

$$(2^{-4j}D^4-2^{-2j}5D^2+4I)u=2^{\frac{j}{2}}\sum_{k\in\mathbb{Z}}C_{jk}\,\delta(2^jx-k) \tag{2.7}$$

where $(C_{jk})_{j,k\in\mathbb{Z}}\in A$. Taking Fourier transform for both sides of (2.7), we get

$$(1+2^{-2j}5\omega^2+2^{-4j}4\omega^4)\hat{u}(\omega)=\sum_{k\in\mathbb{Z}}2^{-\frac{j}{2}}C_{jk}\,e^{-i2^{-j}k\omega}. \tag{2.8}$$

Hence, W_j can again be denoted as

$$W_j=\left\{u\,\Big|\,\hat{u}(\omega)=\frac{2^{-\frac{j}{2}}}{(1+2^{-2j}\omega^2)(4+2^{-2j}\omega^2)}\mu_j(\omega),\ \mu_j\in P(2^{j+1}\pi)\right\}, \tag{2.9}$$

where $P(2^{j+1}\pi)$ denotes the set of all functions with period $2^{j+1}\pi$, and $\{C_{jk}\}_{j,k\in\mathbb{Z}}$ can be denoted as the Fourier coefficients of $\mu_j(\omega)$. We have $(C_{jk})_{j,k\in\mathbb{Z}}\in A$.

3. The Properties of W_j

Lemma 3.1. *Suppose that* $g(x)=\frac{1}{6}\,e^{-|x|}-\frac{1}{12}\,e^{-2|x|}$, *then*

$$\begin{aligned}\sum_{k\in\mathbb{Z}}&|\hat{g}(\omega+2k\pi)|^2\\ &=\frac{1}{1728}\Big[\frac{16(1-12e-e^4+32(3-e+3e^2+e^3)\cos\omega)}{(1+e^2-2e\cos\omega)^2}\\ &\qquad+\frac{38(e^4-1)(1+e^4-2e^2\cos\omega)+24(-2e^2+(1+e^4)\cos\omega)}{(1+e^4-2e^2\cos\omega)^2}\Big],\end{aligned} \tag{3.1}$$

where $\hat{g}(\omega)=\dfrac{1}{(1+\omega^2)(4+\omega^2)}$ *is the Fourier transform of* $g(x)$.

Further we get that $\sum_{k\in\mathbb{Z}} |\,\hat{g}(\omega+2k\pi)|^2$ has the following positive upper boundary and low boundary (using Mathematica 4.0), respectively,

$$M \approx 0.0279948 \quad \text{and} \quad m \approx 0.0082535.$$

Theorem 3.2. *W_j, $j \in \mathbb{Z}$, defined by* (2.5), *have the following properties:*

(i) $W_i \bigcap W_j = \{0\}, \;\; i \neq j, \;\; i,j \in \mathbb{Z}$;
(ii) $\{g(x-k)\}_k \in \mathbb{Z}$ is a Riesz basis of W_0;
(iii) (translation invariance) *If* $u(x) \in W_0$, *then* $u(x-k) \in W_0$, $k \in \mathbb{Z}$;
(iv) (dilatation) *If* $u(x) \in W_j$, *then* $u(2x) \in W_{j+1}$; and
(v) $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$.

Proof. From the definition of W_j, we can easily obtain (iii) and (iv). And (ii) is the result of Lemma 3.1 (see Daubechies [6]). We will prove (i) by contradiction.

Let us assume $u(x) \in W_\ell \bigcap W_j$, $u(x) \neq 0$, $\ell, j \in \mathbb{Z}$, $\ell < j$. By (2.9) we get

$$\hat{u}(\omega) = \frac{2^{-\frac{\ell}{2}}}{(1+2^{-2\ell}\omega^2)(4+2^{-2\ell}\omega^2)}\mu_\ell(\omega) = \frac{2^{-\frac{j}{2}}}{(1+2^{-2j}\omega^2)(4+2^{-2j}\omega^2)}\mu_j(\omega)$$

where $\mu_\ell \in P(2^{\ell+1}\pi)$, $\mu_j \in P(2^{j+1}\pi)$. Since $\ell < j$, so $\mu_\ell(\omega) \in P(2^{j+1}\pi)$. Therefore

$$\frac{(1+2^{-2\ell}\omega^2)(4+2^{-2\ell}\omega^2)}{(1+2^{-2j}\omega^2)(4+2^{-2j}\omega^2)} \in P(2^{j+1}\pi),$$

which yields the contradiction.

In order to prove (v), we first give two lemmas.

Lemma A. *Suppose $\hat{u}(\omega) \in C_*^\infty(\mathbb{R})$, where $C_*^\infty(\mathbb{R}) = \hat{L}_2(\mathbb{R}) \bigcap C^\infty(\mathbb{R})$, $\hat{L}_2(\mathbb{R}) = \{\hat{u}(\omega) | u(x) \in L^2(\mathbb{R})\}$. And let $g(x) = \frac{1}{3}e^{-|x|} - \frac{1}{6}e^{-2|x|}$, then*

$$\begin{aligned}\sum_{k\in\mathbb{Z}} &\hat{u}(2^j(\omega+2k\pi))\,\hat{g}(\omega+2k\pi) \\ &= \frac{i}{24}\Big(2\hat{u}(-2^j i)\cot\frac{\omega+i}{2} - 2\hat{u}(2^j i)\cot\frac{\omega-i}{2} \\ &\qquad + \hat{u}(2^{j+1}i)\cot\frac{\omega-2i}{2} - \hat{u}(-2^{j+1}i)\cot\frac{\omega+2i}{2}\Big), \quad (3.2)\end{aligned}$$

where $\hat{g}$ and $\hat{u}$ denote the Fourier transforms of g and u, respectively, and

$$\hat{g}(\omega) = \frac{1}{(1+\omega^2)(4+\omega^2)}.$$

Proof. Take a square loop C_n with vertex $(n \pm \frac{1}{2})(\pm 1 \pm i)$ on the complex plane, then in the loop C_n function $\pi\coth(\pi z)$ has first order pole at $z = \pm k$, $k = 0, 1, \ldots, n$.

But $z = \pm k,\ k = 0, 1, \dots, n$, are not poles of the function

$$f(z) = \frac{\hat{u}(2^j(\omega + 2z\pi)}{(1 + (2\pi z + \omega)^2)(4 + (2\pi z + \omega)^2)}.$$

For a fixed ω, by choosing n big enough, $f(z)$ has the first order poles at

$$a_1 = -\frac{\omega - i}{2\pi},\ a_2 = -\frac{\omega + i}{2\pi},\ a_3 = -\frac{\omega - 2i}{2\pi},\ a_4 = -\frac{\omega + 2i}{2\pi},$$

then the residues of the function

$$\frac{1}{(1 + (2\pi z + \omega)^2)(4 + (2\pi z + \omega)^2)}$$

are $b_1 = -\frac{i}{12\pi},\ b_2 = \frac{i}{12\pi},\ b_3 = \frac{i}{24\pi},\ b_4 = -\frac{i}{24\pi}$, at $z = a_1, a_2, a_3, a_4$, respectively. By the residue theorem, we have

$$\int_{C_n} \pi \cot(\pi z) f(z)\, dz = 2\pi i\Big(\sum_{k=-n}^{n} f(k) + b_1 + b_2 + b_3 + b_4\Big) \tag{3.3}$$

Since on C_n, $|f(z)| = O(|z|^{-4})$, $|\cot(\pi z)| = O(1)\ as\ |z| \to \infty$,

$$\left|\int_{C_n} \pi \cot(\pi z)\ f(z)\, dz\right| \le \frac{C}{n^3} \to 0\ as\ n \to \infty,$$

where C is a constant. Now taking limits about n, in expression (3.3), we get

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \hat{u}(2^j(\omega + 2k\pi))\, \hat{g}(\omega + 2k\pi) \\
&\quad = \sum_{k\in\mathbb{Z}} \frac{\hat{u}\left(2^j(\omega + 2k\pi)\right)}{(1 + (2\pi k + \omega)^2)(4 + (2\pi k + \omega)^2)} \\
&\quad = \frac{i}{24}\Big(2\hat{u}(-2^j i) \cot\frac{\omega + i}{2} - 2\hat{u}(2^j i)\cot\frac{\omega - i}{2} \\
&\qquad\qquad + \hat{u}(2^{j+1} i)\cot\frac{\omega - 2i}{2} - \hat{u}(-2^{j+1} i)\cot\frac{\omega + 2i}{2}\Big).
\end{aligned}$$

□

Lemma B. $\{g(2^j x - k)\}_{j,\,k\in\mathbb{Z}}$ *is a complete system of* $L^2(\mathbb{R})$.

Proof. $\{g(2^j x - k)\}_{j,\,k\in\mathbb{Z}}$ is a complete system of $L^2(\mathbb{R})$ if and only if

$$\{(g(2^j x - k))^\wedge(\omega)\}$$

is a complete system of $\hat{L}^2(\mathbb{R})$. $C_*^\infty(\mathbb{R})$ is dense in $\hat{L}^2(\mathbb{R})$ obviously, so we only need to show $\{(g(2^j x - k))^\wedge(\omega)\}_{j,\,k\in\mathbb{Z}}$ is a complete system in $C_*^\infty(\mathbb{R})$. In fact, for any $\hat{u}(\omega) \in C_*^\infty(\mathbb{R})$, if

$$\big(\hat{u}(\omega),\ (g(2^j x - k))^\wedge(\omega)\big)_{\hat{L}^2} = 0,$$

then, for $k \in \mathbb{Z}$,

$$\begin{aligned}\big(\hat{u}(\omega),\ (g(2^j x - k))^\wedge(\omega)\big)_{\hat{L}^2} &= 2^{-j}\int_{\mathbb{R}} \hat{u}(\omega)\,\hat{g}(2^{-j}\omega)\,e^{-2^{-j}\omega k i}\,d\omega \\ &= \int_{\mathbb{R}} \hat{u}(2^j\omega)\,\hat{g}(\omega)\,e^{-\omega k i}\,d\omega \\ &= \sum_{\ell\in\mathbb{Z}}\int_{2\pi\ell}^{2\pi(\ell+1)} \hat{u}(2^j\omega)\,\hat{g}(\omega)\,e^{-\omega k i}\,d\omega \\ &= \int_0^{2\pi}\Big[\sum_{\ell\in\mathbb{Z}} \hat{u}(2^j(\omega+2\ell\pi))\,\hat{g}(\omega+2\ell\pi)\Big]\,e^{-\omega k i}\,d\omega \\ &= 0,\end{aligned}$$

thus, for $\omega \in [0, 2\pi]$,

$$\sum_{\ell\in\mathbb{Z}} \hat{u}(2^j(\omega+2\ell\pi))\,\hat{g}(\omega+2\ell\pi)) = 0.$$

From Lemma A, it follows that, for $\omega \in [0, 2\pi]$,

$$\begin{aligned}&2\hat{u}(-2^j i)\cot\frac{\omega+i}{2} - 2\hat{u}(2^j i)\cot\frac{\omega-i}{2} \\ &\qquad + \hat{u}(2^{j+1}i)\cot\frac{\omega-2i}{2} - \hat{u}(-2^{j+1}i)\cot\frac{\omega+2i}{2} = 0. \qquad (3.4)\end{aligned}$$

Since $\cot(\frac{\omega+i}{2})$, $\cot(\frac{\omega-i}{2})$, $\cot(\frac{\omega+2i}{2})$ and $\cot(\frac{\omega-2i}{2})$ are linearly independent, we have

$$\hat{u}(-2^j i) = \hat{u}(2^j i) = 0, \quad j \in \mathbb{Z}. \qquad (3.5)$$

Owing to $\hat{u}(\omega) \in C_*^\infty(\mathbb{R})$, it follows that $\hat{u}(z)$ is an analytic function. Therefore from (3.5) $\hat{u}(\omega) \equiv 0$. This completes the proof of Lemma B. □

Now, we set up to prove (v) in Theorem 3.2.

Using the completeness of $\{g(2^j x - k)\}_{j,k\in\mathbb{Z}}$ in $L^2(\mathbb{R})$. We have, for every $u \in L^2(\mathbb{R})$, there exists $(C_{jk})_{j,k\in\mathbb{Z}} \in A$, such that

$$u(x) = \sum_{j,k\in\mathbb{Z}} C_{jk}\,g(2^j x - k) = \sum_{j\in\mathbb{Z}} u_j, \quad u_j \in W_j \qquad (3.6)$$

Hence $L^2(\mathbb{R}) \subset \bigoplus_{j\in\mathbb{Z}} W_j$. On the other hand, for $u_j \in W_j$, we have

$$u_j(x) = 2^{\frac{j}{2}} \sum_{k\in\mathbb{Z}} C_{jk}\,g(2^j x - k).$$

So that,

$$\hat{u}_j(\omega) = 2^{-\frac{j}{2}} \sum_{k\in\mathbb{Z}} C_{jk}\,e^{-i2^{-j}\omega k}\hat{g}(2^{-j}\omega)$$

$$= 2^{-\frac{j}{2}} \sum_{k\in\mathbb{Z}} C_{jk}\, e^{-i2^{-j}\omega k} \frac{1}{(1+(2^{-j}\omega)^2)(4+(2^{-j}\omega)^2)}$$
$$= \frac{2^{-\frac{j}{2}}\mu_j(\omega)}{(1+(2^{-j}\omega)^2)(4+(2^{-j}\omega)^2)},$$

where $\mu_j(\omega) = \sum_{k\in\mathbb{Z}} C_{jk}\, e^{-i2^{-j}k\pi} \in P(2^{j+1}\pi)$.

Hence:

$$\|u_j(x)\|^2_{\ell^2} = \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{u}_j(\omega)|^2\, d\omega$$
$$= \frac{2^{-j}}{2\pi}\int_{\mathbb{R}} \left|\frac{1}{(1+(2^{-j}\omega)^2)(4+(2^{-j}\omega)^2)}\mu_j(\omega)\right|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{((1+\omega^2)(4+\omega^2))^2}|\mu_j(2^j\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\int_{2\ell\pi}^{2(\ell+1)\pi} \frac{1}{((1+\omega^2)(4+\omega^2))^2}|\mu_j(2^j\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\int_{0}^{2\pi} \frac{1}{((1+(\omega+2\ell\pi)^2)(4+(\omega+2\ell\pi)^2))^2}|\mu_j(2^j\omega)|^2 d\omega.$$

Using Lemma 3.1, we obtain

$$\|u_j(x)\|^2_{\ell^2(\mathbb{R})} \le K\int_0^{2\pi} |\mu_j(2^j\omega)|^2\, d\omega$$
$$= 2^{-j}K\int_0^{2^{j+1}\pi} |\mu_j(\omega)|^2\, d\omega = K\sum_{k\in\mathbb{Z}} C_{jk}^2 < +\infty.$$

Further, from the condition $\sum_{j,k\in\mathbb{Z}} C_{jk}^2 < \infty$, it follows that $u = \sum_j u_j \in L^2(\mathbb{R})$. So,

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j. \qquad \square$$

4. The Definition of Wavelet Analysis of Differential Operator Spline

Let D be a differential operator and $L(D)$ be an m-order polynomial about D with constant coefficients, $g(x-t)$ as the solution of generalized differential equation:

$$L(D)u = \delta(x-t), \quad t\in\mathbb{R}. \tag{4.1}$$

For a partition $\Pi : \{k\}_{k\in\mathbb{Z}}$, let

$$S(\Pi, L) = \Big\{u \,|\, u(x) = \sum_{k\in\mathbb{Z}} C_{0k} g(x-k),\ (c_{jk})_{j,\,k\in\mathbb{Z}} \in A\Big\}. \tag{4.2}$$

And write $W_0 = S(\Pi, L)$. We know that $\sum_{k\in\mathbb{Z}} C_{0k} g(x-k)$ satisfies the following differential equation:

$$L(D)u = \sum_{k\in\mathbb{Z}} C_{0k}\delta(x-k). \tag{4.3}$$

Take Fourier transform for both sides of (4.3), we get

$$[L(D)u]^\wedge(\omega) \equiv L(i\omega)\hat{u}(\omega) = \sum_{k\in\mathbb{Z}} C_{0k} e^{-i\omega k}. \tag{4.4}$$

Therefore, W_0 can again be defined as

$$W_0 = \Big\{u \,|\, \hat{u}(\omega) = \frac{1}{L(i\omega)}\mu_0(\omega),\ \mu_0 \in P(2\pi)\Big\}, \tag{4.5}$$

where $P(2\pi)$ is the set of all functions with period equaling to 2π and the Fourier coefficients $\{C_{0k}\}_{k\in\mathbb{Z}}$ of the function $\mu_0(\omega)$ satisfies $(C_{jk})_{j,\,k\in\mathbb{Z}} \in A$.

For $j \in \mathbb{Z}$, let

$$W_j = \Big\{u \,|\, u(x) = 2^{\frac{j}{2}} \sum_{k\in\mathbb{Z}} C_{jk}\, g(2^j x - k),\ (C_{jk})_{j,\,k\in\mathbb{Z}} \in A\Big\}. \tag{4.6}$$

From (4.1), it follows that $g(2^j x - k)$ satisfies differential equation

$$L(2^{-j}D)u = \delta(2^j x - t), \quad t \in \mathbb{R} \tag{4.7}$$

or

$$u(x) = 2^{\frac{j}{2}} \sum_{k\in\mathbb{Z}} C_{jk}\, g(2^j x - k)$$

satisfies differential equation

$$L(2^{-j}D)u = 2^{\frac{j}{2}} \sum_{k\in\mathbb{Z}} C_{jk}\, \delta(2^j x - k),\ (C_{jk})_{j,\,k\in\mathbb{Z}} \in A. \tag{4.8}$$

Take Fourier transform for both sides of (24),we get

$$L(2^{-j}i\omega)\hat{u}(\omega) = 2^{-\frac{j}{2}} \sum_{k\in\mathbb{Z}} C_{jk}\, e^{-i2^{-j}\omega k}. \tag{4.9}$$

Therefore, W_j can again be denoted as

$$W_j = \Big\{u \,\Big|\, \hat{u}(\omega) = 2^{-\frac{j}{2}} \frac{\mu_j(\omega)}{L(2^{-j}i\omega)},\ \mu_j \in P(2^{j+1}\pi)\Big\}. \tag{4.10}$$

Definition 4.1. Suppose that $g(2^j x - k)$ *and* W_j, $j, k \in \mathbb{Z}$, defined above. If W_0 is a closed subspace of $L^2(\mathbb{R})$ and W_j satisfies the following conditions:

(i) $W_\ell \bigcap W_j = \{0\}$, $\ell \neq j$, $\ell,\ j \in \mathbb{Z}$;
(ii) $\{g(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz base of W_0;
(iii) If $u(x) \in W_0$, then $u(x-k) \in W_0, k \in \mathbb{Z}$;
(iv) If $u(x) \in W_j$, then $u(2x) \in W_{j+1}, j \in \mathbb{Z}$; and
(v) $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$,

then $\{W_j, g\}$ is called a *wavelet analysis* about differential operator $L(D)$, g is called a *wavelet basic function.*

By the Definition 4.1 and Theorem 3.2, we get the following theorem

Theorem 4.1. *Let $L(D) \equiv D^4 - 5D^2 + 4I$. By using the function*

$$g(x-t) = \frac{1}{3}\, e^{-|x-t|} - \frac{1}{6}\, e^{-2|x-t|}$$

where g is a solution of (4.1)*, we get that W_j, (W_j, g), defined by* (4.6)*, is a wavelet analysis of differential operator $L(D) = D^4 - 5D^2 + 4I$.*

5. Theorem of Expanding

In this section, by using the method of differential operator spline wavelet of $L(D) = D^4 - 5D^2 + 4I$, we will discuss the problem of expanding functions. In this paper, we only discuss the expanding of functions which belong to the subset of $L^2(\mathbb{R})$. Let

$$H^1(\mathbb{R}) = \{u(x) \mid u, u' \text{ are absolutely continuous functions and } u, u', u'' \in L^2(\mathbb{R})\}$$

We define the inner product of $H^1(\mathbb{R})$ as follows, for $j = 0, 1, 2, \ldots$,

$$\begin{aligned} &\big(u(x), v(x)\big)_{H_j^1} \\ &= \int_{\mathbb{R}} \left(4 \times 2^j u(x)v(x) - 5 \times 2^{-j} u'(x)v'(x) + 2^{-3j} u''(x)v''(x)\right)\, dx \\ &= \frac{2^j}{2\pi} L(2^{-j} i\omega)\hat{u}(\omega)\overline{\hat{v}(\omega)}\, d\omega \\ &= \frac{2^j}{2\pi} \int_{\mathbb{R}} (1 + (2^{-j}\omega)^2)(4 + (2^{-j}\omega)^2)\hat{u}(\omega)\overline{\hat{v}(\omega)}\, d\omega \end{aligned} \tag{5.1}$$

then $H^1(\mathbb{R})$ is clearly an inner-product space. We denote $H^1(\mathbb{R})$ by $H_j^1(\mathbb{R})$ according to different inner products.

Theorem 5.1. *Let* $g(x) = \frac{1}{6}\, e^{-|x|} - \frac{1}{12}\, e^{-2|x|}$, *and let*

$$\hat{\Phi}(\omega) = \frac{\hat{g}(\omega)}{[\sum_{k\in\mathbb{Z}}(|\hat{g}(\omega+2k\pi)|^2(1+(\omega+2k\pi)^2)(4+(\omega+2k\pi)^2)]^{\frac{1}{2}}},$$

then $\{\Phi(x-k)\}_{k\in\mathbb{Z}}$ *is an orthonormal system of* $H_0^1(\mathbb{R})$.

Proof. Clearly, for every $\ell, k \in \mathbb{Z}$, one has

$$\begin{aligned}
&(\Phi(x-k), \Phi(x-\ell))_{H_0^1(\mathbb{R})} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} (1+\omega^2)(4+\omega^2)\Phi(x-k)^\wedge(\omega)\overline{\Phi(x-\ell)^\wedge(\omega)}\, d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} (1+\omega^2)(4+\omega^2)|\hat{\Phi}(\omega)|^2\, e^{-i(k-\ell)\omega}\, d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(1+\omega^2)(4+\omega^2)|\hat{g}(\omega)|^2 e^{-i(k-\ell)\omega}}{\sum_{k\in\mathbb{Z}}[|\hat{g}(\omega+2k\pi)|^2(1+(\omega+2k\pi)^2)(4+(\omega+2k\pi)^2)]}\, d\omega \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{2n\pi}^{2(n+1)\pi} \frac{(1+\omega^2)(4+\omega^2)|\hat{g}(\omega)|^2 e^{-i(k-\ell)\omega}}{\sum_{k\in\mathbb{Z}}[|\hat{g}(\omega+2k\pi)|^2(1+(\omega+2k\pi)^2)(4+(\omega+2k\pi)^2)]}\, d\omega \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{0}^{2\pi} \frac{|\hat{g}(\omega+2n\pi)|^2}{\sum_{k\in\mathbb{Z}}|\hat{g}(\omega+2k\pi)|^2} e^{-i(k-\ell)\omega}\, d\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k-\ell)\omega}\, d\omega \\
&= \begin{cases} 1, & \ell = k, \\ 0, & \ell \neq k. \end{cases}
\end{aligned}$$

□

Therefore, we obtain

$$\hat{\Phi}(\omega) = \frac{2}{(1+\omega^2)(4+\omega^2)(e-1)}\sqrt{\frac{3(1+e^2-2e\cos\omega)(1+e^4-2e^2\cos\omega)}{(e-1)(1+e)((1+e)^2+2e\cos\omega)}}. \quad (5.2)$$

Corollary 5.1. $\{\Phi(2^j x-k)\}_{k\in\mathbb{Z}}$ *is an orthonormal system of* $H_j^1(\mathbb{R})$.

Lemma 5.2. *Let* $u(x) \in H_j^1(\mathbb{R})$, *then*

$$\big(u, g(2^j x-k)\big)_{H_j^1} = u(2^{-j}k) \quad (5.3)$$

where $g(x) = \dfrac{1}{6}\, e^{-|x|} - \dfrac{1}{12}\, e^{-2|x|}$ *is one solution of equation* (2.1).

Proof. By assumption,

$$\begin{aligned}
&\big(u, g(2^j x-k)\big)_{H_j^1} \\
&= 4\times 2^j \int_{\mathbb{R}} u(x)g(2^j x-k)\, dx - 5\times 2^{-j}\int_{\mathbb{R}} Du(x)Dg(2^j x-k)\, dx
\end{aligned}$$

$$+ 2^{-3j} \int_{\mathbb{R}} D^2u(x)D^2g(2^jx-k)\,dx$$
$$= 4\times 2^j \int_{\mathbb{R}} u(x)g(2^jx-k)\,dx - 5\int_{\mathbb{R}} Du(x)g'(2^jx-k)\,dx$$
$$+ 2^{-j} \int_{\mathbb{R}} D^2u(x)g''(2^jx-k)\,dx$$
$$= 4\times \int_{\mathbb{R}} u(2^{-j}x)g(x-k)\,dx - 5\times 2^{-j}\int_{\mathbb{R}} Du(2^{-j}x)g'(x-k)\,dx$$
$$+ 2^{-2j} \int_{\mathbb{R}} D^2u(2^{-j}x)g''(x-k)\,dx$$
$$= 4\int_{\mathbb{R}} u(2^{-j}x)g(x-k)\,dx - 5\int_{\mathbb{R}} u'(2^{-j}x)g'(x-k)\,dx + \int_{\mathbb{R}} u''(2^{-j}x)g''(x-k)\,dx$$
$$= \int_{\mathbb{R}} u(2^{-j}x)\delta(x-k)\,dx$$
$$= u(2^{-j}k).$$

Therefore, we have $\big(u, g(2^jx-k)\big)_{H_j^1} = u(2^{-j}k)$. □

Similarly, we have the following lemmas.

Lemma 5.3. *Let $u(x) \in H^1(\mathbb{R})$, then*

$$(u(x), \Phi(x-k))_{H_0^1} = [u * h](k) \tag{5.4}$$

where $\hat{\Phi}(\omega)$ is given by (5.2).

$$\hat{h}(\omega) = \frac{2}{(e-1)}\sqrt{\frac{3(1+e^2-2e\cos\omega)(1+e^4-2e^2\cos\omega)}{(e-1)(1+e)((1+e)^2+2e\cos\omega)}}\ , \tag{5.5}$$

and $[u*h](x)$ is a convolution $u(x)$ and $h(x)$.

Lemma 5.4. *Let $u(x) \in H^1(\mathbb{R})$, then*

$$(u(x), \Phi(2^jx-k))_{H_j^1} = \left[u(2^{-j}\cdot) * h(\cdot)\right](k) \tag{5.6}$$

where $h(x)$ is given by (5.5).

Let $P_j : H_j^1(\mathbb{R}) \longrightarrow W_j$ be an orthonormal projection operator, where W_j is taken as a subspace of $H_j^1(\mathbb{R})$:

$$P_ju = \sum_{k\in\mathbb{Z}} (u(x), \Phi(2^jx-k))_{H_j^1}\Phi(2^jx-k). \tag{5.7}$$

We define a sequence of functions r_n according to the following formulas.

$$r_0(x) = u(x),$$

$$
\begin{aligned}
r_1(x) &= r_0(x) - P_0 r_0(x), \\
&\vdots \\
r_n(x) &= r_{n-1}(x) - P_{n-1} r_{n-1}(x).
\end{aligned} \tag{5.8}
$$

Lemma 5.5. *Suppose that $r_n(x)$ is defined by* (5.8)*, then*

$$
r_n(2^{-j}k) = 0, \quad j \le n-1, \quad k \in \mathbb{Z}, \quad n \ge 1. \tag{5.9}
$$

Proof. From $2^{-(j-1)}k = 2^{-j}(2k)$, it is enough to show that the conclusion is valid for $j = n-1$.

By (5.3), we have

$$
\begin{aligned}
& r_n(2^{-(n-1)}k) \\
&= \left(r_n(x),\ g(2^{n-1}x - k)\right)_{H^1_{n-1}} \\
&= \left(r_{n-1}(x) - P_{n-1}r_{n-1}(x),\ g(2^{n-1}x - k)\right)_{H^1_{n-1}} \\
&= \left(r_{n-1}(x),\ g(2^{n-1}x - k)\right)_{H^1_{n-1}} - \left(P_{n-1}r_{n-1}(x),\ g(2^{n-1}x - k)\right)_{H^1_{n-1}},
\end{aligned}
$$

and, moreover

$$
\begin{aligned}
\left(P_{n-1}r_{n-1}(x)\,,\ g(2^{n-1}x - k)\right)_{H^1_{n-1}} &= \left(r_{n-1}(x)\,,\ P_{n-1}g(2^{n-1}x - k)\right)_{H^1_{n-1}} \\
&= \left(r_{n-1}(x)\,,\ g(2^{n-1}x - k)\right)_{H^1_{n-1}},
\end{aligned}
$$

where we used

$$
P_{n-1}g(2^{n-1}x - k) = g(2^{n-1}x - k).
$$

So that $r_n(2^{-(n-1)}k) = 0$, namely,

$$
r_n(2^{-j}k) = 0, \quad j \le n-1, \quad j \in \mathbb{Z}, \quad n \ge 1.
$$

□

Corollary 5.2. *Suppose that P_j is defined by* (5.7) *and r_j by* (5.8)*,*

$$
P_j r_j(x) = \sum_{k\in\mathbb{Z}} \left[r_j(2^{-j}\cdot) * h(\cdot)\right](k)\Phi(2^j x - k). \tag{5.10}
$$

Proof. It follows from (5.6) and (5.7) that

$$
\begin{aligned}
P_j r_j(x) &= \sum_{k\in\mathbb{Z}} (r_j(x), \Phi(2^j x - k))_{H^1_j} \Phi(2^j x - k) && \text{(By (5.7))} \\
&= \sum_{k\in\mathbb{Z}} \left[r_j(2^{-j}\cdot) * h(\cdot)\right](k)\Phi(2^j x - k). && \text{(By (5.6))}
\end{aligned}
$$

This complete the proof. □

Now we set up to establish the expanding theorem of $u(x) \in H^1(\mathbb{R})$.

Theorem 5.6. *Suppose $u(x) \in H^1(\mathbb{R})$, then*

$$u(x) = \sum_{j=0}^{\infty} P_j r_j(x). \tag{5.11}$$

Proof. It is obvious that

$$u(x) = \sum_{\ell=0}^{n-1} P_\ell r_\ell(x) + r_n(x),$$

therefore, we only need to show

$$r_n(x) \to 0 \ as \ n \to \infty,$$

for $x \in \mathbb{R}$.

(1) Take any finite interval $[a, b] \subset \mathbb{R}$ and $x \in (a, b)$, then there exist $N > 0$, k_1, $k_2 \in \mathbb{Z}$ so that when $n > N$, $2^{-(n-1)}k_i \in [a, b]$, $i = 1, 2$. Noting that

$$r_n(2^{-(n-1)}k_i) = 0,$$

we have $x_0 \in [a, b]$, and $r_n'(x_0) = 0$, hence

$$r_n'(x) = \int_{x_0}^{x} r_n''(t)dt$$

and

$$\begin{aligned} |r_n'(x)| &\le \int_{x_0}^{x} |r_n''(t)|dt| \le \int_a^b |r_n''(t)|dt \\ &\le \sqrt{b-a}\sqrt{\int_a^b |r_n''(t)|^2 dt} \le \sqrt{b-a}\sqrt{\int_{\mathbb{R}} |r_n''(t)|^2 dt} \le \sqrt{b-a}\|r_n\|_{H_n^1} \end{aligned}$$

From (5.8), it follows that

$$\|r_n\|_{H_n^1}^2 = \|r_{n+1}\|_{H_n^1}^2 + \|P_n r_n\|_{H_n^1}^2,$$

therefore

$$\|r_{n+1}\|_{H_n^1}^2 \le \|r_n\|_{H_n^1}^2.$$

Further by (5.1), it is also true that

$$\|r_{n+1}\|_{H_{n+1}^1}^2 \le \|r_{n+1}\|_{H_n^1}^2,$$

which means that

$$0 \le \|r_{n+1}\|_{H_{n+1}^1}^2 \le \|r_n\|_{H_n^1}^2,$$

that is, $\|r_n(x)\|_{H_n^1}^2$ is a monotonic decreasing sequence, thus, there exists an integer J such that if $n > J$

$$|r_n'(x)| \le 2\sqrt{b-a}C \tag{5.12}$$

where $\lim_{n\to\infty} \|r_n(x)\| = C$.

(2) For any $\varepsilon > 0$, take $J_0 \in N$. If $n > J_0$, then there is $k \in \mathbb{Z}$ such that $|x - 2^{-(n-1)}k| < \frac{\varepsilon}{2\sqrt{b-aC}}$. From (5.9), (5.12) for $n > J_0$, we have

$$\begin{aligned}|r_n(x)| &= |r_n(x) - r_n(2^{-(n-1)}k)| \\ &= |r_n'(\xi)||x - 2^{-(n-1)}k| < \varepsilon.\end{aligned}$$

Consequently, we obtain $\lim_{n\to\infty} r_n(x) = 0$. □

6. REGULAR ANALYSIS OF THE WAVELET BASIC FUNCTION BASED ON $L(D) = D^4 - 5D^2 + 4I$

Since $g(x) = \dfrac{1}{6}e^{-|x|} - \dfrac{1}{12}e^{-2|x|}$ is a wavelet basic function based on $L(D) \equiv D^4 - 5D^2 + 4I$, the Fourier transform of $g(x)$ is

$$\widehat{g}(\omega) = \frac{1}{(4 + 5\omega^2 + \omega^4)}.$$

Therefore, for every $\varepsilon > 0$, $0 < \varepsilon < 3$

$$\int_{\mathbb{R}} (1 + |\omega|)^{3-\varepsilon}\widehat{g}(\omega)\, d\omega < \infty, \tag{6.1}$$

where $g(x)$ is an even function. Hence, we obtain the following theorem.

Theorem 6.1. *The wavelet analysis $\{W_j, g_{j\in\mathbb{Z}}\}$ based on $L(D) = D^4 - 5D^2 + 4I$ is $(3 - \varepsilon)$-order regular analysis $(0 < \varepsilon < 3)$ with symmetry.*

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